Abstract. We explain that any closed immersion between Shimura varieties defined by morphisms of Shimura data extends to some closed immersion between their projective smooth toroidal compactifications, up to refining the choices of cone decompositions. We also explain that the same holds for many closed immersions between integral models of Shimura varieties and their toroidal compactifications available in the literature.

1. Introduction

Given any closed immersion between Shimura varieties or their integral models defined by some morphism of Shimura data (and some additional data, in the case of integral models), it is natural to ask whether it extends to a closed immersion between their toroidal compactifications. Since the construction of toroidal compactifications depends on the choices of some compatible collections of cone decompositions, part of the question is whether this can be achieved by some good choices of them, which we might want to be refinements of some given ones.

This question is not as trivial as it seems to be. Already in characteristic zero, the analogous question for minimal compactifications is subtle. In fact, in Scholze’s groundbreaking work [27], for Hodge-type Shimura varieties, his “perfectoid minimal compactifications” at infinite levels were first constructed using the closures in the minimal compactifications of Siegel modular varieties, rather than the minimal compactifications of the Hodge-type Shimura varieties themselves; but the morphism from the minimal compactification of the Shimura variety to the closure in...
Acknowledgements. I would like to thank Christopher Skinner for bringing this question to my attention; and to thank Xin Wan and the Morningside Center of Mathematics, and Chia-Fu Yu and the Academia Sinica, for their hospitality during the preparation of this work. I would also like to thank David Loeffler and David Hansen for helpful questions which led to crucial improvements and corrections. Finally, I would like to thank Hélène Esnault and Michael Harris for encouraging me to include an application to the verification of [9, Hypothesis 2.18], and to thank the anonymous referee for helpful comments.

2. Main results

Let us assume we are in one of the following cases:

Assumption 2.1. (1) For each \( i = 0, 1 \), let \((G_i, \mathcal{D}_i)\) be a Shimura datum (see [8, 1.2.1]), where \( \mathcal{D}_i \) is a \( G_i(\mathbb{R}) \)-conjugacy class of a homomorphism \( h_i : \text{Res}_{\mathbb{C}/\mathbb{R}} G_{i, \mathbb{C}} \to G_i(\mathbb{R}) \). Let \( \rho : G_0 \to G_1 \) be an injective homomorphism of algebraic groups over \( \mathbb{Q} \) such that \( (\rho(\mathbb{R}))(D_0) \subset D_1 \). Let \( \mathcal{H}_i \subset \mathcal{G}_i(\mathbb{A}^\infty) \) be neat (see [25, 0.6]) open compact subgroups, for \( i = 0, 1 \), such that \( \mathcal{H}_0 = (\rho(\mathbb{A}^\infty))^{-1}(\mathcal{H}_1) \). Let \( F \) denote a subfield of \( \mathbb{C} \) containing the reflex field of \((G_0, D_0)\) (which then also contains that of \((G_1, D_1)\) by [8, 2.2.1]), and let \( S_0 := \text{Spec}(F) \). For each \( i = 0, 1 \), let \( X_i \) denote the base change to \( F \) of the canonical model of the Shimura variety associated with \((G_i, \mathcal{D}_i)\) at level \( \mathcal{H}_i \). Then we have a canonical morphism \( f : X_0 \to X_1 \) over \( S_0 \), which we assume to be a closed immersion. (This can be achieved up to replacing \( \mathcal{H}_1 \) with a finite index subgroup still containing \((\rho(\mathbb{A}^\infty))(\mathcal{H}_0)\), by [7, 1.15].)

(2) For each \( i = 0, 1 \), let \((O_1, *, L_1, \langle \cdot, \cdot \rangle_1, h_1)\) be an integral PEL datum (see [13, Def. 1.1.1.1]). Assume that \( O_1 \) is a subring of \( O_0 \) preserved by \( *_0 \), that \( *_1 = *_0|_{O_1} \), and that \((L_0, \langle \cdot, \cdot \rangle_0, h_0) \cong (L_1, \langle \cdot, \cdot \rangle_1, h_1)\) as PEL-type \( O_1 \)-lattices (see [16, Def. 1.2.1.3]). For each \( i = 0, 1 \), let \( G_i \) denote the associated group functor over \( \text{Spec}(\mathbb{Z}) \), as in [16, Def. 1.2.1.6], so that we have a canonical injective homomorphism \( \rho : G_0 \to G_1 \) by definition. Let \( F \) denote a subfield of \( \mathbb{C} \) that is a finite extension of the reflex field \( F_0 \) of \((O_0, *, L_0, \langle \cdot, \cdot \rangle_0, h_0)\) (see [16, Def. 1.2.5.4]) (which is also the reflex field of \((G_0 \otimes \mathbb{Q}, G_0(\mathbb{R}) \cdot h_0)\), and hence also that of \( F_1 \) of...
For $i \in \mathbb{Z}$, consider $G_i(\mathbb{A}^\infty_\mathbb{C})$, for $i = 0,1$, such that $\mathcal{H}_0 = (\rho(\mathbb{A}^\infty_\mathbb{C}))^{-1}(\mathcal{H}_1)$. For each $i = 0,1$, let $\mathbf{M}_i$ denote the (smooth) moduli scheme over $\text{Spec}(\mathcal{O}_{F,(\mathcal{O})})$ associated with $(\mathcal{O}_i, \star_i, L_i, \langle \cdot, \cdot \rangle_i, h_i)$ at $\mathcal{H}_i$ (see [10] Def. 1.4.1.4, Thm. 1.4.1.11, and Cor. 7.2.3.10]). By restricting the $\mathcal{O}_0$-endomorphism structures parameterized by $\mathbf{M}_i$ to $\mathcal{O}_1$-endomorphism structures, we obtain a canonical morphism $\mathbf{M}_i \otimes_{\mathcal{O}_{F,(\mathcal{O})}} \mathcal{O}_{F,(\mathcal{O})} \to \mathbf{M}_i \otimes_{\mathcal{O}_{F,(\mathcal{O})}} \mathcal{O}_{F,(\mathcal{O})}$. Then we take $X_0$ and $X_1$ to be open-and-closed subschemes of $\mathbf{M}_i \otimes_{\mathcal{O}_{F,(\mathcal{O})}} \mathcal{O}_{F,(\mathcal{O})}$ and $\mathbf{M}_i \otimes_{\mathcal{O}_{F,(\mathcal{O})}} \mathcal{O}_{F,(\mathcal{O})}$, respectively, such that the above morphism induces a morphism $f : X_0 \to X_1$, which we assume to be a closed immersion.

(3) For $i = 0,1$, suppose that we have integral PEL data $(\mathcal{O}_i, \star_i, L_i, \langle \cdot, \cdot \rangle_i, h_i)$ (for which $p$ might not be good), together with some suitable choices of $(\mathcal{O}_i, \star_i, L_i, \langle \cdot, \cdot \rangle_i, h_i)$ and a shared choice of a collection of auxiliary integral PEL data $(\mathcal{O}_{aux}, \mathcal{E}_{aux}, L_{j,aux}, \langle \cdot, \cdot \rangle_{j,aux}, h_{j,aux})$ (for which $p$ is good), as in [17] Sec. 2 and 4]; and that $(\mathcal{O}_i, \star_i, L_i, \langle \cdot, \cdot \rangle_i, h_i)$ also serves as a choice of an auxiliary integral PEL datum for $(\mathcal{O}_0, \star_0, L_0, \langle \cdot, \cdot \rangle_0, h_0)$ (but without requiring that $p$ is good for either of these two). Then we have morphisms $G_0 \langle \cdot, \cdot \rangle_0 \otimes_{\mathcal{O}_{aux,aux}} G_{j,aux}$, for all $j \in J$. Suppose that we have neat open compact subgroups $\mathcal{H}_0 \subset G(\mathbb{Z})$, $\mathcal{H}_1 \subset G(\mathbb{Z})$, and $H_{j,aux} \subset G_{j,aux}(\mathbb{Z})$ such that $\mathcal{H}_0 = (\rho(\mathbb{A}^\infty_\mathbb{C}))^{-1}(\mathcal{H}_1)$ and that the images of $\mathcal{H}_1$ under $G_1(\mathbb{Z}) \to G_{j,aux}(\mathbb{Z})$ are neat and contained in $H_{j,aux}$ for all $j \in J$. Let $F$ denote a subfield of $\mathbb{C}$ that is a finite extension of the reflex field of $(\mathcal{O}_0, \star_0, L_0, \langle \cdot, \cdot \rangle_0, h_0)$, and hence also those of $(\mathcal{O}_1, \star_1, L_1, \langle \cdot, \cdot \rangle_1, h_1)$ and $(\mathcal{O}_{aux}, \star_{aux}, L_{j,aux}, \langle \cdot, \cdot \rangle_{j,aux}, h_{j,aux})$ for all $j \in J$. With the above data, we have associated moduli problems $\mathbf{M}_{i,aux}$ over $\text{Spec}(\mathcal{F})$, and associated auxiliary moduli problems $\mathbf{M}_{i,aux} \otimes_{\mathcal{O}_{F,(\mathcal{O})}} \mathcal{Q}$ over $\text{Spec}(\mathcal{F})$, which extend to canonical finite morphisms $\mathbf{M}_{i,aux} \otimes_{\mathcal{O}_{F,(\mathcal{O})}} \mathcal{Q}$ over $\text{Spec}(\mathcal{F})$, and as in [17] Sec. 4. Then we take $X_0$ and $X_1$ to be open-and-closed subschemes of $\mathbf{M}_{i,aux}$ and $\mathbf{M}_{i,aux}$, respectively, such that $\mathbf{M}_{i,aux} \to \mathbf{M}_{i,aux}$, induces a morphism $f : X_0 \to X_1$, which we assume to be a closed immersion.

(4) Suppose that we have a morphism of Shimura data $(G_0, D_0) \to (G_1, D_1)$ defined by some injective homomorphism $\rho : G_0 \to G_1$ as in [1], and suppose that we have a Siegel embedding $(G_1, D_1) \to (G_{aux, D_{aux}})$ defined by some injective homomorphism $G_1 \to G_{aux}$, with $G_{aux} \cong GSp_{2g, \mathbb{Q}}$, for some $g \geq 0$. Suppose that we have neat open compact subgroups $\mathcal{H}_0 \subset G_0(\mathbb{A}^\infty_\mathbb{C})$, $\mathcal{H}_1 \subset G_1(\mathbb{A}^\infty_\mathbb{C})$, and $\mathcal{H}_{aux} \subset G_{aux}(\mathbb{A}^\infty_\mathbb{C})$ such that $\mathcal{H}_0 = (\rho(\mathbb{A}^\infty_\mathbb{C}))^{-1}(\mathcal{H}_1)$ and that the image of $\mathcal{H}_1$ under $G_1(\mathbb{A}^\infty_\mathbb{C}) \to G_{aux}(\mathbb{A}^\infty_\mathbb{C})$ is neat and contained in $\mathcal{H}_{aux}$. Let $F$ denote a subfield of $\mathbb{C}$ that is a finite extension of
the reflex field of \((G_0, D_0)\), and hence also that of \((G_1, D_1)\). Let \(X_0\) and \(X_1\) be integral models over \(S_0 = \text{Spec}(\mathbb{C}_{F,p})\) of the Shimura varieties associated with \((G_0, D_0)\) and \((G_1, D_1)\) at levels \(\mathcal{H}_0\) and \(\mathcal{H}_1\), respectively, defined by taking normalizations of the characteristic zero models over \(F\) (which are base changes of the corresponding canonical models to \(F\)) over the Siegel moduli over \(\text{Spec}(\mathbb{Z}_{(p)})\) associated with \((G_{\text{aux}}, D_{\text{aux}})\) and the prime-to-\(p\) level \(\mathcal{H}_{\text{aux}}\), as in [24, Introduction]. Then we have a canonically induced morphism \(f : X_0 \to X_1\) over \(S_0\), which we assume to be a closed immersion.

We shall say that we are in Cases \((1)\), \((2)\), \((3)\), or \((4)\) depending on the case we are in Assumption \(2.1\). In each case, we have good toroidal compactifications \(X_i \hookrightarrow X^\text{tor}_{i, \Sigma_i}\) associated with some compatible collections of cone decompositions \(\Sigma_i\), for \(i = 0, 1\), whose properties we will review in more detail in the next section.

Our main result is the following:

**Theorem 2.2.** Let \(f : X_0 \to X_1\) be as in Assumption \(2.1\). Then there exist toroidal compactifications \(X_i \hookrightarrow X^\text{tor}_{i, \Sigma_i}\), for \(i = 0, 1\), associated with some compatible collections \(\Sigma_i\) of projective smooth cone decompositions (see [23, 24, 26] in Case \((1)\); see [16, Thm. 6.4.1.1 and 7.3.3.4] in Case \((2)\); see [19, Thm. 6.1] in Case \((3)\); and see [24, Thm. 4.1.5 and Rem. 4.1.6] in Case \((4)\)) such that \(f\) extends to a closed immersion \(f^\text{tor}_{\Sigma_0, \Sigma_1} : X^\text{tor}_{0, \Sigma_0} \to X^\text{tor}_{1, \Sigma_1}\). Moreover, if we denote by \(\mathcal{I}_{\Sigma_i}\) the \(\mathcal{O}_{X^\text{tor}_{i, \Sigma_i}}\)-ideal defining the boundary \(X^\text{tor}_{i, \Sigma_i} \setminus X_i\) (with its reduced subscheme structure), for \(i = 0, 1\), then we may require that \(f^\text{tor}_{\Sigma_0, \Sigma_1} : (\mathcal{I}_{\Sigma_1}) \cong \mathcal{I}_{\Sigma_0}\) as \(\mathcal{O}_{X^\text{tor}_{0, \Sigma_0}}\)-ideals. We may require that \(\Sigma_0\) and \(\Sigma_1\) refine any finite number of prescribed compatible collections of cone decompositions.

The proof of Theorem 2.2 will be completed in Section 3.

**Remark 2.3.** (1) In Cases \((2)\) and \((3)\), for example, we can take \(X_i\) to be the schematic closure of the base change to \(\text{Spec}(F)\) of the canonical model of the Shimura variety associated with the Shimura datum \((G_i \otimes \mathbb{Q}, G_i(\mathbb{R}) \cdot h_i)\) (see [14, Sec. 8], [15, Sec. 2], and [23, Sec. 1.2]), for \(i = 0, 1\), when \(G_i \otimes \mathbb{Q}\) is connected and \((G_i \otimes \mathbb{Q}, G_i(\mathbb{R}) \cdot h_i)\) qualifies as a Shimura datum.

(2) In Case \((2)\), in order to show that \(f : X_0 \to X_1\) is indeed a closed immersion, we often have to resort to the moduli interpretations of \(M_{\mathcal{H}_0}\) and \(M_{\mathcal{H}_1}\).

(3) In Case \((3)\), when the levels \(\mathcal{H}_0\) and \(\mathcal{H}_1\) differ at \(p\) from the stabilizers of \(L_0\) and \(L_1\), it is generally more difficult to verify that the morphism \(f : X_0 \to X_1\) defined abstractly by taking normalizations is a closed immersion. Practically, when the levels are parahoric at \(p\) (and satisfies some technical assumptions), we can still define \(X_0\) and \(X_1\) using some explicit moduli problems—see, for example, [17, Ex. 2.4 and 13.12, and Rem. 16.5]. However, we do not (yet) have a method to study higher levels in general.

(4) In Case \((4)\), even in the best understood situation with hyperspecial levels at \(p\) in [13], as far as we know, there is (as yet) no general method for verifying whether the morphism \(f : X_0 \to X_1\) is a closed immersion.

(5) Nevertheless, Theorem 2.2 provides closed immersions \(f^\text{tor}_{\Sigma_0, \Sigma_1} : X^\text{tor}_{0, \Sigma_0} \to X^\text{tor}_{1, \Sigma_1}\), as long as the input \(f : X_0 \to X_1\) is a closed immersion, and we included all four cases (which in theory allows arbitrarily high levels at \(p\)) in Cases \((2)\) and \((3)\) even when the assumption of being a closed immersion cannot be easily verified in general.
Certainly, we expect Theorem 2.2 to extend to integral models of abelian-type Shimura varieties, generalizing those constructed in Cases (2), (3), and (4) in Assumption 2.1, as soon as the their toroidal compactifications are constructed and shown to have desired properties as in Proposition 3.1 and 3.4 below. However, we do not expect it to be any easier to verify that \( f : X_0 \to X_1 \) is indeed a closed immersion.

Remark 2.4. In Theorem 2.2, the main reason to consider the projectivity of the cone decompositions is because, without it, we cannot be sure that the toroidal compactifications we obtained are schemes rather than merely algebraic spaces.

Remark 2.5. In Theorem 2.2, the assertion that \( f^\text{tor}_{X_0, X_1}(I_{\Sigma_1}) \cong I_{\Sigma_0} \) does not follow from the assertion that \( f^\text{tor}_{X_0, X_1} \) is a closed immersion. (See Example 5.1 below.)

Remark 2.6. Since base changes of closed immersions are still closed immersions, by using [20, Thm. 2.3.2], Theorem 2.2 implies similar results for partial toroidal compactifications of well-positioned subschemes of base changes of integral models of Shimura varieties. We shall leave the precise statements to interested readers.

3. Morphisms between toroidal compactifications

In all cases in Assumption 2.1, we have good toroidal and minimal compactifications \( X^\text{tor}_{X_i, \Sigma_i} \to S_0 \) and \( X^\text{min}_{X_i, \Sigma_i} \to S_0 \), for \( i = 0, 1 \), whose qualitative properties we shall summarize as follows, based on the constructions in [4,2,3,26,16,17,19] (as in [21, Prop. 2.2] and [20, Prop. 2.1.2 and 2.1.3, and Cor. 2.1.7] and their proofs):

Proposition 3.1. For each \( i = 0, 1 \), there is a canonical minimal compactification \( J^\text{min}_i : X_i \hookrightarrow X^\text{min}_i \) over \( S_0 \), together with a canonical collection of toroidal compactifications

\[
J^\text{tor}_{X_i, \Sigma_i} : X_i \hookrightarrow X^\text{tor}_{X_i, \Sigma_i}
\]

over \( S_0 \), labeled by certain compatible collections \( \Sigma_i \) of cone decompositions, satisfying the following properties:

(1) For each \( \Sigma_i \), there is a proper surjective structural morphism

\[
f^\text{tor}_{X_i, \Sigma_i} : X^\text{tor}_{X_i, \Sigma_i} \to X^\text{min}_{X_i, \Sigma_i},
\]

compatible with \( J^\text{min}_{X_i, \Sigma_i} \) and \( J^\text{tor}_{X_i, \Sigma_i} \) in the sense that \( J^\text{min}_{X_i, \Sigma_i} = f^\text{tor}_{X_i, \Sigma_i} \circ J^\text{tor}_{X_i, \Sigma_i} \).

(2) The scheme \( X^\text{min}_{X_i, \Sigma_i} \) admits a stratification by locally closed subschemes \( Z_i \) flat over \( S_0 \), each of which is isomorphic to a finite quotient of an analogue of \( X_i \). (Nevertheless, in Cases (2) and (3), we can still identify each \( Z_i \) with an analogue of \( X_i \).)

(3) Each \( \Sigma_i \) is a set \( \{ \Sigma_{Z_i} \} \) of cone decompositions \( \Sigma_{Z_i} \) with the same index set as that of the strata of \( X^\text{min}_{X_i, \Sigma_i} \). (In [19], the elements of this index set was called cusp labels.) For simplicity, we shall suppress such cusp labels and denote the associated objects with subscripts given by the strata \( Z_i \).

(4) For each stratum \( Z_i \), the cone decomposition \( \Sigma_{Z_i} \) is a cone decomposition of some \( P_{Z_i} \), where \( P_{Z_i} \) is the union of the interior \( P_{Z_i}^+ \) of a homogenous self-adjoint cone (see [3, Ch. 2]) and its rational boundary components, which is admissible with respect to some arithmetic group \( \Gamma_{Z_i} \) acting on \( P_{Z_i} \) (and hence also on \( \Sigma_{Z_i} \)). Then \( \Sigma_{Z_i} \) has a subset \( \Sigma_{Z_i}^+ \) forming a cone...
decomposition of $P^+_Z$. If $\tau$ is a cone in $\Sigma^+_i$ that is not in $\Sigma^+_Z$, then there exist a stratum $Z'_i$ of $X^+_i$, whose closure in $X^+_i$ contains $Z_i$, and a cone $\tau'$ in $\Sigma^+_Z$, whose $\Gamma^+_Z$-orbit is uniquely determined by the $\Gamma^+_Z$-orbit of $\tau$.

We may and shall assume that $\Sigma_i$ is smooth, and that, for each $Z_i$ and each $\sigma \in \Sigma^+_i$, the stabilizer $\Gamma^+_{Z_i,\sigma}$ of $\sigma$ in $\Gamma^+_i$ is trivial.

(5) For each $\Sigma_i$, the associated $X^+_{i,\Sigma_i}$ admits a stratification by locally closed subschemes $Z_{i,[\sigma]}$ flat over $S_0$, labeled by the strata $Z_i$ of $X^+_{i}$ and the orbits $[\sigma] \in \Sigma^+_i/\Gamma^+_i$. The stratifications of $X^+_{i,\Sigma_i}$ and $X^+_{i}$ are compatible with each other in a precise sense, which we summarize as follows: The preimage of a stratum $Z_i$ of $X^+_{i}$ is the (set-theoretic) disjoint union of the strata $Z_{i,[\sigma]}$ of $X^+_{i,\Sigma_i}$ with $[\sigma] \in \Sigma^+_i/\Gamma^+_i$. If $\tau$ is a face of a representative $\sigma$ of $[\sigma]$, which is identified (as in the property [3] above) with the $\Gamma^+_Z$-orbit $[\tau']$ of some cone $\tau'$ in $\Sigma^+_Z$, where $Z'_i$ is a stratum whose closure in $X^+_i$ contains $Z_i$, then $Z_{i,[\sigma]}$ is contained in the closure of $Z'_{i,[\tau']}$. 

(6) For each stratum $Z_i$ of $X^+_{i}$, there is a proper surjective morphism 

$$C_{Z_i} \rightarrow Z_i$$

(whose precise description is not important for our purpose), together with a morphism 

$$\Xi_{Z_i} \rightarrow C_{Z_i}$$

of schemes which is a torsor under the pullback of a split torus $E_{Z_i}$ with some character group $S_{Z_i}$ over $\text{Spec}(\mathbb{Z})$, so that we have 

$$\Xi_{Z_i} \cong \text{Spec}(\mathcal{O}_{C_{Z_i}} \left( \bigoplus_{\ell \in S_{Z_i}} \Psi_{Z_i}(\ell) \right)),$$

for some invertible sheaves $\Psi_{Z_i}(\ell)$. (Each $\Psi_{Z_i}(\ell)$ can be viewed as the subsheaf of $(\Xi_{Z_i} \rightarrow C_{Z_i}), \mathcal{O}_{\Xi_{Z_i}}$, on which $E_{Z_i}$ acts via the character $\ell \in S_{Z_i}$.) This character group $S_{Z_i}$ admits a canonical action of $\Gamma_{Z_i}$, and its $\mathbb{R}$-dual $S^\vee_{Z_i,\mathbb{R}} := \text{Hom}_{\mathbb{Z}}(S_{Z_i}, \mathbb{R})$ canonically contains the above $P_{Z_i}$ and $P^+_{Z_i}$ as subsets with compatible $\Gamma_{Z_i}$-actions.

(7) For each $\sigma \in \Sigma_i$, consider the canonical pairing $(\cdot, \cdot) : S_{Z_i} \times S^\vee_{Z_i,\mathbb{R}} \rightarrow \mathbb{R}$ and $\sigma^\vee := \{ \ell \in S_{Z_i} : (\ell, y) > 0, \forall y \in \sigma \}$, $\sigma^\prime := \{ \ell \in S_{Z_i} : (\ell, y) = 0, \forall y \in \sigma \}$, and $\sigma^\prime : = \{ \ell \in S_{Z_i} : (\ell, y) = 0, \forall y \in \sigma \} \cong \sigma^\prime / \sigma^\prime$. Then we have the affine toroidal embedding

$$\Xi_{Z_i} \hookrightarrow \Xi_{Z_i}(\sigma) := \text{Spec}(\mathcal{O}_{C_{Z_i}} \left( \bigoplus_{\ell \in \sigma^\prime} \Psi_{Z_i}(\ell) \right)),$$

The scheme $\Xi_{Z_i}(\sigma)$ has a closed subscheme $\Xi_{Z_i,\sigma}$ defined by the ideal sheaf corresponding to $\bigoplus_{\ell \in \sigma^\prime} \Psi_{Z_i}(\ell)$, so that

$$\Xi_{Z_i,\sigma} \cong \text{Spec}(\mathcal{O}_{C_{Z_i}} \left( \bigoplus_{\ell \in \sigma^\prime} \Psi_{Z_i}(\ell) \right)),$$

Then $\Xi_{Z_i}(\sigma)$ admits a natural stratification by locally closed subschemes $\Xi_{Z_i,\tau}$ (i.e., the closed subscheme as above of the open subscheme $\Xi_{Z_i}(\tau)$ of $\Xi_{Z_i}(\sigma)$), where $\tau$ runs over all the faces of $\sigma$ in $\Sigma_i$. 

(8) For each given $\Sigma_i$, and for each $Z_i$, consider the full toroidal embedding

$$\Xi_{Z_i,\Sigma_i} = \bigcup_{\sigma \in \Sigma_i} \Xi_{Z_i}(\sigma)$$
defined by the cone decomposition $\Sigma_Z$, (cf. [16 Thm. 6.1.2.8 and Sec. 6.2.5]), and consider the formal completion

$$\mathfrak{X}_{Z, \Sigma} := (\Xi_{Z, \Sigma})^\wedge \bigcup_{\tau \in \Sigma^+_Z} \Xi_{Z, \tau}$$

of $\Xi_{Z, \Sigma}$ along its closed subscheme $\bigcup_{\tau \in \Sigma^+_Z} \Xi_{Z, \tau}$. Consider, for each $\sigma \in \Sigma^+_Z$, the formal completion

$$\mathfrak{X}^\sigma_{Z, \sigma} := (\Xi_{Z, \sigma})^\wedge \bigcup_{\tau \in \Sigma^+_Z, \tau \prec \sigma} \Xi_{Z, \tau}$$

of $\Xi_{Z, \sigma}$ along its closed subscheme $\bigcup_{\tau \in \Sigma^+_Z, \tau \prec \sigma} \Xi_{Z, \tau}$. Then $\mathfrak{X}_{Z, \Sigma}$ admits an open covering by $\mathfrak{X}^\sigma_{Z, \sigma}$ for $\sigma$ running through elements of $\Sigma^+_Z$, and we have canonical flat morphisms $\mathfrak{X}^\sigma_{Z, \sigma} \hookrightarrow \mathfrak{X}_{Z, \Sigma} \to X^\sigma_{i, Z}$ (of locally ringed spaces) inducing isomorphisms

$$\mathfrak{X}^\sigma_{Z, \sigma} \sim (X^\sigma_{i, \Sigma})^\wedge \bigcup_{\tau \in \Sigma^+_Z, \tau \prec \sigma} Z_{i, \tau}$$

and

$$\mathfrak{X}_{Z, \Sigma} / \Gamma_Z \sim (X^\sigma_{i, \Sigma})^\wedge \bigcup_{\tau \in \Sigma^+_Z / \Gamma_Z} Z_{i, \tau}.$$  

More precisely, for each $\sigma \in \Sigma^+_Z$, and for each affine open formal subscheme $\mathfrak{M} = \text{Spf}(R)$ of $\mathfrak{X}^\sigma_{Z, \sigma}$, under the canonically induced (flat) morphisms $W := \text{Spec}(R) \to X^\sigma_{i, \Sigma}$ and $\text{Spec}(R) \to \Xi_{Z, \sigma}$ induced by (3.2), the stratification of $W$ induced by that of $X^\sigma_{i, \Sigma}$ coincides with the stratification of $W$ induced by that of $\Xi_{Z, \sigma}$. In particular, the preimages of $X_i$ and $\Xi_Z$ coincide as an open subscheme $W^0$ of $W$.

As for the morphism $f : X_0 \to X_1$, we have the following:

**Proposition 3.4.** Assume slightly more generally (than in Assumption [2.1]) that $(\rho(\Delta^\infty))(\mathcal{H}_0) \subset \mathcal{H}_1$ and hence that the morphism $f : X_0 \to X_1$ is finite. Then there exists a canonical finite morphism

$$f_{\min} : X^\min_0 \to X^\min_1$$

such that $f_{\min} \circ J_{0, \min}^0 = J_{1, \min}^1 \circ f$ over $S_0$, together with a canonical collection of proper morphisms

$$f^\tor_{S_0, \Sigma_1} : X^\tor_{S_0, \Sigma_1} \to X^\tor_{1, \Sigma_1}$$

such that $f^\tor_{S_0, \Sigma_1} \circ J_{0, \min}^0 = J^\tor_{1, \Sigma_1} \circ f$ and $f_{\min} \circ f^\tor_{S_0, \Sigma_1} = f^\tor_{1, \Sigma_1} \circ f_{\min}$ over $S_0$, labeled by certain pairs $(\Sigma_0, \Sigma_1)$ of compatible collections of cone decompositions that are compatible with each other in a sense that we shall explain below, satisfying the following properties:

1. For each stratum $Z_0$ of $X^\min_0$, there exists a (unique) stratum $Z_1$ of $X^\min_1$ such that $f_{\min}(Z_0) \subset Z_1$ (as subsets of $X^\min_1$). Moreover, $Z_0$ is both open and closed in $(f_{\min})^{-1}(Z_1)$, and $f_{\min}$ induces a finite morphism $Z_0 \to Z_1$.

2. Over any $Z_0 \to Z_1$ as above, we have a finite morphism

$$C_{Z_0} \to C_{Z_1},$$
over which we have a finite morphism

\[ \Xi_{Z_0} \to \Xi_{Z_1}, \]

which induces a finite morphism \( \Xi_{Z_0} \hookrightarrow \Xi_{Z_1} \times_{C_{Z_1}} C_0 \) which is equivariant with the pullback of a group homomorphism of tori

\[ E_{Z_0} \to E_{Z_1}, \]

with finite kernel over Spec(\( \mathbb{Z} \)) that is dual to a homomorphism

\[ S_{Z_1} \to S_{Z_0} \]

of character groups with finite cokernel. The \( \mathbb{R} \)-dual of this last homomorphism is an injective homomorphism \( S_{Z_1}^{\vee, \mathbb{R}} \hookrightarrow S_{Z_0}^{\vee, \mathbb{R}} \) of \( \mathbb{R} \)-vector spaces, inducing a Cartesian diagram of injective maps

\[
\begin{array}{ccc}
P_{Z_0} & \hookrightarrow & P_{Z_1} \\
\downarrow & & \downarrow \\
P_{Z_0} & \hookrightarrow & P_{Z_1} \\
\end{array}
\]

All the above maps from objects associated with \( Z_0 \) to the corresponding ones associated with \( Z_1 \) are equivariant with a canonical homomorphism \( \Gamma_{Z_0} \to \Gamma_{Z_1} \). If \( \ell_1 \in S_{Z_1} \) is mapped to \( \ell_0 \in S_{Z_0} \) under \( S_{Z_1} \to S_{Z_0} \), then the invertible sheaf \( \Psi_{Z_1}(\ell_1) \) over \( C_{Z_1} \) is canonically isomorphic to the pullback of the invertible sheaf \( \Psi_{Z_1}(\ell) \) over \( C_{Z_1} \), under the above morphism \( C_{Z_2} \to C_{Z_1} \).

When \( \mathcal{H}_0 = \left( \rho(\mathbb{A}^\infty) \right)^{-1}(\mathcal{H}_1) \), the homomorphism \( S_{Z_1} \to S_{Z_0} \) is surjective, and hence the dual homomorphism \( E_{Z_0} \to E_{Z_1} \) is a closed immersion.

(3) If the image of \( \sigma \in \Sigma_{Z_0} \) under \( P_{Z_0} \to P_{Z_1} \) is contained in some \( \tau \in \Sigma_{Z_1} \), then we have a canonical morphism

\[ \Xi_{Z_0}(\sigma) = \text{Spec}_{\sigma \in C_{Z_0}} \left( \bigoplus_{\ell_0 \in \sigma^+} \Psi_{Z_0}(\ell_0) \right) \to \Xi_{Z_1}(\tau) = \text{Spec}_{\tau \in C_{Z_1}} \left( \bigoplus_{\ell \in \tau^+} \Psi_{Z_1}(\ell) \right) \]

extending \( \Xi_{Z_0} \to \Xi_{Z_1} \), and inducing a canonical morphism \( \Xi_{Z_0}(\sigma) \to \Xi_{Z_1}(\tau) \times_{C_{Z_1}} C_0 \) which is equivariant with the pullback of \( E_{Z_0} \to E_{Z_1} \). Moreover, there is an induced morphism

\[ \Xi_{Z_0,\sigma} = \text{Spec}_{\sigma \in C_{Z_0}} \left( \bigoplus_{\ell_0 \in \sigma^+} \Psi_{Z_0}(\ell_0) \right) \to \Xi_{Z_1,\tau} = \text{Spec}_{\tau \in C_{Z_1}} \left( \bigoplus_{\ell \in \tau^+} \Psi_{Z_1}(\ell) \right). \]

(4) We say that the collections \( \Sigma_0 = \{ \Sigma_{Z_0} \}_{Z_0} \) and \( \Sigma_1 = \{ \Sigma_{Z_1} \}_{Z_1} \) are compatible with each other or simply compatible if, when \( Z_0 \) is mapped to \( Z_1 \) as above, the image of each \( \sigma \in \Sigma_{Z_0}^+ \) under the map \( P_{Z_0} \to P_{Z_1} \) is contained in some \( \tau \in \Sigma_{Z_1}^+ \). We say that \( \Sigma_0 \) is induced by \( \Sigma_1 \) if each \( \sigma \in \Sigma_{Z_0}^+ \) is exactly the preimage of some \( \tau \in \Sigma_{Z_1}^+ \). (If \( \Sigma_0 \) is induced by \( \Sigma_1 \), then they are necessarily compatible.)

(5) The morphism \( f : X_0 \to X_1 \) extends to a proper (resp. finite) morphism \( f^{\text{tor}}_{\Sigma_0,\Sigma_1} : X_0^{\text{tor}}_{\Sigma_0} \to X_1^{\text{tor}}_{\Sigma_1} \) as above if and only if \( \Sigma_0 \) and \( \Sigma_1 \) are compatible (resp. \( \Sigma_0 \) is induced by \( \Sigma_1 \)). When \( \Sigma_0 \) and \( \Sigma_1 \) are compatible, if the image of \( \sigma \in \Sigma_{Z_0}^+ \) under \( P_{Z_0} \to P_{Z_1} \) is contained in \( \tau \in \Sigma_{Z_1}^+ \), then the morphism \( f^{\text{tor}}_{\Sigma_0,\Sigma_1} \) induces a morphism \( Z_{0,[\sigma]} \to Z_{1,[\tau]} \) (which is not necessarily proper), which can be canonically identified with the morphism \( \Xi_{Z_0,\sigma} \to \Xi_{Z_1,\tau} \) above.
For each \( \tau \in \Sigma_{Z_1}^+ \), the preimage of \( Z_{1,[\tau]} \) is the (set-theoretic) disjoint union of the strata \( Z_{0,[\sigma]} \) labeled by \( \sigma \in \Sigma_{Z_0}^+ \) that are mapped into \( \tau \) under \( P_{Z_0}^+ \hookrightarrow P_{Z_1}^+ \). If there is a unique such \( \sigma \), which is the case exactly when \( \sigma \) is the preimage of \( \tau \), then the induced morphism \( Z_{0,[\sigma]} \to Z_{1,[\tau]} \) is finite.

(6) Suppose that \( \Sigma_0 \) and \( \Sigma_1 \) are compatible. Then there is a proper morphism

\[
\Xi_{Z_0,\Sigma_{Z_0}} \to \Xi_{Z_1,\Sigma_{Z_1}},
\]

whose formal completion gives a proper morphism

\[(3.5) \quad \mathfrak{X}_{Z_0,\Sigma_{Z_0}} \to \mathfrak{X}_{Z_1,\Sigma_{Z_1}}.\]

These two morphisms are equivariant with the homomorphism \( \Gamma_{Z_0} \to \Gamma_{Z_1} \) and induces a proper morphism \( \mathfrak{X}_{Z_0,\Sigma_{Z_0}}/\Gamma_{Z_0} \to \mathfrak{X}_{Z_1,\Sigma_{Z_1}}/\Gamma_{Z_1} \), which can be identified (via isomorphisms as in (3.3)) with

\[
(\mathfrak{X}_{0,\Sigma_0})_{(\tau)} \leftarrow \bigcup_{[\sigma] \in \Sigma_{Z_0}^+} (\mathfrak{X}_{1,\Sigma_1})_{(\tau)}^{\Sigma_{Z_1}^+} \leftarrow \bigcup_{[\sigma] \in \Sigma_{Z_0}^+} \Xi_{Z_0,\Sigma_{Z_0}}^{\Sigma_{Z_1}^+}. \tag{3.5}
\]

If the image of \( \sigma \in \Sigma_{Z_0}^+ \) under \( P_{Z_0}^+ \to P_{Z_1}^+ \) is contained in some \( \tau \in \Sigma_{Z_1}^+ \), we have an induced morphism \( \mathfrak{X}_{0,\Sigma_0} \to \mathfrak{X}_{1,\Sigma_1} \), which can be identified (via isomorphisms as in (3.3)) with

\[
(\mathfrak{X}_{0,\Sigma_0})_{(\tau)} = \bigcup_{\sigma \in \Sigma_{Z_0}^+} (\mathfrak{X}_{1,\Sigma_1})_{(\sigma)}^{\Sigma_{Z_1}^+} \leftarrow \bigcup_{\sigma \in \Sigma_{Z_0}^+} \Xi_{Z_0,\Sigma_{Z_0}}^{\Sigma_{Z_1}^+}. \tag{3.5}
\]

For a fixed \( \tau \in \Sigma_{Z_1}^+ \), the pullback of (3.5) to the open formal subscheme \( \mathfrak{X}_{1,\Sigma_1}^\tau \) on the target gives a proper morphism

\[
\bigcup_{\sigma \in \Sigma_{Z_0}^+} \mathfrak{X}_{0,\Sigma_{Z_0}}^{\Sigma_{Z_1}^+} \to \mathfrak{X}_{1,\Sigma_1}^\tau. \tag{3.6}
\]

Suppose moreover that \( \Sigma_0 \) is induced by \( \Sigma_1 \). Then both morphisms (3.5) and (3.6) are finite. For each \( \tau \in \Sigma_{Z_1}^+ \), as above, with \( \sigma \in \Sigma_{Z_0}^+ \) the preimage of \( \tau \), which is the unique element in \( \Sigma_{Z_0}^+ \) such that \( (P_{Z_0}^+ \to P_{Z_1}^+)(\sigma) \subset \tau \); and for each affine open formal subscheme \( \mathfrak{M}_1 = \text{Spf}(R_1) \) of \( \mathfrak{X}_{1,\Sigma_1}^\tau \), let

\[
\mathfrak{M}_0 = \text{Spf}(R_0)
\]

denote its pullback to \( \mathfrak{X}_{0,\Sigma_0}^{\Sigma_{Z_1}^+} \). Under the morphisms \( W_1 := \text{Spec}(R_1) \to \mathfrak{X}_{1,\Sigma_1}^\tau, W_2 := \text{Spec}(R_0) \to \mathfrak{X}_{0,\Sigma_0}^{\Sigma_{Z_1}^+}, \) and \( W_0 := \mathfrak{X}_{0,\Sigma_0}^{\Sigma_{Z_1}^+} \) induced by morphisms as in (3.3), the preimages of \( X_1 \) and \( \Xi_{Z_1} \) coincide as an open subscheme \( W_1^0 \) of \( W_1 \), and their further preimages in \( W_0 \) coincide with the preimages of \( \Xi_{Z_0} \) and \( \Xi_{Z_1} \) as an open subscheme \( W_0^0 \).

**Proof.** Except for the first assertion in (5), these follow from the same arguments as in [24] Sec. 2.1.28 and 4.1.12 [which are based on 26 Sec. 4.16, 6.25, and 12.4] and [11] Sec. 3.3) in Cases (1) and (4), and as in [19] Sec. 8–11 and [17] the proof of Prop. 2.1.3] in Cases (2) and (3). As for the first assertion in (5), it follows from the universal or functorial properties of toroidal compactifications in terms of the associated cone decompositions, as in [24] [3] Ch. II, Sec. 7, [26] Prop. 6.25, [16] Thm. 6.4.1.1(6), [19] Thm. 6.1(6)], and [24] Prop. 4.1.13. \( \square \)

**Corollary 3.7.** In Proposition 3.4, suppose that \( \Sigma_0 \) is induced by \( \Sigma_1 \). Let \( Z_1 \) be a stratum of \( X_1^{\text{min}} \) and let \( \{Z_{0,j}\} \) be all the strata of \( X_0^{\text{min}} \) such that \( f_{\text{min}}(Z_{0,j}) \subset Z_1 \) (as subsets of \( X_1^{\text{min}} \)). Consider any \( \tau \in \Sigma_{Z_1}^+ \). For each \( j \), let

\[
\sigma_j := (P_{Z_0,j}^+ \hookrightarrow P_{Z_1}^+)^{-1}(\tau) \in \Sigma_{Z_0,j}^+. \]
Then the pullback of the finite morphism
\[ f_{\Sigma_0, \Sigma_1}^{tor} : X_{0, \Sigma_0}^{tor} \to X_{1, \Sigma_1}^{tor} \]
under the composition of the canonical morphisms
\[ X_{1, \Sigma_1}^{tor} \xrightarrow{\sim} (X_{1, \Sigma_1}^{tor})^\wedge_{\tau' \in \Sigma_{Z_1}^+} \cup_{\tau \subset \tau'} z_{1, \Sigma_1} \]
as in (3.2) and \( (X_{1, \Sigma_1}^{tor})^\wedge_{\tau' \in \Sigma_{Z_1}^+} \cup_{\tau \subset \tau'} z_{1, \Sigma_1} \to X_{1, \Sigma_1}^{tor} \) can be identified with the finite morphism
\[ \prod_j X_{0, j}^0, \sigma_j \to X_{1, \Sigma_1}^{tor} \]
(defined by combining morphisms as in (3.6)).

**Proof.** This follows from (1) and (6) of Proposition 3.4. \( \square \)

**Corollary 3.8.** In Corollary 3.7, with any \( \tau \in \Sigma_{Z_1}^+ \) there inducing \( \sigma_j \in \Sigma_{Z_0, j}^+ \), for each \( j \), we have a commutative diagram of canonical morphisms
\[ \begin{array}{ccc}
E_{Z_0, j} & \xrightarrow{\subset} & E_{Z_0, j}(\sigma_j) \\
\downarrow & & \downarrow \\
E_{Z_1} & \xrightarrow{\subset} & E_{Z_1}(\tau)
\end{array} \]
over Spec(\( \mathbb{Z} \)), in which the horizontal morphisms are affine toroidal embeddings, which are open immersions, and where the vertical morphisms are finite. Let \( x_1 \) be any point of \( X_{1, \Sigma_1}^{tor} \) that lies on the stratum \( Z_{1, \tau} \). Then, étale locally at \( x_1 \), the commutative diagram
\[ \begin{array}{ccc}
X_0 & \overset{f_{\Sigma_0, \Sigma_1}}{\longrightarrow} & X_{0, \Sigma_0}^{tor} \\
\downarrow f & & \downarrow f_{\Sigma_0, \Sigma_1}^{tor} \\
X_1 & \overset{f_{\Sigma_1}}{\longrightarrow} & X_{1, \Sigma_1}^{tor}
\end{array} \]
can be identified with a commutative diagram
\[ \begin{array}{ccc}
\prod_j (E_{Z_0, j} \times_{\text{Spec}(\mathbb{Z})} C_{Z_0, j}) & \xrightarrow{\subset} & \prod_j (E_{Z_0, j}(\sigma_j) \times_{\text{Spec}(\mathbb{Z})} C_{Z_0, j}) \\
\downarrow & & \downarrow \\
E_{Z_1} \times_{\text{Spec}(\mathbb{Z})} C_{Z_1} & \xrightarrow{\subset} & E_{Z_1}(\tau) \times_{\text{Spec}(\mathbb{Z})} C_{Z_1}
\end{array} \]
induced by taking fiber products of some translations of the vertical morphisms in the diagram (3.9) by sections of \( E_{Z_1} \) and of the canonical morphisms \( C_{Z_0, j} \to C_{Z_1} \).

More precisely, there exists an étale neighborhood
\[ \overline{U}_1 \to X_{1, \Sigma_1}^{tor} \]
of \( x_1 \) and an étale morphism
\[ \overline{U}_1 \to E_{Z_1}(\tau) \times_{\text{Spec}(\mathbb{Z})} C_{Z_1}, \]
(3.11)
which induce by pullback under the finite morphisms $f_{j | X_{0, \Sigma_\tau}^{\text{tor}} : X_{0, \Sigma_\tau}^{\text{tor}} \to X_{0, \Sigma_\tau}^{\text{tor}}}$ and

$\prod_j (E_{Z_{0, j}}(\sigma_j) \times C_{Z_{0, j}}) \to E_{Z, \tau} | \times C_{Z, \tau}$ (as in (3.10)) some étale morphisms

$U_0 \to X_{0, \Sigma_\tau}^{\text{tor}}$ and $U_0 \to \prod_j (E_{Z_{0, j}}(\sigma_j) \times C_{Z_{0, j}})$, respectively, such that the preimage $U_1$ of $X_1$ in $U_1$ coincides with the preimage of $E_{Z_1}$, and such that the preimage $U_0$ of $U_1$ in $U_0$ coincides with the preimages of $X_0$ and of $\prod_j (E_{Z_{0, j}} \times C_{Z_{0, j}})$.

Moreover, we may require in the above that the pullback of $X_{0, \Sigma_\tau}^{\text{tor}} - X_0$ (with its reduced subscheme structure) to $U_0$ coincides (as a subscheme) with the pullback of $\prod_j (E_{Z_{0, j}}(\sigma_j)^+ \times C_{Z_{0, j}})$, where $E_{Z_{0, j}}(\sigma_j)^+ := \bigcup_{\sigma \in \Sigma_{Z_{0, j}, j}^\tau, \tau < \sigma} E_{Z_{0, j}, \sigma}$ (with its reduced subscheme structure), for each $j$; and that the pullback of $X_{1, \Sigma_\tau}^{\text{tor}} - X_1$ to $U_1$ coincides (as a subscheme) with the pullback of $E_{Z, \tau}(E_{Z, \tau})^+ \times C_{Z, \tau}$, where

$E_{Z, \tau}(\tau)^+ := \bigcup_{\tau' \in \Sigma_{Z, \tau}^\tau, \tau' < \tau} E_{Z, \tau', \tau}$ (with its reduced subscheme structure).

**Proof.** These follow from Corollary 3.7 and Artin’s approximation (see [1] Thm. 1.12, and the proof of the corollaries in Sec. 2) as in the proofs of [21] Prop. 2.2(9) and Cor. 2.4, [20] Cor. 2.1.7, and [22] Prop. 5.1], which are applicable because we only need to approximate finitely many formal schemes finite over $X_{Z, \tau}$, and because the formation of Henselizations of semi-local rings is compatible with base change under finite morphisms by [10] IV-4, 18.6.8; and from the fact that all the torus torsors are already Zariski locally trivial, as in the proof of [21] Lem. 2.3].

(Not the torus torsors might be trivialized by incompatible sections, and hence we need to allow the group homomorphisms $E_{Z_{0, j}} \to E_{Z_1}$ to be translated by some possibly different sections of $E_{Z_1}$, when there are more than one $j$.)

**Remark 3.12.** In Proposition 3.4 and in Corollaries 3.7 and 3.8 we only need the weaker assumption that $(\rho(\mathbb{A}^\infty))(\mathcal{H}_0) \subset \mathcal{H}_1$. When $\mathcal{H}_0 = (\rho(\mathbb{A}^\infty))^{-1}(\mathcal{H}_1)$, we already know in Proposition 3.4 that the morphism $E_{Z_{0, j}} \to E_{Z_1}$ in (3.9) is a closed immersion, without assuming that $f$ is a closed immersion; but it is generally not true that the morphism $E_{Z_{0, j}}(\sigma_j) \to E_{Z_1}(\sigma)$ is a closed immersion when $E_{Z_{0, j}} \to E_{Z_1}$ is (cf. Remark 4.1 below), regardless of whether $f$ is.

We shall reinstate the full Assumption 2.1 from now on.

4. CONDITIONS ON CONE DECOMPOSITIONS

Motivated by Corollary 3.8 with the goal of proving Theorem 2.2 in mind, we would like to show the existence of collections $\Sigma_0$ and $\Sigma_1$ such that $\Sigma_0$ is induced by $\Sigma_1$ as in Proposition 3.4 and such that, for each $\sigma \in \Sigma_{Z_0, \tau}$ that is the preimage under $P_{Z_0}^\tau \to P_{Z_1}^\tau$ of some $\tau \in \Sigma_{Z_1}^\tau$, the canonical morphism $E_{Z_0}(\sigma) \to E_{Z_1}(\tau)$ (cf. (3.9)) is a closed immersion.

**Remark 4.1.** This condition of being a closed immersion is not satisfied in general. For example, it is possible to choose the linear algebraic data such that $S_{Z_1} \cong \mathbb{Z}^{\mathbb{Z}_2} \to S_{Z_2} \cong \mathbb{Z}^{\mathbb{Z}_2}$ corresponds to the projection to the first two factors, in which case $S_{Z_0, \mathbb{R}} \cong \mathbb{R}^{\mathbb{Z}_2} \to S_{Z_1, \mathbb{R}} \cong \mathbb{R}^{\mathbb{Z}_2}$ is the inclusion of the first two coordinates, and such that we have the following:
Remark 4.2. In fact, in Remark 4.1 even the induced map $E_{Z_0}(\sigma)(\mathbb{C}) \to E_{Z}(\tau)(\mathbb{C})$ on $\mathbb{C}$-points is not injective: For $? = \pm 1$, if $x_0 : \mathbb{Z}[\sigma'] \to \mathbb{C}$ is the ring homomorphism sending $(-1,1)$ and $(1,0)$ in $\sigma'$ to 0 and $?$, respectively, then the induced homomorphism $y : \mathbb{Z}[\tau'] \to \mathbb{C}$ sends $(-1,1,0)$, $(0,1,0)$, and $(2,0,1)$ to 0, 0, and 1, respectively. That is, both the $\mathbb{C}$-points defined by $x_1$ and $x_{-1}$ are sent to the same $\mathbb{C}$-point defined by $y$. This shows that, already in characteristic zero, the induced morphism $E_{Z_0}(\sigma) \to E_{Z}(\tau)$ is not universally injective, and hence cannot induce a universal homeomorphism between the source and its image in the target. Moreover, for any rational polyhedral cone $\sigma' \subset \sigma$, the induced morphism $E_{Z_0}(\sigma') \to E_{Z}(\tau)$ is not universally injective either.

Nevertheless, we have the following:

Lemma 4.3. For any rational polyhedral cones $\sigma \subset S_{Z_0,\mathbb{R}}^\vee$ and $\tau \subset S_{Z_1,\mathbb{R}}^\vee$ such that $\tau = (S_{Z_0,\mathbb{R}} \hookrightarrow S_{Z_1,\mathbb{R}}^\vee)(\sigma)$, the canonical morphism $E_{Z_0}(\sigma) \cong \text{Spec}(\mathbb{Z}[\sigma']) \to E_{Z_1}(\tau) \cong \text{Spec}(\mathbb{Z}[\tau'])$ is a closed immersion.

Proof. Given an arbitrary $\ell_0 \in \sigma'$, take any lift $\ell_1$ of it in $S_{Z_1}$, which exists because $S_{Z_1} \to S_{Z_0}$ is surjective. Given an arbitrary $y_1 \in \tau$, by assumption, there exists some $y_0 \in \sigma$ such that $y_1 = (S_{Z_0,\mathbb{R}} \hookrightarrow S_{Z_1,\mathbb{R}}^\vee)(y_0)$, and so that $\langle \ell_1, y_1 \rangle = \langle \ell_0, y_0 \rangle \geq 0$. Consequently, $\ell_1 \in \tau'$, and $\tau' \to \sigma'$ is surjective, as desired. \qed

Lemma 4.4. In Lemma 4.3 let us identify $S_{Z_0,\mathbb{R}}^\vee$ with a subspace of $S_{Z_1,\mathbb{R}}^\vee$ for simplicity, so that $\tau = \sigma$ under this identification; and let $S' := S_{Z_1}^\vee \cap (\mathbb{R} \cdot \sigma)$ and $S := \text{Hom}_{\mathbb{Z}}(S', \mathbb{Z})$, so that we have surjective homomorphisms $S_{Z_1} \to S_{Z_0} \to S$ corresponding to injective homomorphisms of tori $E \to E_{Z_0} \to E_{Z_1}$. For the sake of clarity, let us denote by $\varsigma$ the same cone $\sigma$ in $S_{Z_1}^\vee = \mathbb{R} \cdot \sigma$. Let $E$, $E_{Z_0}^\natural$, and $E_{Z_1}^\natural$ be the split tori over $\text{Spec}(\mathbb{Z})$ with character groups $S$, $S_{Z_0}^\perp := \ker(S_{Z_0} \to S)$, and $S_{Z_1}^\perp := \ker(S_{Z_1} \to S)$, respectively. Let us pick any splitting $S_{Z_1} \cong S \oplus S_{Z_1}^\perp$ (as $\mathbb{Z}$-modules) which induces a splitting $S_{Z_0} \cong S \oplus S_{Z_0}^\perp$. Then these splittings are dual to compatible fiber products $E_{Z_1} \cong E \times_{\text{Spec}(\mathbb{Z})} E_{Z_0}^\natural$ and $E_{Z_0} \cong E \times_{\text{Spec}(\mathbb{Z})} E_{Z_1}^\natural$, respectively; and the canonical injective homomorphism $E_{Z_0} \to E_{Z_1}$ factors as a fiber product of the identity homomorphism of $E$ with the canonical injective homomorphism $E_{Z_0}^\natural \to E_{Z_1}^\natural$ dual to $S_{Z_1}^\perp \to S_{Z_0}^\perp$. Moreover, these splittings extend to compatible fiber products $E_{Z_1}(\tau) \cong E(\varsigma) \times_{\text{Spec}(\mathbb{Z})} E_{Z_0}^\natural$, and $E_{Z_0}(\sigma) \cong E(\varsigma) \times_{\text{Spec}(\mathbb{Z})} E_{Z_1}^\natural$, respectively; and the canonical closed immersion $E_{Z_0}(\sigma) \to E_{Z_1}(\tau)$ factors as the fiber product of the identity morphism of $E(\varsigma)$ with the same injective group homomorphism $E_{Z_0}^\natural \to E_{Z_1}^\natural$ as above. Furthermore, any closed immersion $E_{Z_0}(\sigma) \hookrightarrow E_{Z_1}(\tau)$ that is a translation of the canonical one by some section of $E_{Z_1}$, can be identified with the product of an isomorphism $E(\varsigma) \cong E(\varsigma)$ that is the translation of the identity.
morphism on \( E(\varsigma) \) by some section of \( E \) with a closed immersion \( E \) \( E_{Z_0}^+ \to E_{Z_1}^+ \) that is the translation of the canonical one by some section of \( E_{Z_1}^+ \).

Proof. These follow from the identification \( \tau' = (S_{Z_1} \to S_{Z_0})^{-1}(\sigma') \) in the proof of Lemma 4.3 and from the various definitions introduced in this lemma.

**Lemma 4.5.** In Lemma 4.3 let \( E_{Z_0}(\sigma)^+ := \bigcup_{\sigma' \in \Sigma_{Z_0}, \tau' \subset \tau} E_{Z_0,\sigma'} \) and \( E_{Z_1}(\tau)^+ := \bigcup_{\tau' \in \Sigma_{Z_1}, \tau \subset \tau'} E_{Z_1,\tau'} \), as reduced closed subschemes of \( E_{Z_0}(\sigma) \) and \( E_{Z_1}(\tau) \), respectively. Then the canonical morphism \( E_{Z_0}(\sigma) \to E_{Z_1}(\tau) \) induces a canonical morphism \( E_{Z_0}(\sigma)^+ \to E_{Z_1}(\tau)^+ \) and a canonical isomorphism \( E_{Z_0}(\sigma)^+ \cong E_{Z_1}(\tau)^+ \times E_{Z_1,\tau}(\sigma) \). If we denote by \( \mathcal{I}_\sigma \) (resp. \( \mathcal{I}_\tau \)) the \( \mathcal{O}_{E_{Z_0}(\sigma)} \)-ideal (resp. \( \mathcal{O}_{E_{Z_1}(\tau)} \)-ideal) defining \( E_{Z_0}(\sigma)^+ \) (resp. \( E_{Z_1}(\tau)^+ \)), then \( \mathcal{I}_\sigma \cong (E_{Z_0}(\sigma) \to E_{Z_1}(\tau))^{\mathcal{I}_\tau} \) as \( \mathcal{O}_{E_{Z_0}(\sigma)} \)-ideals.

Proof. If we replace \( \sigma \) with a face \( \sigma' \in \Sigma_{Z_0}^+ \) of it, then the image of \( \sigma' \) in \( S_{Z_1,\tau} \) is a face \( \tau' \in \Sigma_{Z_1}^+ \) of \( \tau \). Consider the reduced closed subscheme \( E(\varsigma)^+ := \bigcup_{\sigma' \in \Sigma_{Z_0}^+ \tau' \subset \tau} E_{\varsigma'} \) of \( E(\varsigma) \), where \( \varsigma' \) denotes the face of \( \varsigma \) corresponding to the face \( \sigma' \) of \( \sigma \). Let \( \mathcal{I}_{\varsigma} \) denote the \( \mathcal{O}_{E(\varsigma)} \)-ideal defining \( E(\varsigma)^+ \). Then both \( \mathcal{I}_\sigma \) and \( \mathcal{I}_\tau \) are pullbacks of \( \mathcal{I}_{\varsigma} \) under the first projections in the compatible fiber products \( E_{Z_1}(\tau) \cong E(\varsigma)^+ \times E_{Z_1}^+ \)

These justify the following:

**Definition 4.6.** We say that two compatible collections \( \Sigma_0 \) and \( \Sigma_1 \) of cone decompositions as in Proposition 3.4.4 are strictly compatible with each other or simply strictly compatible if, for each \( Z_0 \to Z_1 \) as in Proposition 3.4.4, the image of each \( \sigma \in \Sigma_{Z_0}^+ \) under \( P_{Z_0}^+ \to P_{Z_1}^+ \) is exactly some \( \tau \in \Sigma_{Z_1}^+ \).

**Remark 4.7.** Certainly, if \( \Sigma_0 \) and \( \Sigma_1 \) are strictly compatible as in Definition 4.6 then \( \Sigma_0 \) is induced by \( \Sigma_1 \), and they are compatible, as in Proposition 3.4.4.

**Lemma 4.8.** Under the assumption that \( f : X_0 \to X_1 \) is a closed immersion, the morphism \( \prod_j (E_{Z_0,j}(\sigma_j) \times C_{Z_0,j}) \to E_{Z_1}(\tau) \times C_{Z_1} \) in Corollary 3.8 is a closed immersion over the open image of \( U_1 \) under (3.11). Since \( E_{Z_0,j} \) and \( E_{Z_1} \) are separated group schemes with identity sections which are closed immersions, \( C_{Z_0,j} \to C_{Z_1} \) (and hence \( Z_{Z_0,j} \to Z_{Z_1} \)) are also closed immersions over the further image of \( U_1 \) in \( C_{Z_1} \), for all \( j \). Moreover, if \( \Sigma_0 \) and \( \Sigma_1 \) are strictly compatible as in Definition 4.6 then the morphism \( \prod_j (E_{Z_0,j}(\sigma_j) \times C_{Z_0,j}) \to E_{Z_1}(\tau) \times C_{Z_1} \) in Corollary 3.8 is also a closed immersion over the open image of \( U_1 \) under (3.11).

Proof. The first two assertions follow immediately from Corollary 3.8. By Lemma 4.3 the morphism \( E_{Z_0,j}(\sigma_j) \times C_{Z_0,j} \to E_{Z_1}(\tau) \times C_{Z_1} \) is a closed immersion over the image of \( U_1 \), for each \( j \). It remains to show that any point \( x \) in the image of \( \prod_j (E_{Z_0,j}(\sigma_j) \times C_{Z_0,j}) \to E_{Z_1}(\tau) \times C_{Z_1} \) lies on at most one of the images of the above closed immersions. Suppose that there are two distinct indices \( j \) and \( j' \),
together with points \( y \) and \( y' \) of \( E_{\mathbb{Z}_0,i}(\sigma_j) \times \mathcal{O}_{\mathbb{Z},i} \) and \( E_{\mathbb{Z}_0,j}(\sigma_{j'}) \times \mathcal{O}_{\mathbb{Z},j} \), respectively, which are mapped to the point \( x \) of \( E_{\mathbb{Z}_i}(\tau) \times \mathcal{O}_{\mathbb{Z},i} \). Then \( x, y, \) and \( y' \) have the same image \( z \) in \( \mathbb{Z}_1 \), which is also in the images of the closed immersions from \( \mathbb{Z}_0,i \) and \( \mathbb{Z}_0,j \), and we obtain (by pullback to \( z \)) closed immersions \( \phi_j : E_{\mathbb{Z}_0,i}(\sigma_j)_z \to E_{\mathbb{Z}_1}(\tau)_z \) and \( \phi_{j'} : E_{\mathbb{Z}_0,j}(\sigma_{j'})_z \to E_{\mathbb{Z}_1}(\tau)_z \) over \( z \), which are translations of the canonical ones by sections of \( (E_{\mathbb{Z}_1})_z \), with overlapping images. Hence, by Lemma 4.3 in the notation there, \( \phi_j \) and \( \phi_{j'} \) are, respectively, fiber products over \( z \) of some isomorphisms \( E(\zeta)_z \cong E(\zeta)_z \) that are translations of the identity morphism of \( E(\zeta)_z \) by sections of \( E_z \) with closed immersions \( \psi_j : (E_{\mathbb{Z}_0,i})_z \to (E_{\mathbb{Z}_1})_z \) and \( \psi_{j'} : (E_{\mathbb{Z}_0,j})_z \to (E_{\mathbb{Z}_1})_z \) that are translations of the canonical ones by some sections of \( (E_{\mathbb{Z}_1})_z \). Thus, the images of \( \psi_j \) and \( \psi_{j'} \) overlap, exactly because the images of \( \phi_j \) and \( \phi_{j'} \) do, regardless of the above translations of the identity morphism of \( E(\zeta)_z \) by sections of \( E_z \); and hence the images of the closed immersions \( (E_{\mathbb{Z}_0,i})_z \to (E_{\mathbb{Z}_1})_z \) and \( (E_{\mathbb{Z}_0,j})_z \to (E_{\mathbb{Z}_1})_z \) given by the restrictions of \( \phi_j \) and \( \phi_{j'} \) also overlap, contradicting the first assertion of this proposition, as desired. \( \square \)

By Corollary 3.8 and Lemmas 4.5 and 4.8 we obtain the following:

**Proposition 4.9.** If there exist compatible collections \( \Sigma_0 \) and \( \Sigma_1 \) that are strictly compatible as in Definition 4.6, then the induced morphism \( f_{\Sigma_0,\Sigma_1} : X_{\Sigma_0,\Sigma_1}^\text{tor} \to X_{\Sigma_1,\Sigma_1}^\text{tor} \) as in Proposition 3.4 is a closed immersion extending \( f : \mathcal{X}_0 \to \mathcal{X}_1 \). Moreover, if we denote by \( \mathcal{I}_{\Sigma_0,\Sigma_1} \), the \( \mathcal{O}_{\mathcal{X}_{\Sigma_1,\Sigma_1}}^\text{tor} \)-ideal defining the boundary \( X_{\Sigma_1,\Sigma_1}^\text{tor} = \mathcal{X}_1 \) (with its reduced subscheme structure), for \( i = 0, 1 \), then we have \( f_{\Sigma_0,\Sigma_1}^* (\mathcal{I}_{\Sigma_1}) \cong \mathcal{I}_{\Sigma_0} \) as \( \mathcal{O}_{\mathcal{X}_{\Sigma_0,\Sigma_0}}^\text{tor} \)-ideals.

In order to prove Theorem 2.2 it remains to establish the following:

**Proposition 4.10.** There exist compatible collections \( \Sigma_0 \) and \( \Sigma_1 \) that are strictly compatible as in Definition 4.6, which we may assume to be projective and smooth and satisfy the condition that, for \( i = 0, 1 \), and for each \( Z_i \) and each \( \sigma \in \Sigma_{\mathbb{Z}_i}^+ \), the stabilizer \( \Gamma_{Z_i,\sigma} \) of \( \sigma \) in \( \Gamma_{Z_i} \) is trivial. Moreover, we may assume that \( \Sigma_0 \) and \( \Sigma_1 \) refine any finite number of prescribed compatible collections of cone decompositions.

**Proof.** Let us temporarily ignore the assumption on projectivity and smoothness, and take \( \Sigma_0 \) to be induced by \( \Sigma_1 \) as in Proposition 3.1 (cf. [11, Sec. 3.3]). Note that, given any \( Z_i \) and any \( \tau \in \Sigma_{\mathbb{Z}_i}^+ / \Gamma_{\mathbb{Z}_i} \), there exist only finitely many \( Z_0 \) mapped to \( Z_1 \); and for each such \( Z_0 \), there exist only finitely many \( \sigma \in \Sigma_{\mathbb{Z}_0}^+ / \Gamma_{\mathbb{Z}_0} \) mapped to \( \tau \) under the map \( \Sigma_{\mathbb{Z}_0}^+ / \Gamma_{\mathbb{Z}_0} \to \Sigma_{\mathbb{Z}_1}^+ / \Gamma_{\mathbb{Z}_1} \) (simply because the sets of all possible \( Z_0 \) or \( \sigma \) are finite). Since \( \Sigma_0 \) is induced by \( \Sigma_1 \), for any \( \tau \in \Sigma_{\mathbb{Z}_1}^+ \), representing some \( \tau \) as above, each \( \sigma \) that is mapped to \( \tau \) as above is represented by some \( \sigma \in \Sigma_{\mathbb{Z}_0}^+ \), that is the preimage of \( \tau \) under the injection \( \mathcal{S}_{\mathbb{Z}_0} \to \mathcal{S}_{\mathbb{Z}_1} \) as in Proposition 3.4. In this case, the image of \( \sigma \) is the intersection of \( \tau \) with the image of \( \mathcal{S}_{\mathbb{Z}_0} \to \mathcal{S}_{\mathbb{Z}_1} \). As a result, up to refining each such \( \tau \) by intersections with finitely many hyperplanes, and up to refining all the finitely many \( \sigma \) involved accordingly, we may assume that \( \Sigma_0 \) and \( \Sigma_1 \) are strictly compatible (but still not necessarily projective and smooth). We may also refine both of them, and assume that they refine any finite number of prescribed compatible collections and satisfy the condition in the end of the first sentence of the proposition. Finally, up to further refinements, we may also assume that \( \Sigma_0 \) and \( \Sigma_1 \) are both projective and smooth, because as soon as \( \Sigma_0 \) and \( \Sigma_1 \) are
strictly compatible and satisfy the last condition of the proposition, any further refinements will remain so; and because, when \( \Sigma_0 \) and \( \Sigma_1 \) are strictly compatible, both the projectivity and smoothness of \( \Sigma_1 \) are automatically inherited by \( \Sigma_0 \), and hence it suffices to refine \( \Sigma_1 \). (However, note that such an inheritance is not true in general, when \( \Sigma_0 \) is merely induced by \( \Sigma_1 \).) \( \Box \)

The proof of Theorem 2.2 is now complete.

5. Some examples

Example 5.1. In Case \( [1] \), suppose that \( G_0 = GL_{2,\mathbb{Q}} \) and \( G_1 := GL_{2,\mathbb{Q}} \times GL_{2,\mathbb{Q}} \), where the two structure morphisms in the fiber product are both the determinant homomorphism. Then \( G_1 \) is naturally a subgroup scheme of \( G_0 \times G_0 \), and the diagonal morphism of \( G_0 \) factors through a homomorphism \( \rho : G_0 \rightarrow G_1 \). Let \( \mathcal{H}_+ \) and \( \mathcal{H}_- \) denote the Poincaré upper and lower half-planes, respectively, and let \( i \) denote the \( \sqrt{-1} \) in \( \mathcal{H}_+ \). Let \( h_0 : \text{Res}_{\mathbb{C}/\mathbb{R}} G_{m,\mathbb{C}} \rightarrow G_{0,\mathbb{R}} = GL_{2,\mathbb{R}} \) be defined by \( a + bi \mapsto (a \quad -b \choose b \quad a) \), and let \( h_1 \) the composition of \( h_0 \) with \( \rho \). Then \( G_0(\mathbb{R}) \cdot h_0 = \mathcal{H}_+ \), and \( G_1(\mathbb{R}) \cdot h_1 = (\mathcal{H}_+ \times \mathcal{H}_+) \cap (\mathcal{H}_- \times \mathcal{H}_-) \). Let \( \mathcal{H}_0 \subset G_0(\mathbb{A}^\infty) = GL_2(\mathbb{A}^\infty) \) be a principal congruence subgroup of some level \( n \geq 3 \), and let \( \mathcal{H}_1 := (\mathcal{H}_0 \times \mathcal{H}_0) \cap G_1(\mathbb{A}^\infty) \). Then \( \mathcal{X}_0 \) is the modular curve of level \( n \) over \( S_0 = \text{Spec}(\mathbb{Q}) \), and \( \mathcal{X}_1 \) is an open-and-closed subscheme of \( \mathcal{X}_0 \times \mathcal{X}_0 \). In this case, the morphism \( f : \mathcal{X}_0 \rightarrow \mathcal{X}_1 \) is the closed immersion induced by the diagonal morphism of \( \mathcal{X}_0 \), and all possible maps \( \mathcal{P}_{\mathcal{X}_0} \rightarrow \mathcal{P}_{\mathcal{X}_1} \) can be identified with either \( \{0\} \rightarrow \{0\} \) or the diagonal map \( \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^2_{\geq 0} \). There is a unique choice of \( \Sigma_0 \), and \( \mathcal{X}_{0,\Sigma_0}^{\text{tor}} \) is the usual compactified modular curve. Let \( \Sigma'_1 \) denote the compatible collection of cone decompositions for \( \mathcal{X}_1 \) induced by \( \Sigma_0 \times \Sigma_0 \), which is given by either \( \{0\} \) or the faces of the whole cone \( \mathbb{R}^2_{\geq 0} \). Then \( \mathcal{X}_{0,\Sigma_0}^{\text{tor}} \times \mathcal{X}_{0,\Sigma_0}^{\text{tor}} \) is an open-and-closed subscheme of \( \mathcal{X}_{0,\Sigma_0}^{\text{tor}} \times \mathcal{X}_{0,\Sigma_0}^{\text{tor}} \), and the morphism \( f_{0,\Sigma_0}^{\text{tor}} : \mathcal{X}_{0,\Sigma_0}^{\text{tor}} \rightarrow \mathcal{X}_{1,\Sigma'_1}^{\text{tor}} \) is the closed immersion induced by the diagonal morphism of \( \mathcal{X}_{0,\Sigma_0}^{\text{tor}} \). However, \( \Sigma_0 \) and \( \Sigma'_1 \) are not strictly compatible, and the pullback of \( I_{\Sigma'_1} \) is \( I_{\Sigma'_0}^{\mathbb{Z}_2} \) rather than \( I_{\Sigma_0} \) (which means the image of \( f_{0,\Sigma_0}^{\text{tor}} \) does not meet the boundary of \( \mathcal{X}_{1,\Sigma'_1}^{\text{tor}} \), transversely). (See Remark 2.3.) Nevertheless, by Theorem 2.2 there exists a refinement \( \Sigma'_1 \) such that \( f_{0,\Sigma_0}^{\text{tor}} : \mathcal{X}_{0,\Sigma_0}^{\text{tor}} \rightarrow \mathcal{X}_{1,\Sigma'_1}^{\text{tor}} \) is a closed immersion and such that the pullback of \( I_{\Sigma'_1} \) is \( I_{\Sigma'_0}^{\mathbb{Z}_2} \). In practice, the difference between \( \Sigma'_1 \) and its refinement \( \Sigma_1 \) is given by some subdivisions of cones of the form \( \mathbb{R}^2_{\geq 0} \), which correspond to blowups at (possibly nonreduced) closed subschemes supported at the products of cusps, after which the image of \( f_{0,\Sigma_0}^{\text{tor}} \) meets the boundary of \( \mathcal{X}_{1,\Sigma'_1}^{\text{tor}} \) transversally.

Example 5.2. In Case \( [2] \), suppose that we have the following:

1. \( O_0 = \mathbb{Z} \times \mathbb{Z} \) and \( O_1 = \mathbb{Z} \) is diagonally embedded in \( O_0 \), and \( \ast_0 \) and \( \ast_1 \) are trivial.
2. \( L_1 = \mathbb{Z}^{\otimes 4} \), with the first (resp. second) factor of \( O_0 = \mathbb{Z} \times \mathbb{Z} \) acting naturally on the first and third (resp. second and fourth) factors of \( L_1 = \mathbb{Z}^{\otimes 4} \) and trivially on the remaining factors.
3. Let \( \langle \cdot , \cdot \rangle : L_1 \times L_1 \rightarrow \mathbb{Z}(1) \) be the self-dual pairing defined by composing the standard symplectic pairing \( \langle (x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) \rangle \mapsto x_1 y_3 + x_2 y_4 - x_3 y_1 - x_4 y_2 \) with a fixed choice of isomorphism \( 2\pi i : \mathbb{Z} \rightarrow \mathbb{Z}(1) \), where \( i \) is the \( \sqrt{-1} \) in \( \mathcal{H}_+ \) as in Example 5.1 and let \( h_1(a + bi) \) act on
Then \((G_0 \otimes \mathbb{Q}, G_0(\mathbb{R}) \cdot h_0)\) is the same as the \((G_1, G_1(\mathbb{R}) \cdot h_1)\) in Example 5.1, and
\((G_1 \otimes \mathbb{Q}, G_1(\mathbb{R}) \cdot h_1) = (\text{GSp}_{4, \mathbb{Q}}, \mathcal{H}_{2, \pm})\), where \(\mathcal{H}_{2, \pm}\) is the union of the Siegel upper and lower half-spaces of genus two. In both cases, the reflex field is \(\mathbb{Q}\), so that we can take \(F = \mathbb{Q}\), and there are no bad primes for the integral PEL data.

Let \(\mathcal{H}_0 \subset G_0(\mathbb{Z}^>)\) be any neat open compact subgroup, which can be assumed to be of the form \(\mathcal{H}_1 \cap G_0(\mathbb{Z}^>)\) for some neat open compact subgroup of \(G_1(\mathbb{Z}^>)\) (cf. the proofs of [15], Lem. 2.1.1.18 or [17], Lem. 4.16). Then the moduli problem defined by \((\mathcal{O}_1, \star_1, L_1, \langle \cdot, \cdot \rangle_1, h_1)\) and \(\mathcal{H}_1\) is a smooth integral model \(X_1\) of the Siegel threefold over \(\mathbb{Z}_0 = \text{Spec}(\mathbb{Z}(\mathcal{O}))\), parameterizing principally polarized abelian surfaces with symplectic level-\(\mathcal{H}_1\) structures; and the moduli problem defined by \((\mathcal{O}_0, \star_0, L_0, \langle \cdot, \cdot \rangle_0, h_0)\) and \(\mathcal{H}_0\) is the closed moduli subscheme \(X_0\) of \(X_1\) parameterizing principally polarized abelian surfaces of the form \((E_1 \times E_2, \lambda_1 \times \lambda_2)\), where \((E_1, \lambda_1)\) and \((E_2, \lambda_2)\) are canonically principally polarized elliptic curves, with level-\(\mathcal{H}_1\) structures satisfying some conditions. At the level of connected components, up to replacing \(\mathcal{H}_0\) with a finite index subgroup, \(X_0\) can be viewed as the product of two smooth integral models of modular curves. In this case, we have a closed immersion \(f : X_0 \hookrightarrow X_1\), and Theorem [2.2] guarantees the existence of some closed immersion of toroidal compactifications \(f_{\mathcal{O}_0, \mathcal{H}_0}^{\text{tor}} : X_{0, \mathcal{O}_0}^{\text{tor}} \to X_{1, \mathcal{H}_1}^{\text{tor}}\) extending \(f\), defined by some collections \(\Sigma_0\) and \(\Sigma_1\) of cone decompositions that are strictly compatible.

The map \(\overline{\mathcal{P}}_{\mathcal{O}_0}^+ := (\mathcal{P}^+_{\mathcal{O}_0} - \{0\})/\mathbb{R}_{>0} \to \overline{\mathcal{P}}_{\mathcal{H}_1}^+ := (\mathcal{P}^+_{\mathcal{H}_1} - \{0\})/\mathbb{R}_{>0}\) can be from the empty set to the empty set; from a single point to a single point; or from the vertical half-line in \(\mathbb{R}_{>0}\) to \(\mathcal{H}_+\) (up to some identifications). In the last case, \(\Gamma_{\mathcal{O}_0}\) acts trivially on \(\mathbb{R}_{>0}\) because of neatness, while \(\Gamma_{\mathcal{H}_1}\) acts via a neat congruence subgroup of \(\text{SL}_2(\mathbb{Z})\) on \(\mathcal{H}_+\) (with trivial stabilizers). Then \(\Sigma_0^+\) gives a subdivision of \(\mathbb{R}_{>0}\), while \(\Sigma_1^+\) gives a triangularization of \(\mathcal{H}_+\) that is compatible with \(\Gamma_{\mathcal{H}_1}\) and descends to a triangularization of \(\mathcal{H}_+/(\Gamma_{\mathcal{H}_1})\). Note that any nontrivial subdivision of \(\mathbb{R}_{>0}\) means, when we view the connected components of \(X_0\) as products of those of two smooth integral models of modular curves, we have (possibly repeated) blowups at some products of cusps. (This is the end of Example 5.2)

**Example 5.3.** In Case (2), suppose that we have the following:

1. \(n \geq 2\) is any integer.
2. \(K\) is an imaginary quadratic extension of \(\mathbb{Q}\), with maximal order \(\mathcal{O}_K\).
3. \(\mathcal{O}_0 = \mathcal{O}_K \times \mathcal{O}_K\) and \(\mathcal{O}_1 = \mathcal{O}_K\) is diagonally embedded in \(\mathcal{O}_0\), and \(\star_0\) and \(\star_1\) are the complex conjugations (simultaneously on both factors of \(\mathcal{O}_0\)).
4. \(L_1 = \mathcal{O}_K^{\mathbb{Z}^n+1}\), with the first (resp. second) factor of \(\mathcal{O}_0 = \mathcal{O}_K \times \mathcal{O}_K\) acting naturally on the first \(n\) factors (resp. last factor) of \(L_1 = \mathcal{O}_K^{\mathbb{Z}^n+1}\) and trivially on the remaining factors.
5. \(\varepsilon \in \text{Diff}^{\mathbb{Z}_0/k} \mathcal{O}_K\) be any element in the inverse different that is invariant under the complex conjugation, and let \(\langle \cdot, \cdot \rangle_1 : L_1 \times L_1 \to \mathbb{Z}(1)\) be the pairing defined by composing the pairing \((x_1, x_2, \ldots, x_{n+1}, y_1, y_2, \ldots, y_2, y_3, y_4)\) \(\mapsto (2n\sqrt{-1}) \text{Tr}_{\mathcal{O}_K/\mathcal{O}_K} (\varepsilon (-x_1 y_1 + x_2 y_2 + \cdots + x_{n+1} y_{n+1}))\) with a fixed choice of isomorphism \(2n\sqrt{-1} : \mathbb{Z} \to \mathbb{Z}(1)\), and let \(h_1(z)\) act on \(L_{1, \mathbb{R}} \cong \mathbb{C}^{\mathbb{Z}^n+1}\) via
the left multiplication by the complex conjugate \( \overline{z} \) on the first factor, and by \( z \) itself on the remaining factors. Then \( G_0 \otimes \mathbb{R} \cong \mathbb{G}(U_{n-1,1} \times U_1) \cong \mathbb{G}U_{n-1,1} \times \mathbb{G}U_1 \), where the two structure morphisms in the fiber product are similitude homomorphisms; and \( G_1 \otimes \mathbb{R} \cong \mathbb{G}U_{n,1} \).

In both cases, the reflex field is \( K \) because \( n \geq 2 \), so that we can take \( F = K \); and the bad primes are those ramified in \( K \) and divides \( \text{Tr}_{K/2}(\varepsilon) \), and we can take \( \square \) to be any set of rational primes that are not bad. Let us choose \( H_0 \) and \( H_1 \) suitably, so that we have smooth integral models \( X_0 \) and \( X_1 \) over \( S_0 \), with a closed immersion \( f : X_0 \to X_1 \), which can be interpreted as mapping a smooth integral model of a \( GU_{n-1,1} \) Shimura variety to a smooth integral model of a \( GU_{n,1} \) Shimura variety defined by taking fiber products of the universal abelian scheme with some CM elliptic curves (which explains the \( U_1 \) part). (It is perhaps better to work with abelian type Shimura varieties and arrange \( G_0 \otimes \mathbb{R} \to G_1 \otimes \mathbb{R} \) to be \( U_{n-1,1} \to U_{n,1} \), but the difference is on the centers and hence unimportant for our purpose.)

By Theorem 2.2, there exists some closed immersion of toroidal compactifications \( f_{\Sigma_0,\Sigma_1}^\text{tor} : X_{0,\Sigma_0}^\text{tor} \to X_{1,\Sigma_1}^\text{tor} \), extending \( f \), defined by some strictly compatible collections \( \Sigma_0 \) and \( \Sigma_1 \) of cone decompositions. But note that we have no choice to make for \( \Sigma_0 \) and \( \Sigma_1 \). All possible maps \( \mathbb{P}Z_0 \to \mathbb{P}Z_1 \) can be identified with either \( \{0\} \to \{0\} \) or \( \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \), and in all cases the cone decompositions are uniquely determined and trivial (and satisfy all the usual conditions we impose). Hence, Theorem 2.2 just says that the canonical morphism \( f^\text{tor} : X_0^\text{tor} \to X_1^\text{tor} \) between smooth integral models of toroidal compactifications over \( S_0 \), where all the collections of cone decompositions are now justifiably omitted from the notation, is a closed immersion.

Nevertheless, such a discussion is not completely meaningless. The fact that smooth toroidal compactifications of \( X_0 \) and \( X_1 \) uniquely exist is well known, but the fact that closed immersions \( f : X_0 \to X_1 \) extend to closed immersions \( f^\text{tor} : X_0^\text{tor} \to X_1^\text{tor} \) is probably less so. Also, as soon as we have such a \( f^\text{tor} \), we can consider the closed immersion \( \text{Id}_{X_0,f} : X_0^\text{tor} \to X_0^\text{tor} \times X_1^\text{tor}^\text{tor} \), which extends to the closed immersion \( \text{Id}_{X_0^\text{tor},f^\text{tor}} : X_0^\text{tor} \to X_0^\text{tor} \times X_1^\text{tor}^{\text{tor}} \), which provides the justification for some usual geometric considerations related to the Gan–Gross–Prasad conjecture.

We have similar assertions in Case \( 3 \). (This is the end of Example 5.3.)

**Example 5.4.** In Case \( 1 \), suppose that \( G_0 \) is the special orthogonal group over \( \mathbb{Q} \) defined by a quadratic space \( V_0 \) of signature \((n-1,2)\) at \( \infty \), for some \( n \geq 2 \), and let \( G_1 \) be the special orthogonal group over \( \mathbb{Q} \) defined by \( V_1 := (\mathbb{Q} \cdot e) \perp V_0 \), where the quadratic form is defined to have value \(+1\) on the additional basis vector \( e \). Then \( G_0 \otimes \mathbb{R} \cong SO_{n-1,2} \) and \( G_1 \otimes \mathbb{R} \cong SO_{n,2} \). Let \( i \) be the same \( \sqrt{-1} \) as in Example 5.1.

Up to suitable choices of the above isomorphisms, we can arrange that \( h_0 \) and \( h_1 \) are defined by mapping \( G_{m,C} \to SO_{2,R} \) : \( r(\cos \theta + i \sin \theta) \mapsto (\cos \theta \sin 2\theta, -\sin \theta \cos 2\theta) \) into the second factors of the diagonally embedded compact subgroups \( SO_{n-1,1,R} \times SO_{2,R} \) and \( SO_{n,R} \times SO_{2,R} \) of \( SO_{n-1,2} \) and \( SO_{n,2} \), respectively. Then the reflex fields of both Shimura data \((G_0, G_0(R) \cdot h_0)\) and \((G_1, G_1(R) \cdot h_1)\) are \( \mathbb{Q} \), and we can take \( F \) to be \( \mathbb{Q} \) (or any field extension in \( \mathbb{Q} \)). Let \( \mathcal{H}_0 \) and \( \mathcal{H}_1 \) be chosen such that \( f : X_0 \to X_1 \) is a closed immersion over \( S_0 = \text{Spec}(F) \). Then \( \mathbb{P}Z_0 \to \mathbb{P}Z_1 \) can be either from the empty set to the empty set; from a single point to a single point; or from the hyperbolic \((n-1)\)-space to the hyperbolic \( n \)-space (equivariant
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Finally, as an application, let us verify [9, Hypothesis 2.18]. As explained in [9], this allows for a substantial simplification of the proof of the main theorems in [9].

Let us explain how [9, Hypothesis 2.18] fits into our setting. In Case (i), suppose that $G_1 = \text{GSp}_{2g},$ for some $g \geq 0,$ so that $\rho : G_0 \to G_1$ induces a Siegel embedding $(G_0, D_0) \to (G_1, D_1),$ making $(G_0, D_0)$ a Hodge-type Shimura datum. We shall fix the choice of a rational prime $p > 0,$ and assume that the base field $F = C$ is the completion of an algebraic closure of $\mathbb{Q}_p.$ Let $\mathcal{H}^p_0 \subset G_1(\mathbb{A}^{\infty-p})$ be a neat open compact subgroup, and let $\mathcal{H}^p_0 := (\rho(\mathbb{A}^{\infty-p}))^{-1}(\mathcal{H}^p_0).$ For each $r \geq 0,$ consider the principal congruence subgroup $\mathcal{H}_{1,p}^{(r)} := \ker(\text{GSp}_{2g}(\mathbb{Z}_p) \to \text{GSp}_{2g}(\mathbb{Z}/p^r))$ at $p,$ and let $\mathcal{H}_{r}^{(r)} := \mathcal{H}_{1,p}^{(r)} \mathcal{H}_{1,p}^{(r-1)},$ and $\mathcal{H}_{0,p}^{(r)} := (\rho(\mathbb{Q}_p))^{-1}(\mathcal{H}_{1,p}^{(r)})$ and $\mathcal{H}_{0}^{(r)} := \mathcal{H}_{0,p}^{(r)} \mathcal{H}_{0,p}^{(r-1)} = (\rho(\mathbb{A}^{\infty-p}))^{-1}(\mathcal{H}_{1,p}^{(r)}).$ Then we have morphisms between the associated Shimura varieties $f(r) : X_0^{(r)} \to X_i^{(r)}$ at levels $\mathcal{H}_{0,p}^{(r)}$ and $\mathcal{H}_{0}^{(r)},$ which are compatible with each other when we vary $r.$ We shall similarly denote other objects at $\mathcal{H}_{0,p}^{(r)}$ and $\mathcal{H}_{0}^{(r)}$ with superscripts $^\bullet(r).$ By [13, Lem. 2.1.2], up to replacing $\mathcal{H}_{0,p}^{(r)}$ with a finite index subgroup, we may assume that $f(r)$ is a closed immersion, for all $r \geq 0.$

By Proposition 4.10, there exist collections $\{\Xi_0^{(0)}\}$ and $\{\Xi_i^{(0)}\}$ for $\{X_0^{(0)}\}$ and $\{X_i^{(0)}\},$ respectively, that are strictly compatible with each other as in Definition 4.6, which we assume to be projective and smooth and satisfy the condition that, for $i = 0, 1,$ and for each $\Xi_i^{(0)}$ and each $\sigma \in \Sigma_2^{(0)},$ the stabilizer $\Gamma_{\Sigma_i^{(0)},\sigma}$ of $\sigma$ in $\Gamma_{\Sigma_i^{(0)}}$ is trivial.

Note that Proposition 3.4 can also be applied to morphisms between Shimura varieties associated with the same Shimura datum, but with possibly different levels. For each $r \geq 0,$ let $\{\Sigma_0^{(r)}\}$ and $\{\Sigma_i^{(r)}\}$ denote the induced collections at levels $\mathcal{H}_{0,p}^{(r)}$ and $\mathcal{H}_{0}^{(r)},$ respectively. Then they are projective and satisfy the analogue of the above condition on stabilizers, and are strictly compatible with each other. Since the levels $\mathcal{H}_{0,p}^{(r)}$ at $p$ are principal, and since $\mathcal{H}_{0,p}^{(r)} = (\rho(\mathbb{Q}_p))^{-1}(\mathcal{H}_{1,p}^{(r)}),$ the canonical homomorphisms $\mathcal{S}_{\Xi_0^{(0)}} \to \mathcal{S}_{\Xi_i^{(r)}}$ can be identified with $\mathcal{S}_{\Xi_0^{(0)}} \to \frac{1}{p^r}\mathcal{S}_{\Xi_i^{(0)}},$ for $i = 1, 2.$ In particular, the smoothness condition on one decompositions remains the same when we vary $r \geq 0.$ Thus, $\{\Sigma_0^{(r)}\}$ and $\{\Sigma_i^{(r)}\}$ are also smooth, and we have verified all the conditions we would like to impose on these collections. By Proposition 3.4, the canonical morphisms $X_{\Sigma_i^{(r)},\text{tor}} \to X_{\Sigma_i^{(r)}},$ for $i = 1, 2$ and $r \geq r' \geq 0,$ are all finite. Note that each such finite morphism is automatically flat (by [10, IV-3, 15.4.2 c')$\Rightarrow b$]) and therefore universally open (by [10, IV-2, 2.4.6]), because both its source and target are smooth and of the same equi-dimension.

For simplicity, we shall omit the subscripts $^\bullet(r)$ in the following. We shall change the font from $X$ to $\mathcal{X}$ when we denote the associated adic spaces.
The case \( i = 0 \) of the following proposition verifies [9] Hypothesis 2.18:

**Proposition 6.1.** For \( i = 0, 1 \), there is a perfectoid space \( \mathcal{X}_{i}^{(\infty), \text{tor}} \) over \( C \) such that \( \mathcal{X}_{i}^{(\infty), \text{tor}} \sim \varprojlim_{r} \mathcal{X}_{i}^{(r), \text{tor}} \), where \( \sim \) has the same meaning as in [28] Def. 2.4.1.

**Proof.** We shall imitate the proof of [27] Thm. 4.1.1(i). Recall that the assertion \( \mathcal{X}_{i}^{(\infty), \text{tor}} \sim \varprojlim_{r} \mathcal{X}_{i}^{(r), \text{tor}} \) means there are compatible morphisms \( \mathcal{X}_{i}^{(\infty), \text{tor}} \to \mathcal{X}_{i}^{(r), \text{tor}} \) inducing a homeomorphism of topological spaces \( |\mathcal{X}_{i}^{(\infty), \text{tor}}| \sim \varprojlim_{r} |\mathcal{X}_{i}^{(r), \text{tor}}| \), as well as an open covering of \( \mathcal{X}_{i}^{(\infty), \text{tor}} \) by affinoid adic spaces \( \text{Spa}(\mathcal{R}_{i}^{(\infty)}, \mathcal{R}_{i}^{(\infty), +}) \) inducing a homomorphism \( \varprojlim_{r} \mathcal{R}_{i}^{(r)} \to \mathcal{R}_{i}^{(\infty)} \) with dense image, where the direct limit runs over all \( r \geq 0 \) and all affinoid open subspaces \( \text{Spa}(\mathcal{R}_{i}^{(r)}, \mathcal{R}_{i}^{(r), +}) \subset \mathcal{X}_{i}^{(r), \text{tor}} \) through which the compositions of \( \text{Spa}(\mathcal{R}_{i}^{(\infty)}, \mathcal{R}_{i}^{(\infty), +}) \to \mathcal{X}_{i}^{(\infty), \text{tor}} \xrightarrow{\varepsilon} \mathcal{X}_{i}^{(r), \text{tor}} \) factor.

By [25] Cor. A.19 and its proof, the above holds when \( i = 1 \), and we may assume that each member \( \text{Spa}(\mathcal{R}_{1}^{(\infty)}, \mathcal{R}_{1}^{(\infty), +}) \) in the open covering of \( \mathcal{X}_{1}^{(\infty), \text{tor}} \) is a perfectoid and is the preimage of some \( \text{Spa}(\mathcal{R}_{1}^{(r)}, \mathcal{R}_{1}^{(r), +}) \), for all sufficiently large \( r \). For each \( r \geq 0 \), since \( f^{(r), \text{tor}} : \mathcal{X}_{1}^{(r), \text{tor}} \to \mathcal{X}_{1}^{(\infty), \text{tor}} \) is a closed immersion by Proposition 4.9 (and the constructions of \( \Sigma_{0}^{(r)} \) and \( \Sigma_{1}^{(r)} \)), the associated morphism \( \mathcal{X}_{1}^{(r), \text{tor}} \to \mathcal{X}_{1}^{(\infty), \text{tor}} \) is a closed immersion of adic spaces, and we have \( \mathcal{X}_{1}^{(r), \text{tor}} \times \text{Spa}(\mathcal{R}_{i}^{(r)}, \mathcal{R}_{i}^{(r), +}) \cong \text{Spa}(\mathcal{R}_{0}^{(r)}, \mathcal{R}_{0}^{(r), +}) \), where \( R_{0}^{(r)} \to R_{0}^{(\infty)} \) is surjective. Therefore, by [27] Lem. 2.2.2, we have an affinoid perfectoid space \( \mathcal{X}_{1}^{(r), \text{tor}} \times \text{Spa}(\mathcal{R}_{1}^{(\infty)}, \mathcal{R}_{1}^{(\infty), +}) \cong \text{Spa}(\mathcal{R}_{0}^{(\infty), (r)}, \mathcal{R}_{0}^{(\infty), (r), +}) \), where \( \mathcal{R}_{0}^{(\infty), (r)} \) has dense image, for all sufficiently large \( r \). Let \( R_{0}^{(\infty), +} \) denote the \( p \)-adic completion of \( \varprojlim_{r} \mathcal{R}_{0}^{(\infty), (r), +} \), where the direct limit runs over all sufficiently large \( r \) such that \( \mathcal{R}_{0}^{(\infty), (r), +} \) is defined. Then the \( p \)-th power homomorphism \( R_{0}^{(\infty), +}/p \to R_{0}^{(\infty), +}/p \) is surjective because the \( p \)-th power homomorphisms \( R_{0}^{(\infty), (r), +}/p \to R_{0}^{(\infty), (r), +}/p \) are, and hence \( R_{0}^{(\infty)} := R_{0}^{(\infty), +}[1/p] \) is a perfectoid C-algebra, by [12] Prop. 3.6.2. By construction, for all \( r \), we have compatible homomorphisms \( R_{0}^{(r)} \to R_{0}^{(\infty)} \), and the composition of the homomorphisms \( \varprojlim_{r} R_{1}^{(r)} \to R_{1} \) has dense image and factors through \( \varprojlim_{r} R_{0}^{(r)} \to R_{0}^{(\infty)} \). Thus, the affinoid perfectoid space \( \text{Spa}(R_{1}^{(\infty)}, R_{1}^{(\infty), +}) \) satisfies \( \text{Spa}(R_{0}^{(\infty)}, R_{0}^{(\infty), +}) \sim \varprojlim_{r} \text{Spa}(R_{0}^{(r)}, R_{0}^{(r), +}) \). By gluing such \( \text{Spa}(R_{0}^{(\infty)}, R_{0}^{(\infty), +}) \) using [28] Prop. 2.4.3 and 2.4.5] over an open covering of \( \mathcal{X}_{1}^{(\infty), \text{tor}} \) by affinoid perfectoid spaces \( \text{Spa}(R_{1}^{(\infty)}, R_{1}^{(\infty), +}) \) as above, we obtain a perfectoid space \( \mathcal{X}_{1}^{(\infty), \text{tor}} \) over \( C \) such that \( \mathcal{X}_{1}^{(0), \text{tor}} \sim \varprojlim_{r} \mathcal{X}_{1}^{(r), \text{tor}} \), as desired. \( \square \)

**References**

17. , Compactifications of PEL-type Shimura varieties in ramified characteristics, Forum Math. Sigma 4 (2016), e1, 98.

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