CLOSED IMMERSEOS OF TOROIDAL COMPACTIFICATIONS
OF SHIMURA VARIETIES

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Abstract. We explain that any closed immersion between Shimura varieties
defined by morphisms of Shimura data extends to some closed immersion be-
tween their projective smooth toroidal compactifications, up to refining the
choices of cone decompositions. We also explain that the same holds for
many closed immersions between integral models of Shimura varieties and
their toroidal compactifications available in the literature.

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1. Introduction

Given any closed immersion between Shimura varieties or their integral mod-
els defined by some morphism of Shimura data (and some additional data, in the
case of integral models), it is natural to ask whether it extends to a closed immer-
sion between their toroidal compactifications. Since the construction of toroidal
compactifications depend on the choices of some compatible collections of cone de-
compositions, part of the question is whether this can be achieved by some good
choices of them, or whether we can prescribe at least part of the choices.

This question is not as trivial as it seems to be. Already in characteristic zero,
the analogous question for minimal compactifications is subtle. In fact, in Scholze’s
groundbreaking work [29], for Hodge-type Shimura varieties, his perfectoid mini-
mal compactifications at infinite levels were first constructed using the closures in
the minimal compactifications of Siegel modular varieties, rather than the minimal
compactifications of the Hodge-type Shimura varieties themselves; and it was ex-
plained later in [30] that the associated diamonds are nevertheless isomorphic and

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define the same associated perfectoid spaces at infinite levels. However, in the case of toroidal compactifications, if the ambient toroidal compactification is prescribed, then the closure of the Shimura subvariety is generally not normal; and in this case the morphism from the toroidal compactification of the Shimura subvariety is generally not universally injective either, and so even the passage to diamonds might not help. (See Remark 4.1 for a related counter-example.)

What can be shown is that, under some reasonable assumptions, there exist some compatible collections of cone decompositions, up to refinements, such that the morphisms between the associated toroidal compactifications are indeed closed immersions (see Theorem 2.2). Also, we can prescribe certain cone decompositions for the Shimura subvariety, up to raising the level of the ambient Shimura variety and refining the cone decompositions there accordingly (see Theorem 2.3). We expect these to be not only useful for studying cycles of Shimura varieties defined by special subvarieties—see Section 6 for some examples—but also for generalizing some constructions in the appendix of [27], from the case of Siegel modular varieties to those of more general Hodge-type or even abelian-type Shimura varieties.

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2. Main results

Let us assume we are in one of the following cases:

Assumption 2.1.  
(1) For each $i = 0, 1$, let $(G_i, D_i)$ be a Shimura datum (see [9 1.2.1]), where $D_i$ is a $G_i(\mathbb{R})$-conjugacy class of a homomorphism $h_i : \text{Res}_{\mathbb{C}/\mathbb{R}} G_{m, \mathbb{C}} \to G_i(\mathbb{R})$. Let $\rho : G_0 \to G_1$ be an injective homomorphism of algebraic groups over $\mathbb{Q}$ such that $(\rho(\mathbb{R}))(D_0) \subset D_1$. Let $\mathcal{H}_i \subset G_i(\mathbb{A}^\infty)$ be neat (see [28 0.6]) open compact subgroups, for $i = 0, 1$, such that $\mathcal{H}_0 = (\rho(\mathbb{A}^\infty))^{-1}(\mathcal{H}_1)$. Let $F$ denote a subfield of $\mathbb{C}$ containing the reflex field of $(G_0, D_0)$ (which then also contains that of $(G_1, D_1)$ by [9 2.2.1]), and let $S_0 := \text{Spec}(F)$. For each $i = 0, 1$, let $X_i$ denote the base change to $F$ of the canonical model of the Shimura variety associated with $(G_i, D_i)$ at level $\mathcal{H}_i$. Then we have a canonical morphism $f : X_0 \to X_1$ over $S_0$, which we assume to be a closed immersion. (This can be achieved up to replacing $\mathcal{H}_1$ with a finite index subring still containing $(\rho(\mathbb{A}^\infty))(\mathcal{H}_0)$, by [8 1.15]).

(2) For each $i = 0, 1$, let $(\mathcal{O}_i, \ast_i, L_i, \langle \cdot, \cdot \rangle_i, h_i)$ be an integral PEL data (see [20 Def. 1.1.1.1]). Assume that $\mathcal{O}_1$ is a subring of $\mathcal{O}_0$ preserved by $\ast_0$, that $\ast_1 = \ast_0|_{\mathcal{O}_1}$, and that $(L_0, \langle \cdot, \cdot \rangle_0, h_0)$ is PEL-type $\mathcal{O}_1$-lattices (see [18 Def. 1.2.1.3]). For each $i = 0, 1$, let $G_i$ denote the associated group functor over $\text{Spec}(\mathbb{Z})$, as in [18 Def. 1.2.1.6], so that we have a canonical injective homomorphism $\rho : G_0 \to G_1$ by definition. Let $F$ denote a subfield of $\mathbb{C}$ that is a finite extension of the reflex field $F_0$ of $(\mathcal{O}_0, \ast_0, L_0, \langle \cdot, \cdot \rangle_0, h_0)$ (see [18 Def. 1.2.5.4]) which is also the reflex field of $(G_0 \otimes_{\mathbb{Z}} \mathbb{Q}, G_0(\mathbb{R}) \cdot h_0)$, and hence also that of $F_1$ of $(\mathcal{O}_1, \ast_1, L_1, \langle \cdot, \cdot \rangle_1, h_1)$ or $(G_1 \otimes_{\mathbb{Q}} \mathbb{Q}) G_1(\mathbb{R}) \cdot h_1)$, by [9 2.2.1]). Let $\square$ be a set of rational primes (see [18 Notation and Conventions]) that are good (see [18 Def. 1.4.1.1]) for both $(\mathcal{O}_i, \ast_i, L_i, \langle \cdot, \cdot \rangle_i, h_i)$, for $i = 0, 1$, and let
For a morphism \( f \) and \( M \) a canonical morphism. By restricting the 1.4.1.11, and Cor. 7.2.3.10\] features parameterized by \( M \) \(( (G, i) \), either of these two finite extension of the reflex field of \( \mathcal{H}_1 \) contained in \( \mathcal{H}_0 \). Suppose that we have a Siegel embedding defined by some injective homomorphism \( \rho \). Suppose that we have neat open compact subgroups \( H \) auxiliary integral PEL for which \( \mathcal{H}_1 \) denotes the \( (\text{smooth}) \) moduli scheme over \( \text{Spec}(O_{F,(\mathcal{C})}) \) associated with \( (O_1, i, L_1, \langle \cdot, \cdot \rangle, h_1) \) at \( H \) (see [18, Def. 1.4.1.4, Thm. 1.4.1.11, and Cor. 7.2.3.10]). By restricting the \( O_{0,\text{end}} \)-structures parameterized by \( M \) to \( O_{1,\text{end}} \)-structures, we obtain a canonical morphism \( M_{\mathcal{H}_0} \otimes O_{F,(\mathcal{C})} \to M_{\mathcal{H}_1} \otimes O_{F,(\mathcal{C})} \). Then we take \( X_0 \) and \( X_1 \) to be open and closed subschemes of \( M_{\mathcal{H}_0} \otimes O_{F,(\mathcal{C})} \) and \( M_{\mathcal{H}_1} \otimes O_{F,(\mathcal{C})} \), respectively, such that the above morphism induces a morphism \( f : X_0 \to X_1 \), which we assume to be a closed immersion.

(3) For \( i = 0, 1 \), suppose that we have integral PEL data \( (O_i, *_i, L_i, \langle \cdot, \cdot \rangle_i, h_i) \) (for which \( p \) might not be good) together with some suitable choices of \( (O_i, *_i, L_i, \langle \cdot, \cdot \rangle_i, h_i) \) and a shared choice of a collection of auxiliary integral PEL data \( (\text{Spec}(O_{\mathcal{H}_0} \otimes O_{F,(\mathcal{C})} \to M_{\mathcal{H}_0} \otimes O_{F,(\mathcal{C})}) = H \); in [18, Sections 2 and 4]; and that \( (O_1, *_1, L_1, \langle \cdot, \cdot \rangle_1, h_1) \) also serves as a choice of auxiliary integral PEL datum for \( (O_0, *_0, L_0, \langle \cdot, \cdot \rangle_0, h_0) \) (but without requiring that \( p \) is good for either of these two). Then we have morphisms \( G_0 \to G_1 \to G_{\mathcal{H}_1} \). Suppose that we have neat open compact subgroups \( \mathcal{H}_0 \subset \mathcal{G}(\mathbb{Z}) \), \( \mathcal{H}_1 \subset \mathcal{G}(\mathbb{Z}) \), and \( \mathcal{H}_{j,\text{aux}} \subset \mathcal{G}_{j,\text{aux}}(\mathbb{Z}) \) such that \( \mathcal{H}_0 = (\rho(\mathbb{A})^{-1})(\mathcal{H}_1) \) and such that the image of \( \mathcal{H}_1 \) under \( G_1(\mathbb{Z}) \to G_{j,\text{aux}}(\mathbb{Z}) \) is neat and contained in \( \mathcal{H}_{j,\text{aux}} \), for all \( j \in J \). Let \( F \) denote a subfield of \( \mathcal{C} \) that is a finite extension of the reflex field of \( (O_0, *_0, L_0, \langle \cdot, \cdot \rangle_0, h_0) \), and hence also those of \( (O_1, *_1, L_1, \langle \cdot, \cdot \rangle_1, h_1) \) and \( (O_{\mathcal{H}_1} \otimes O_{F,(\mathcal{C})} \to M_{\mathcal{H}_0} \otimes O_{F,(\mathcal{C})}) \), together with canonical finite morphisms \( \mathcal{H}_1 \to \mathcal{H}_{j,\text{aux}} \), \( j \in J \). With the above data, we have associated moduli problems \( M_{\mathcal{H}_0} \) and \( M_{\mathcal{H}_1} \) over \( \text{Spec}(F) \), and associated auxiliary moduli problems \( M_{\mathcal{H}_{j,\text{aux}}} \) over \( S_0 = \text{Spec}(O_{F,(p)}) \), for all \( j \in J \). Then we take \( X_0 \) and \( X_1 \) to be open and closed subschemes of \( M_{\mathcal{H}_0} \) and \( M_{\mathcal{H}_1} \), respectively, such that \( M_{\mathcal{H}_0} \to M_{\mathcal{H}_1} \) induces a morphism \( f : X_0 \to X_1 \), which we assume to be a closed immersion.

(4) Suppose that we have a morphism of Shimura data \( (G_0, D_0) \to (G_1, D_1) \) defined by some injective homomorphism \( \rho : G_0 \to G_1 \) as in [11], and suppose that we have a Siegel embedding \( (G_1, D_1) \subset (G_{\text{aux}}, D_{\text{aux}}) \) defined by some injective homomorphism \( G_1 \to G_{\text{aux}}, \) with \( G_{\text{aux}} \cong GSp_{2g,Q} \), for some \( g \geq 0 \). Suppose that we have neat open compact subgroups \( \mathcal{H}_0 \subset G_0(\mathbb{A}), \mathcal{H}_1 \subset G_1(\mathbb{A}), \) and \( \mathcal{H}_{\text{aux}} \subset G_{\text{aux}}(\mathbb{A}) \) such that \( \mathcal{H}_0 = (\rho(\mathbb{A}^{-1}))(\mathcal{H}_1) \) and such that the image of \( \mathcal{H}_1 \) under \( G_1(\mathbb{A}) \to G_{\text{aux}}(\mathbb{A}) \) is neat and contained in \( \mathcal{H}_{\text{aux}} \). Let \( F \) denote a subfield of \( \mathcal{C} \) that is a finite extension of the reflex field of \( (G_0, D_0) \), and hence also that of \( (G_1, D_1) \). Let \( X_0 \) and \( X_1 \) be integral models over \( S_0 = \text{Spec}(O_{F,(p)}) \) of the Shimura varieties associated with \( (G_0, D_0) \) and \( (G_1, D_1) \) at levels \( H_0 \) and \( H_1 \), respectively, defined by taking normalizations of the characteristic zero models over \( F \) (which are base
changes of the corresponding canonical models to $F$) over the Siegel moduli over $\text{Spec}(\mathbb{Z}(p))$ associated with $(G_{\text{aux}}, D_{\text{aux}})$ and the prime-to-$p$ level $\mathcal{H}_{\text{aux}}$, as in [25, Introduction]. Then we have a canonically induced morphism $f : X_0 \to X_1$ over $S_0$, which we assume to be a closed immersion.

We shall say that we are in Cases (1), (2), (3), or (4) depending on the case we are in Assumption 2.1. In each case, we have good toroidal compactifications $X_i \hookrightarrow X_{i, \text{tor}}$, associated with some compatible collections of cone decompositions $\Sigma_i$, for $i = 0, 1$, whose properties we will review in more detail in the next section.

Our main results are the following two theorems:

**Theorem 2.2.** Let $f : X_0 \to X_1$ be as in Assumption 2.1. Then there exist toroidal compactifications $X_i \hookrightarrow X_{i, \text{tor}}$, for $i = 0, 1$, associated with some compatible collections $\Sigma_i$ of projective smooth cone decompositions (see [2, 3, 28] in Case (1); see [18, Thm. 6.4.1.1 and 7.3.3.4] in Case (2); see [21, Thm. 6.1] in Case (3); and see [20, Thm. 4.1.5 and Rem. 4.1.6] in Case (4)) such that $f$ extends to a closed immersion $f_{\Sigma_0, \Sigma_1} : X_{0, \text{tor}} \to X_{1, \text{tor}}$. We may require $\Sigma_0$ and $\Sigma_1$ to refine any finite number of prescribed compatible collections of cone decompositions.

**Theorem 2.3.** Let $f : X_0 \to X_1$ be as in Assumption 2.1, with a toroidal compactification $X_0 \hookrightarrow X_{0, \text{tor}}$ associated with some prescribed compatible collection $\Sigma_0$ of smooth cone decompositions. Then, up to replacing $\mathcal{H}_1$ with a finite index subgroup (without changing $\mathcal{H}_0$), which does not affect the assumption that $f : X_0 \to X_1$ is a closed immersion (see Remark 2.5 below), there exist some toroidal compactifications $X_1 \hookrightarrow X_{1, \text{tor}}$, associated with some collection of smooth cone decomposition $\Sigma_1$ such that $f$ extends to a closed immersion $f_{\Sigma_0, \Sigma_1}^\text{tor} : X_{0, \text{tor}} \to X_{1, \text{tor}}$. If there exists some projective $\Sigma'_1$ such that $f$ extends to a finite morphism $f_{\Sigma_0, \Sigma'_1}^\text{tor} : X_{0, \text{tor}} \to X_{1, \Sigma'_1}$ (i.e., $\Sigma_0$ is induced by $\Sigma'_1$), then we may assume in the above that $\Sigma_1$ is projective smooth and refines $\Sigma'_1$ (or rather, when $\mathcal{H}_1$ is raised to a higher level, the compatible collection induced by $\Sigma'_1$ at the higher level).

The proofs of Theorems 2.2 and 2.3 will be given in the next few subsections, and will be finished in Sections 4 and 5, respectively.

**Remark 2.4.**

(1) In Cases (2) and (3), for example, we can take $X_i$ to be the schematic closure of the based change to $\text{Spec}(F)$ of the canonical model of the Shimura variety associated with the Shimura datum $(G_i \otimes \mathbb{Z}, G_i(\mathbb{R}) \cdot h_i)$ (see [16, Section 8], [17, Section 2], and [25, Section 1.2]), for $i = 0, 1$, when $G_i \otimes \mathbb{Z}$ is connected and so $(G_i \otimes \mathbb{Q}, G_i(\mathbb{R}) \cdot h_i)$ qualifies as a Shimura datum.

(2) In Case (2), in order to show that $f : X_0 \to X_1$ is indeed a closed immersion, we often have to resort to the moduli interpretations of $\mathcal{M}_{\mathcal{H}_0}$ and $\mathcal{M}_{\mathcal{H}_1}$.

(3) In Case (3), when the levels $\mathcal{H}_0$ and $\mathcal{H}_1$ differ at $p$ from the stabilizers of $L_0$ and $L_1$, it is generally much more difficult to verify that the morphism $f : X_0 \to X_1$ defined abstractly by taking normalizations is a closed immersion. Practically, when the levels are parahoric at $p$, we can still work with some explicit alternative definitions of $X_0$ and $X_1$ using certain moduli problems with additional structures, and resort to local calculations or the theory of local models—see, for example, [19, Ex. 2.4 and 13.12, and Rem. 16.5]. However, we do not (yet) have a method to study higher levels in general.
(4) In Case (4), even in the best understood situation with hyperspecial levels at \( p \) in [15], as far as we know, there is (as yet) no general method for verifying whether the morphism \( f : X_0 \to X_1 \) is a closed immersion.

(5) Nevertheless, Theorems 2.2 and 2.3 provide closed immersions \( f_{\Sigma_0, \Sigma_1}^{\text{tor}} : X_{0,\Sigma_0}^{\text{tor}} \to X_{1,\Sigma_1}^{\text{tor}} \) as long as the input \( f : X_0 \to X_1 \) is a closed immersion, and we included all four cases (which in theory allows arbitrarily high levels at \( p \) in Cases (3) and (4)) even when the assumption of being a closed immersion cannot be easily verified in general.

(6) Certainly, we expect both Theorems 2.2 and 2.3 to extend to integral models of abelian-type Shimura varieties, generalizing those constructed in Cases (2), (3), and (4) in Assumption 2.1, as soon as the their toroidal compactifications are constructed and shown to have desired properties as in Proposition 3.1 and 3.4 below. However, we do not expect it to be any easier to verify that \( f : X_0 \to X_1 \) is indeed a closed immersion.

**Remark 2.5.** In Theorem 2.3, when we replace \( H_1 \) with a finite index subgroup \( H_1' \), if we denote the corresponding models by \( X_1 \) and \( X_1' \) for simplicity, then we have finite morphisms \( X_0 \to X_1' \to X_1 \) whose composition is a closed immersion by assumption, and the first morphism \( X_0 \to X_1' \) is automatically a closed immersion. There is no need to verify the assumption of being a closed immersion again.

**Remark 2.6.** In Theorem 2.2 and 2.3, the projectivity of the cone decompositions is practically important because, without it, we cannot be sure that the toroidal compactifications we obtained are schemes rather than merely algebraic spaces.

## 3. Morphisms between toroidal compactifications

In all cases in Assumption 2.1 we have good toroidal and minimal compactifications \( X_i^{\text{tor}} \to S_0 \) and \( X_i^{\text{min}} \to S_0 \), for \( i = 0, 1 \), whose qualitative properties we shall summarize as follows, based on the constructions in [4, 2, 3, 28, 18, 19, 21] (as in [23, Prop. 2.2] and [22, Prop. 2.1.2 and 2.1.3, and Cor. 2.1.7] and their proofs):

**Proposition 3.1.** For each \( i = 0, 1 \), there is a canonical minimal compactification \( J_i^{\text{min}} : X_i \hookrightarrow X_i^{\text{min}} \) over \( S_0 \), together with a canonical collection of toroidal compactifications \( J_i^{\text{tor}} : X_i \hookrightarrow X_i^{\text{tor}} \) over \( S_0 \), labeled by certain compatible collections \( \Sigma_i \) of cone decompositions, satisfying the following properties:

1. For each \( \Sigma_i \), there is a proper surjective structural morphism \( f_{i, \Sigma_i}^{\text{tor}} : X_i^{\text{tor}} \to X_i^{\text{min}} \)
   compatible with \( J_i^{\text{min}} \) and \( J_i^{\text{tor}} \) in the sense that \( J_i^{\text{min}} = f_{i, \Sigma_i}^{\text{tor}} \circ J_i^{\text{tor}} \).
2. The scheme \( X_i^{\text{min}} \) admits a stratification by locally closed subschemes \( Z_i \) flat over \( S_0 \), each of which is isomorphic to a finite quotient of an analogue of \( X_i \). (Nevertheless, in Cases (2) and (3), we can still identify each \( Z_i \) with an analogue of \( X_i \).)
3. Each \( \Sigma_i \) is a set \( \{ \Sigma_{Z_i} \} \) of cone decompositions \( \Sigma_{Z_i} \) with the same index set as that of the strata of \( X_i^{\text{min}} \). (In [18], the elements of this index set was
called *cusp labels*.) For simplicity, we shall suppress such cusp labels and denote the associated objects with subscripts given by the strata $Z_i$.

(4) For each stratum $Z_i$, the cone decomposition $\Sigma_i$ is a cone decomposition of some $P_{Z_i}$, where $P_{Z_i}$ is the union of the interior $P_{Z_i}^+$ of a homogenous self-adjoint cone (see [1, Chapter 2]) and its rational boundary components, which is admissible with respect to some arithmetic group $\Gamma_i$, acting on $P_{Z_i}$ (and hence also on $\Sigma_i$). Then $\Sigma_i$ has a subset $\Sigma_i^+$, forming a cone decomposition of $P_{Z_i}^+$. If $\tau$ is a cone in $\Sigma_i$, that is not in $\Sigma_i^+$, then there exist a stratum $Z_i'$ of $X_i^\min$, whose closure in $X_i^\min$ contains $Z_i$, and a cone $\tau'$ in $\Sigma_i^+$, whose $\Gamma_i$-orbit is uniquely determined by the $\Gamma_i$-orbit of $\tau$.

We may and we shall assume that $\Sigma_i$ is smooth, and that, for each $Z_i$ and $\sigma \in \Sigma_i^+$, its stabilizer $\Gamma_i,\sigma$ in $\Gamma_i$ is trivial.

(5) For each $\Sigma_i$, the associated $X_i^\tor$ admits a stratification by locally closed subschemes $Z_{i,\sigma}$ flat over $S_0$, labeled by the strata $Z_i$ of $X_i^\min$ and the orbits $[\sigma] \in \Sigma_i^+/\Gamma_i$. The stratifications of $X_i^\tor$ and $X_i^\min$ are compatible with each other in a precise sense, which we summarize as follows: The preimage of a stratum $Z_i$ of $X_i^\min$ is the (set-theoretic) disjoint union of the strata $Z_{i,\sigma}$ of $X_i^\tor$, with $[\sigma] \in \Sigma_i^+/\Gamma_i$. If $\tau$ is a face of a representative $\sigma$ of $[\sigma]$, which is identified (as in the property [1] above) with the $\Gamma_i$-orbit $[\tau']$ of some cone $\tau'$ in $\Sigma_i^+$, where $Z_i'$ is a stratum whose closure in $X_i^\min$ contains $Z_i$, then $Z_{i,\sigma}$ is contained in the closure of $Z_i'$.

(6) For each stratum $Z_i$ of $X_i^\min$, there is a proper surjective morphism

$$C_{Z_i} \rightarrow Z_i$$

(whose precise description is not important for our purpose), together with a morphism

$$\Xi_{Z_i} \rightarrow C_{Z_i}$$

of schemes which is a torsor under the pullback of a split torus $E_{Z_i}$ with some character group $S_{Z_i}$ over $\text{Spec}(\mathbb{Z})$, so that we have

$$\Xi_{Z_i} \cong \text{Spec}_{C_{Z_i}} \left( \bigoplus_{\ell \in S_{Z_i}} \Psi_{Z_i}(\ell) \right),$$

for some invertible sheaves $\Psi_{Z_i}(\ell)$. (Each $\Psi_{Z_i}(\ell)$ can be viewed as the subsheaf of $(\Xi_{Z_i} \rightarrow C_{Z_i})_*\mathcal{O}_{\Xi_{Z_i}}$ on which $E_{Z_i}$ acts via the character $\ell \in S_{Z_i}$.)

This character group $S_{Z_i}$ admits a canonical action of $\Gamma_i$, and its $\mathbb{R}$-dual $S_{Z_i,\mathbb{R}} := \text{Hom}_{\mathbb{Z}}(S_{Z_i}, \mathbb{R})$ canonically contains the above $P_{Z_i}$ and $P_{Z_i}^+$ as subsets with compatible $\Gamma_i$-actions.

(7) For each $\sigma \in \Sigma_i$, consider the canonical pairing $(\cdot, \cdot) : S_{Z_i} \times S_{Z_i,\mathbb{R}} \rightarrow \mathbb{R}$ and $\sigma^\vee := \{ \ell \in S_{Z_i} : (\ell, y) \geq 0, \forall y \in \sigma \}$, $\sigma_0^\vee := \{ \ell \in S_{Z_i} : (\ell, y) > 0, \forall y \in \sigma \}$, and $\sigma^\perp := \{ \ell \in S_{Z_i} : (\ell, y) = 0, \forall y \in \sigma \} \cong \sigma^\vee/\sigma_0^\vee$. Then we have the affine toroidal embedding

$$\Xi_{Z_i} \hookrightarrow \Xi_{Z_i}(\sigma) := \text{Spec}_{C_{Z_i}} \left( \bigoplus_{\ell \in \sigma^\vee} \Psi_{Z_i}(\ell) \right).$$
The scheme $\Xi_{Z_i}(\sigma)$ has a closed subscheme $\Xi_{Z_i,\sigma}$ defined by the ideal sheaf corresponding to $\bigoplus_{\ell \in \sigma^+} \Psi_{Z_i}(\ell)$, so that

$$\Xi_{Z_i,\sigma} = \text{Spec}_{\sigma_{Z_i}} \left( \bigoplus_{\ell \in \sigma^+} \Psi_{Z_i}(\ell) \right).$$

Then $\Xi_{Z_i}(\sigma)$ admits a natural stratification by locally closed subschemes $\Xi_{Z_i,\tau}$ (i.e., the closed subscheme as above of the open subscheme $\Xi_{Z_i}(\tau)$ of $\Xi_{Z_i}(\sigma)$), where $\tau$ runs over all the faces of $\sigma$ in $\Sigma_{Z_i}$.

(8) For each given $\Sigma_i$, and for each $Z_i$, consider the full toroidal embedding

$$\Xi_{Z_i,\Sigma_i} = \bigcup_{\sigma \in \Sigma_i} \Xi_{Z_i}(\sigma)$$

defined by the cone decomposition $\Sigma_{Z_i}$ (cf. [18, Thm. 6.1.2.8 and Section 6.2.5]), and consider the formal completion $X_{Z_i,\Sigma_i} := (\Xi_{Z_i,\Sigma_i})^\wedge$ of $\Xi_{Z_i,\Sigma_i}$ along its closed subscheme $\bigcup_{\tau \in \Sigma_i} \Xi_{Z_i,\tau}$. Consider, for each $\sigma \in \Sigma^+_i$, the formal completion $X_{Z_i,\sigma}^\wedge := (\Xi_{Z_i}(\sigma))^\wedge$ of $\Xi_{Z_i}(\sigma)$ along its closed subscheme $\Xi_{Z_i}(\sigma)^+ := \bigcup_{\tau \in \Sigma^+_i, \tau \subset \pi} \Xi_{Z_i,\tau}$. Then $X_{Z_i,\Sigma_i}$ admits an open covering by $X_{Z_i,\sigma}$ for $\sigma$ running through elements of $\Sigma^+_i$, and we have canonical flat morphisms $X_{Z_i,\sigma} \hookrightarrow X_{Z_i,\Sigma_i} \to X_{i,\Sigma_i}^\text{tor}$ (of locally ringed spaces) inducing isomorphisms

$$X_{Z_i,\sigma}^\wedge \sim (X_{i,\Sigma_i}^\text{tor})^\wedge \bigcup_{\tau \in \Sigma^+_i, \tau \subset \pi} Z_{i,\tau} \in \Sigma_i$$

and

$$X_{Z_i,\Sigma_i}/\Gamma_{Z_i} \sim (X_{i,\Sigma_i}^\text{tor})^\wedge \bigcup_{\tau \in \Sigma^+_i/\Gamma_{Z_i}} Z_{i,\tau} \in \Sigma_i.$$

More precisely, for each $\sigma \in \Sigma^+_i$, and for each affine open formal subscheme $\mathfrak{M} = \text{Spf}(R)$ of $X_{Z_i,\sigma}^\wedge$, under the canonically induced (flat) morphisms $W := \text{Spec}(R) \to X_{i,\Sigma_i}^\text{tor}$ and $\text{Spec}(R) \to \Xi_{Z_i}(\sigma)$ induced by (3.2), the stratification of $W$ induced by that of $X_{i,\Sigma_i}^\text{tor}$ coincides with the stratification of $W$ induced by that of $\Xi_{Z_i}(\sigma)$. In particular, the preimages of $X_i$ and $\Xi_{Z_i}$ coincide as an open subscheme $W^0$ of $W$.

As for the morphism $f : X_0 \to X_1$, we have the following:

**Proposition 3.4.** Assuming slightly more generally (than in Assumption 2.1) that $(\rho(A^{\infty}))(H_0) \subset H_1$ and hence that the morphism $f : X_0 \to X_1$ is finite, there exists a canonical finite morphism

$$f_{\text{min}} : X_{0,\Sigma_0}^\text{min} \to X_{1,\Sigma_1}^\text{min}$$

such that $f_{\text{min}} \circ J_{0,\text{min}} = J_{1,\text{min}} \circ f$ over $S_0$, together with a canonical collection of proper morphisms

$$f_{\Sigma_0,\Sigma_1}^\text{tor} : X_{0,\Sigma_0}^\text{tor} \to X_{1,\Sigma_1}^\text{tor}$$
such that $f^\text{tor}_{\Sigma_0,\Sigma_1} \circ f^\text{tor}_{\Sigma_0,\Sigma_0} = f^\text{tor}_{\Sigma_1,\Sigma_1} \circ f$ and $f^\text{min} \circ f^\text{tor}_{\Sigma_0,\Sigma_0} = f^\text{tor}_{\Sigma_1,\Sigma_1} \circ f^\text{tor}_{\Sigma_0,\Sigma_1}$ over $S_0$, labeled by certain pairs $(\Sigma_0, \Sigma_1)$ of compatible collections of cone decompositions that are compatible with each other in a sense that we shall explain below, satisfying the following properties:

1. For each stratum $Z_0$ of $X^\text{min}_0$, there exists a (unique) stratum $Z_1$ of $X^\text{min}_1$ such that $f^\text{min}(Z_0) \subset Z_1$ (as subsets of $X^\text{min}_1$). Moreover, $Z_0$ is both open and closed in $(f^\text{min})^{-1}(Z_1)$, and $f^\text{min}$ induces a finite morphism $Z_0 \to Z_1$.

2. Over any $Z_0 \to Z_1$ as above, we have a finite morphism

$$C_{Z_0} \to C_{Z_1},$$

over which we have a finite morphism

$$\Xi_{Z_0} \to \Xi_{Z_1},$$

which induces a finite morphism $\Xi_{Z_0} \to \Xi_{Z_1} \times C_0$ which is equivariant with a group homomorphism of tori

$$E_{Z_0} \to E_{Z_1},$$

with finite kernel over $\text{Spec}(Z)$ that is dual to a homomorphism

$$S_{Z_1} \to S_{Z_0}$$

of character groups with finite cokernel. The $\mathbb{R}$-dual of this last homomorphism is an injective homomorphism $S_{Z_1}^\vee \to S_{Z_0}^\vee$ of $\mathbb{R}$-vector spaces, inducing a Cartesian diagram of injective maps

$$\begin{array}{ccc}
P^+_Z & \to & P^+_\mathbb{Z} \\
\downarrow & & \downarrow \\
P_{Z_0} & \to & P_{\mathbb{Z}}.
\end{array}$$

All the above maps from objects associated with $Z_0$ to the corresponding ones associated with $Z_1$ are equivariant with a canonical homomorphism $\Gamma_{Z_0} \to \Gamma_{Z_1}$. If $\ell_1 \in S_{Z_1}$ is mapped to $\ell_0 \in S_{Z_0}$ under $S_{Z_1} \to S_{Z_0}$, then the invertible sheaf $\Psi_{Z_1}(\ell_0)$ over $C_{Z_0}$ is canonically isomorphic to the pullback of the invertible sheaf $\Psi_{Z_1}(\ell)$ over $C_{Z_1}$ under the above morphism $C_{Z_0} \to C_{Z_1}$.

When $\mathcal{H}_0 = (\rho(\mathbb{A}^\infty))^{-1}(\mathcal{H}_1)$, the homomorphism $S_{Z_1} \to S_{Z_0}$ is surjective, and hence the dual homomorphism $E_{Z_0} \to E_{Z_1}$ is a closed immersion.

3. If the image of $\sigma \in \Sigma_{Z_0}$ under $P_{Z_0} \to P_{\mathbb{Z}}$ is contained in some $\tau \in \Sigma_{Z_1}$, then we have a canonical morphism

$$\Xi_{Z_0}(\sigma) = \text{Spec}_{C_{Z_0}} \left( \bigoplus_{\ell \in \sigma} \Psi_{Z_0}(\ell) \right) \to \Xi_{Z_1}(\tau) = \text{Spec}_{C_{Z_1}} \left( \bigoplus_{\ell \in \tau} \Psi_{Z_1}(\ell) \right)$$

extending $\Xi_{Z_0} \to \Xi_{Z_1}$, and inducing a canonical morphism

$$\Xi_{Z_0}(\sigma) \to \Xi_{Z_1}(\tau) \times C_0$$

which is equivariant with $E_{Z_0} \to E_{Z_1}$.

Moreover, there is an induced morphism

$$\Xi_{Z_0}(\sigma) = \text{Spec}_{C_{Z_0}} \left( \bigoplus_{\ell \in \sigma} \Psi_{Z_0}(\ell_0) \right) \to \Xi_{Z_1}(\tau) = \text{Spec}_{C_{Z_1}} \left( \bigoplus_{\ell \in \tau} \Psi_{Z_1}(\ell_1) \right).$$

4. We say that the compatible collections $\Sigma_0 = \{\Sigma_{Z_0}\}_{Z_0}$ and $\Sigma_1 = \{\Sigma_{Z_1}\}_{Z_1}$ are compatible with each other if, when $Z_0$ is mapped to $Z_1$ as above, the image of each $\sigma \in \Sigma^+_Z$ under the map $P^+_Z \to P^+_\mathbb{Z}$...
\( P_2^+ \) is contained in some \( \tau \in \Sigma^+_Z \). We say that \( \Sigma_0 \) is induced by \( \Sigma_1 \) if each \( \sigma \in \Sigma^+_Z \) is exactly the preimage of some \( \tau \in \Sigma^+_Z \). (If \( \Sigma_0 \) is induced by \( \Sigma_1 \), then they are necessarily compatible.)

(5) The morphism \( f : X_0 \to X_1 \) extends to a proper (resp. finite) morphism \( f^\text{tor} : X^\text{tor}_{0, \Sigma_1} \to X^\text{tor}_{1, \Sigma_1} \) as above if and only if \( \Sigma_0 \) and \( \Sigma_1 \) are compatible (resp. \( \Sigma_0 \) is induced by \( \Sigma_1 \)). When \( \Sigma_0 \) and \( \Sigma_1 \) are compatible, if the image of \( \sigma \in \Sigma^+_Z \) under \( P^+_Z \to P^+_Z \) is contained in \( \tau \in \Sigma^+_Z \), then the morphism \( f^\text{tor}_{\Sigma_0, \Sigma_1} \) induces a morphism \( Z_0[\sigma] \to Z_1[\tau] \) (which is not necessarily proper), which can be canonically identified with the morphism \( \Xi_{Z_0, \sigma} \to \Xi_{Z_1, \tau} \) above.

For each \( \tau \in \Sigma^+_Z \), the preimage of \( Z_1[\tau] \) is the (set-theoretic) disjoint union of the strata \( Z_0[\sigma] \) labeled by \( \sigma \in \Sigma^+_Z \) that are mapped into \( \tau \) under \( P^+_Z \to P^+_Z \). If there is a unique such \( \sigma \), which is the case exactly when \( \sigma \) is the preimage of \( \tau \), then the induced morphism \( Z_0[\sigma] \to Z_1[\tau] \) is finite.

(6) Suppose that \( \Sigma_0 \) and \( \Sigma_1 \) are compatible. Then there is a proper morphism

\[
\Xi_{Z_0, \Sigma_0} \to \Xi_{Z_1, \Sigma_1},
\]

whose formal completion gives a proper morphism

\[
(3.5) \quad X^\text{tor}_{Z_0, \Sigma_0} \to X^\text{tor}_{Z_1, \Sigma_1}.
\]

These two morphisms are equivariant with the homomorphism \( \Gamma_{Z_0} \to \Gamma_{Z_1} \) and induces a proper morphism \( X^\text{tor}_{Z_0, \Sigma_0} / \Gamma_{Z_0} \to X^\text{tor}_{Z_1, \Sigma_1} / \Gamma_{Z_1} \), which can be identified (via isomorphisms as in (3.3)) with

\[
\left( X^\text{tor}_{0, \Sigma_0} \right)^\land \left. \bigcup_{\sigma \in \times_{Z_0} \Sigma_0} Z_0[\sigma] \right\} \to \left( X^\text{tor}_{1, \Sigma_1} \right)^\land \left. \bigcup_{\tau \in \times_{Z_1} \Sigma_1} Z_1[\tau] \right\}.
\]

If the image of \( \sigma \in \Sigma^+_Z \) under \( P^+_Z \to P^+_Z \) is contained in some \( \tau \in \Sigma^+_Z \), we have an induced morphism \( X^\text{tor}_{2, \sigma} \to X^\text{tor}_{1, \tau} \), which can be identified (via isomorphisms as in (3.2)) with

\[
\left( X^\text{tor}_{0, \Sigma_0} \right)^\land \left. \bigcup_{\sigma' \in \Sigma^+_Z, \sigma' \subset \sigma} Z_1[\sigma'] \right\} \to \left( X^\text{tor}_{1, \Sigma_1} \right)^\land \left. \bigcup_{\tau' \in \Sigma^+_Z, \tau' \subset \tau} Z_1[\tau'] \right\}.
\]

For a fixed \( \tau \in \Sigma^+_Z \), the pullback of (3.5) to the open formal subscheme \( X^\text{tor}_{1, \tau} \) on the target gives a proper morphism

\[
(3.6) \quad \bigcup_{\sigma \in \Sigma^+_Z, (P_0^+ \to P_1^+)(\sigma) \subset \tau} X^\text{tor}_{Z_0, \sigma} \to X^\text{tor}_{Z_1, \tau}.
\]

Suppose moreover that \( \Sigma_0 \) is induced by \( \Sigma_1 \). Then both morphisms (3.5) and (3.6) are finite. For \( \tau \in \Sigma^+_Z \) as above, with \( \sigma \in \Sigma^+_Z \) the preimage of \( \tau \), which is the unique element in \( \Sigma^+_Z \) such that \( (P_0^+ \to P_1^+)(\sigma) \subset \tau \); and for each affine open formal subscheme \( \mathfrak{M}_1 = \text{Spf}(R_1) \) of \( X^\text{tor}_{Z_1, \sigma} \), let \( \mathfrak{M}_0 = \text{Spf}(R_0) \) denote its pullback to \( X^\text{tor}_{Z_0, \sigma} \). Under the morphisms \( W_1 := \text{Spec}(R_1) \to X^\text{tor}_{Z_1, \sigma}, W_1 \to \Xi_{Z_1, \tau}, W_0 := \text{Spec}(R_0) \to X^\text{tor}_{Z_0, \sigma}, \) and \( W_0 \to \Xi_{Z_2, \tau} \) induced by morphisms as in (3.2), the preimages of \( W_1 \) and \( \Xi_{Z_1} \) coincide as an open subscheme \( W_0^1 \) of \( W_1 \), and their further preimages in \( W_0 \) coincide with the preimages of \( W_0 \) and \( \Xi_{Z_0} \) as an open subscheme \( \Xi_{Z_2}^0 \).

Proof. Except for the first assertion in (5), these follow from the same arguments as in Sections 2.1.28 and 4.1.12 (which are based on Sections 4.16, 6.25, and 12.4 and Section 3.3) in Cases [1] and [4], and as in Sections 8–11 and
the proof of Prop. 2.1.3] in Cases (2) and (3). As for the first assertion in (5), it follows from the universal or functorial properties of toroidal compactifications in terms of the associated cone decompositions, as in [23 Chapter II, Section 7], [28 Prop. 6.25], [18 Thm. 6.4.1.1(6)], [21 Thm. 6.1(6)], and [26 Prop. 4.1.13]. □

Corollary 3.7. In Proposition 3.4, suppose that $\Sigma_0$ is induced by $\Sigma_1$. Let $Z_1$ be a stratum of $X^{\text{min}}_{Z_1}$, and let $\{Z_{0,1}\}$ be all the strata of $X^\text{min}_{Z_0}$ such that $f^\text{min}(Z_{0,j}) \subset Z_1$ (as subsets of $Z_1$). Consider any $\tau \in \Sigma_1^{+}$. For each $j$, let

$$\sigma_j := (P_{Z_{0,1}}^+ \hookrightarrow P_{Z_1}^+)^{-1}(\tau) \in \Sigma_0^{+}.$$  

Then the pullback of the finite morphism

$$f_{Z_{0,1}}^{\text{tor}} : X_{Z_{0,1}}^{\text{tor}} \to X_{Z_{1,1}}^{\text{tor}}$$

under the composition of the canonical morphisms $X_{Z_{1,1}}^{\text{tor}} \to (X_{1,1}^{\text{tor}})^{\wedge}_{\tau' \in \Sigma_1^{+}} Z_{1,1}(\tau') \to X_{1,1}^{\text{tor}}$ can be identified with the finite morphism

$$\prod_j X_{Z_{0,1},\sigma_j}^{\text{tor}} \to X_{Z_{1,1}}^{\text{tor}}$$

(defined by combining morphisms as in (3.6)).

Proof. This follows from (1) and (6) of Proposition 3.4. □

Corollary 3.8. In Corollary 3.7, with any $\tau \in \Sigma_1^{+}$ there inducing $\sigma_j \in \Sigma_0^{+}$, for each $j$, we have a commutative diagram of canonical morphisms

$$E_{Z_{0,1}} \xrightarrow{\rho_{Z_{0,1}}} X_{Z_{0,1}}^{\text{tor}} \xrightarrow{f_{Z_{0,1},Z_{1,1}}} X_{Z_{1,1}}^{\text{tor}}$$

over $\text{Spec}(\mathbb{Z})$, in which the horizontal morphisms are affine toroidal embeddings, which are open immersions, and where the vertical morphisms are finite. Let $x_1$ be any point of $X_{Z_{1,1}}^{\text{tor}}$ that lies on the stratum $Z_{1,1}(\tau)$. Then, étale locally at $x_1$, the commutative diagram

$$\xymatrix{ X_0^{\text{tor}} \ar[r]^{f_{Z_{0,1},Z_{1,1}}} \ar[d]_{f} & X_1^{\text{tor}} \ar[d]^{f_{Z_{0,1},Z_{1,1}}} \\
X_0^{\text{tor}}_{Z_0} \ar[r] & X_1^{\text{tor}}_{Z_1} }$$

can be identified with a commutative diagram

$$\prod_j (E_{Z_{0,1}} \times_{\text{Spec}(\mathbb{Z})} C_{Z_{0,1}}) \xrightarrow{\prod_j (E_{Z_{0,1}}(\sigma_j) \times_{\text{Spec}(\mathbb{Z})} C_{Z_{0,1}})} \prod_j (E_{Z_{1,1}}(\tau) \times_{\text{Spec}(\mathbb{Z})} C_{Z_{1,1}})$$

over $\text{Spec}(\mathbb{Z})$. The pullback of the finite morphism

$$f_{Z_{1,1}}^{\text{tor}} : X_{Z_{1,1}}^{\text{tor}} \to X_{Z_{1,1}}^{\text{tor}}$$

under the composition of the canonical morphisms $X_{Z_{1,1}}^{\text{tor}} \to (X_{1,1}^{\text{tor}})^{\wedge}_{\tau' \in \Sigma_1^{+}} Z_{1,1}(\tau') \to X_{1,1}^{\text{tor}}$ can be identified with the finite morphism

$$\prod_j X_{Z_{0,1},\sigma_j}^{\text{tor}} \to X_{Z_{1,1}}^{\text{tor}}$$

(defined by combining morphisms as in (3.6)).

Proof. This follows from (1) and (6) of Proposition 3.4. □
induced by taking fiber products of some translations of the vertical morphisms in the diagram (3.9) by sections of \( E_{Z_1} \), and of the canonical morphisms \( C_{Z_{0,j}} \to C_{Z_1} \). More precisely, there exists an étale neighborhood \( \overline{U}_1 \to X^*_1, \Sigma_1 \)

of \( x_1 \) and an étale morphism

\[
(3.11) \quad \overline{U}_1 \to E_{Z_1}(\tau) \times_{\text{Spec}(\mathbb{Z})} C_{Z_1},
\]

which induce by pullback under the finite morphisms \( f^\text{tor}_0 : X^\text{tor}_0, \Sigma_0 \to X^\text{tor}_1, \Sigma_1 \) and \( \prod_j (E_{Z_{0,j}}(\sigma_j) \times_{\text{Spec}(\mathbb{Z})} C_{Z_{0,j}}) \to E_{Z_1}(\tau) \times_{\text{Spec}(\mathbb{Z})} C_{Z_1} \) (as in (3.10)) some étale morphisms \( \overline{U}_0 \to X^\text{tor}_0, \Sigma_0 \) and \( \overline{U}_0 \to \prod_j (E_{Z_{0,j}}(\sigma_j) \times_{\text{Spec}(\mathbb{Z})} C_{Z_{0,j}}) \), respectively, such that the preimage \( U_1 \) of \( X_1 \) in \( \overline{U}_1 \) coincides with the preimage of \( E_{Z_1} \), and such that the preimage \( U_0 \) of \( U_1 \) in \( \overline{U}_0 \) coincides with the preimages of \( E_0 \) and of \( \prod_j (E_{Z_{0,j}} \times_{\text{Spec}(\mathbb{Z})} C_{Z_{0,j}}) \).

Proof. These follow from Corollary 3.7 by Artin’s approximation (see [1], Thm. 1.12, and the proof of the corollaries in Section 2]) as in the proofs of [23], Prop. 2.2(9) and Cor. 2.4, [22], Cor. 2.1.7, and [24], Prop. 5.1, which is applicable because we only need to approximate finitely many formal schemes finite over \( \mathcal{X}_0^\text{tor} \), and from the fact that all the torus torsors are Zariski locally trivial, as in the proof of [23], Lem. 2.3. (Note that the torus torsors might be trivialized by incompatible sections, and hence we need to allow the group homomorphisms \( E_{Z_{0,j}} \to E_{Z_1} \) to be translated by some possibly different sections of \( E_{Z_1} \), when there are more than one \( j \).)

\[ \square \]

Remark 3.12. In Proposition 3.4 and in Corollaries 3.7 and 3.8, we only need the weaker assumption that \( (\rho(A^{\infty}))(\mathcal{H}_0) \subset \mathcal{H}_1 \). When \( \mathcal{H}_0 = (\rho(A^{\infty}))^{-1}(\mathcal{H}_1) \), we already know in Proposition 3.4(2) that the morphism \( E_{Z_{0,j}} \to E_{Z_1} \) in (3.9) is a closed immersion, without assuming that \( f \) is a closed immersion; but it is generally not true that the morphism \( E_{Z_{0,j}}(\sigma_j) \to E_{Z_1}(\tau) \) is a closed immersion when \( E_{Z_{0,j}} \to E_{Z_1} \) is (cf. Remark 4.1 below), regardless of whether \( f \) is.

We shall reinstate the full Assumption 2.1 from now on.

4. Conditions on cone decompositions

Motivated by Corollary 3.8, with the goal of proving Theorem 2.2 in mind, we would like to show the existence of collections \( \Sigma_0 \) and \( \Sigma_1 \) such that \( \Sigma_0 \) is induced by \( \Sigma_1 \) as in Proposition 3.4(1) and such that, for each \( \sigma \in \Sigma_0^+ \) that is the preimage under \( \mathbf{P}^+_z \to \mathbf{P}^+_Z \) of some \( \tau \in \Sigma_1^+ \), the canonical morphism \( E_{Z_0}(\sigma) \to E_{Z_1}(\tau) \) (cf. (3.9)) is a closed immersion.

Remark 4.1. This condition of being a closed immersion is not satisfied in general. For example, it is possible to choose the linear algebraic data such that \( \mathbf{S}_Z \cong \mathbb{Z}^{\oplus 3} \to \mathbf{S}_Z \cong \mathbb{Z}^{\oplus 2} \) corresponds to the projection to the first two factors, in which case \( \mathbf{S}_Z^\vee \cong \mathbb{R}^{\oplus 2} \to \mathbf{S}_Z^\vee_{Z_{0,R}} \cong \mathbb{R}^{\oplus 3} \) is the inclusion of the first two coordinates, and such that we have the following:

- \( \tau \subset \mathbf{S}_Z^\vee_{Z_{0,R}} \) is \( \mathbb{R}_{>0} \)-spanned by \( \{(0,0,1),(-1,0,2),(1,1,-2)\} \), in which case \( \tau^\vee \) is \( \mathbb{Z}_{>0} \)-spanned by the \( \mathbb{Z} \)-basis \( \{(-1,1,0),(0,1,0),(2,0,1)\} \) of \( \mathbb{Z}^{\oplus 3} \).
Remark 4.2. In fact, in Remark 4.1 even the induced map \( E_{Z_0}(\sigma)(C) \to E_{Z_1}(\tau)(C) \) on \( \mathbb{C} \)-points is not injective: For \( ? = \pm 1 \), if \( x_1 : \mathbb{Z}[\sigma^\vee] \to \mathbb{C} \) is the ring homomorphism sending \((-1, 1)\) and \((1, 0)\) in \( \sigma^\vee \) to 0 and \( ? \), respectively, then the induced homomorphism \( y : \mathbb{Z}[\tau^\vee] \to \mathbb{C} \) sends \((-1, 1, 0), (0, 1, 0), \) and \((2, 0, 1)\) to 0, 0, and 1, respectively. That is, both the \( \mathbb{C} \)-points defined by \( x_1 \) and \( x_{-1} \) are sent to the same \( \mathbb{C} \)-point defined by \( y \). This shows that, already in characteristic zero, the induced morphism \( E_{Z_0}(\sigma) \to E_{Z_1}(\tau) \) is not universally injective, and hence cannot induce a universal homeomorphism between the source and its image in the target. Therefore, when finite morphisms between toroidal compactifications of Shimura varieties fail to be closed immersions, the passage to the associated diamonds might not help as in the case of minimal compactifications in [40 paragraph preceding Cor. 10.2.3] (which addressed a subtlety in the statement of [29 Thm. 4.1.1]).

Nevertheless, we have the following:

Lemma 4.3. For any \( \sigma \in S^+_Z \) and \( \tau = (S^\vee_{Z_0, \mathbb{R}} \to S^\vee_{Z_1, \mathbb{R}})(\sigma) \in \Sigma^Z_{\mathbb{R}} \), the canonical morphism \( E_{Z_0}(\sigma) \cong \text{Spec}(\mathbb{Z}[\sigma^\vee]) \to E_{Z_1}(\tau) \cong \text{Spec}(\mathbb{Z}[\tau^\vee]) \) is a closed immersion.

Proof. Given an arbitrary \( \ell_0 \in \sigma^\vee \), take any lift \( \ell_1 \) of it in \( S_{Z_1} \), which exists because \( S_{Z_1} \to S_{Z_0} \) is surjective. Given an arbitrary \( y_1 \in \tau \), by assumption, there exists some \( y_0 \in \sigma \) such that \( y_1 = (S^\vee_{Z_0, \mathbb{R}} \leftarrow S^\vee_{Z_1, \mathbb{R}})(y_0) \), and so that \( \langle \ell_1, y_1 \rangle = \langle \ell_0, y_0 \rangle \geq 0 \). Consequently, \( \ell_1 \in \tau^\vee \), and \( \tau^\vee \to \sigma^\vee \) is surjective, as desired. \( \square \)

Lemma 4.4. In Lemma 4.3 let us identify \( S^\vee_{Z_0, \mathbb{R}} \) with a subspace of \( S^\vee_{Z_1, \mathbb{R}} \) for simplicity, so that \( \tau = \sigma \) under this identification; and let \( S^\vee := S^\vee_{Z_1} \cap (\mathbb{R} \cdot \sigma) \) and \( S := \text{Hom}_{\mathbb{Z}}(S^\vee, \mathbb{Z}) \), so that we have surjective homomorphisms \( S_{Z_1} \to S_{Z_0} \to S \) corresponding to injective homomorphisms of tori \( E \to E_{Z_0} \to E_{Z_1} \). For the sake of clarity, let us denote by \( \varsigma \) the same cone \( \sigma \in S^\vee_\mathbb{R} = \mathbb{R} \cdot \sigma \). Let \( E, E^\perp_{Z_0}, \) and \( E^\perp_{Z_1} \) be the split tori over \( \text{Spec}(\mathbb{Z}) \) with character groups \( S, S^\perp_{Z_1} := \ker(S_{Z_0} \to S) \), and \( S^\perp_{Z_1} := \ker(S_{Z_1} \to S) \), respectively. Let us pick any splitting \( S_{Z_1} \cong S \oplus S^\perp_{Z_1} \) (as \( \mathbb{Z} \)-modules) which induces a splitting \( S_{Z_0} \cong S \oplus S^\perp_{Z_0} \). Then these splittings are dual to compatible fiber products \( E_{Z_1} \cong E \times_{\text{Spec}(\mathbb{Z})} E^\perp_{Z_1} \) and \( E_{Z_0} \cong E \times_{\text{Spec}(\mathbb{Z})} E^\perp_{Z_0} \), respectively, and the canonical injective homomorphism \( E_{Z_0} \to E_{Z_1} \) factors as a fiber product of the identity homomorphism of \( E \) with the canonical injective homomorphism \( E^\perp_{Z_0} \to E^\perp_{Z_1} \) dual to \( S^\perp_{Z_1} \to S^\perp_{Z_0} \). Moreover, these splittings extend to compatible fiber products \( E_{Z_1}(\tau) \cong E(\varsigma) \times E^\perp_{Z_1} \) and \( E_{Z_0}(\sigma) \cong E(\varsigma) \times E^\perp_{Z_0} \), respectively, and the canonical closed immersion \( E_{Z_0}(\sigma) \to E_{Z_1}(\tau) \) factors as a fiber product of the identity morphism of \( E(\varsigma) \) with the same injective group homomorphism \( E^\perp_{Z_0} \to E^\perp_{Z_1} \) as above. Furthermore, any closed immersion \( E_{Z_0}(\sigma) \to E_{Z_1}(\tau) \) that is a translation of the canonical one by some section of \( E_{Z_1} \) can be identified with the product of the identity morphism on \( E(\sigma) \) with a closed immersion \( E^\perp_{Z_0} \to E^\perp_{Z_1} \), that is the translation of the canonical one by some section of \( E^\perp_{Z_1} \).
Proof. These follow from the identification \( \tau' = (S_{Z_j} \to S_{Z_0})^{-1}(\sigma') \) in the proof of Lemma 4.3 and from the various definitions introduced in this lemma. \( \square \)

These justify the following:

**Definition 4.5.** We say that two compatible collections \( \Sigma_0 \) and \( \Sigma_1 \) of cone decompositions as in Proposition 3.4(4) are **strictly compatible** with each other or simply **strictly compatible** if, for each \( Z_0 \to Z_1 \) as in Proposition 3.4(1), the image of each \( \sigma \in \Sigma_{Z_0}^{+} \) under \( P_{Z_0}^{+} \to P_{Z_1}^{+} \) is exactly some \( \tau \in \Sigma_{Z_1}^{+} \).

**Remark 4.6.** Certainly, if \( \Sigma_0 \) and \( \Sigma_1 \) are strictly compatible as in Definition 4.5 then \( \Sigma_0 \) is induced by \( \Sigma_1 \), and they are compatible, as in Proposition 3.4(4).

**Lemma 4.7.** Under the assumption that \( f : X_0 \to X_1 \) is a closed immersion, the morphism \( \prod_j (E_{Z_0,j} \times C_{Z_0,j}) \to E_{Z_1} \times C_{Z_1} \) in Corollary 3.8 is a closed immersion over the open image of \( \overline{U}_1 \) under (3.11). Since \( E_{Z_0,j} \) and \( E_{Z_1} \) are separated group schemes with identity sections which are closed immersions, \( C_{Z_0,j} \to C_{Z_1} \) (and hence \( \Xi_{Z_0,j} \to \Xi_{Z_1} \)) are also closed immersions over the further image of \( \overline{U}_1 \) in \( C_{Z_1} \), for all \( j \). Moreover, if \( \Sigma_0 \) and \( \Sigma_1 \) are strictly compatible as in Definition 4.5 then the morphism \( \prod_j (E_{Z_0,j}(\sigma_j) \times C_{Z_0,j}) \to E_{Z_1}(\tau) \times C_{Z_1} \) in Corollary 3.8 is also a closed immersion over the open image of \( \overline{U}_1 \) under (3.11).

**Proof.** The first two assertions follow immediately from Corollary 3.8. By Lemma 4.3, the morphism \( E_{Z_0,j}(\sigma_j) \times C_{Z_0,j} \to E_{Z_1}(\tau) \times C_{Z_1} \) is a closed immersion over the image of \( \overline{U}_1 \), for each \( j \). It remains to show that any point \( x \) in the image of \( \prod_j (E_{Z_0,j}(\sigma_j) \times C_{Z_0,j}) \to E_{Z_1}(\tau) \times C_{Z_1} \) lies on at most one of the images of the above closed immersions. Suppose that there are two distinct indices \( j \) and \( j' \), together with points \( y \) and \( y' \) of \( E_{Z_0,j}(\sigma_j) \times C_{Z_0,j} \) and \( E_{Z_0,j'}(\sigma_{j'}) \times C_{Z_0,j'} \), respectively, which are mapped to the point \( x \) of \( E_{Z_1}(\tau) \times C_{Z_1} \). Then \( x, y, \) and \( y' \) have the same image \( z \) in \( C_{Z_1} \), which is also in the images of the closed immersions from \( C_{Z_0,j} \) and \( C_{Z_0,j'} \), and we obtain closed immersions \( E_{Z_0,j}(\sigma_j)z \to E_{Z_1}(\tau)z \) and \( E_{Z_0,j'}(\sigma_{j'})z \to E_{Z_1}(\tau)z \), which are translations of the canonical ones by sections of \( E_{Z_1} \), with overlapping images. Hence, by Lemma 4.4 in the notation there, the images of the closed immersions \( (E_{Z_0,j})z \to (E_{Z_1})z \) and \( (E_{Z_0,j'})z \to (E_{Z_1})z \) overlap, and hence the images of the closed immersions \( (E_{Z_0,j})z \to (E_{Z_1})z \) and \( (E_{Z_0,j'})z \to (E_{Z_1})z \) also overlap, contradicting the first assertion of this proposition. \( \square \)

By Corollary 3.8 and Lemma 4.7, we obtain the following:

**Proposition 4.8.** If there exist compatible collections \( \Sigma_0 \) and \( \Sigma_1 \) that are strictly compatible as in Definition 4.5, then the induced morphism \( f_{\text{tor}}^{\text{tor}} : X_{0,\Sigma_0}^{\text{tor}} \to X_{1,\Sigma_1}^{\text{tor}} \) as in Proposition 3.4 is a closed immersion extending \( f : X_0 \to X_1 \).

In order to prove Theorem 2.2, it remains to establish the following:

**Proposition 4.9.** There exist compatible collections \( \Sigma_0 \) and \( \Sigma_1 \) that are strictly compatible as in Definition 4.5 which we may assume to be projective and smooth and satisfy the condition that, for \( i = 0, 1 \), and for each \( Z_i \) and each \( \sigma \in \Sigma_i^{+} \), the stabilizer \( \Gamma_{Z_i,\sigma} \) of \( \sigma \) in \( \Gamma_{Z_i} \) is trivial. Moreover, we may assume that \( \Sigma_0 \) and \( \Sigma_1 \) refine any finite number of prescribed compatible collections of cone decompositions.
Proof. Let us temporarily ignore the assumption on projectivity and smoothness, and take \( \Sigma_0 \) to be induced by \( \Sigma_1 \) as in Proposition 3.4(4) (cf. [12, Section 3.3]). Note that, given any \( Z_1 \) and any \( [\tau] \in \Sigma_{Z_1}^+ / \Gamma_{Z_1} \), there exist only finitely many \( Z_0 \) mapped to \( Z_1 \); and for each such \( Z_0 \), there exist only finitely many \( [\sigma] \in \Sigma_{Z_0}^+ / \Gamma_{Z_0} \) mapped to \( [\tau] \) under the map \( \Sigma_{Z_0}^+ / \Gamma_{Z_0} \to \Sigma_{Z_1}^+ / \Gamma_{Z_1} \) (simply because the sets of all possible \( Z_0 \) or \([\sigma]\) are finite). As a result, up to refining \( \tau \) by intersections with finitely many hyperplanes, and up to refining all the finitely many \( \sigma \) involved accordingly, we may assume that \( \Sigma_0 \) and \( \Sigma_1 \) are strictly compatible (but still not necessarily projective and smooth). We may also refine both of them, and assume that they refine any finite number of prescribed compatible collections and satisfy the condition in the end of the first sentence of the proposition. Finally, up to further refinements, we may also assume that \( \Sigma_0 \) and \( \Sigma_1 \) are both projective and smooth, because as soon as \( \Sigma_0 \) and \( \Sigma_1 \) are strictly compatible and satisfy the last condition of the proposition, any further refinements will remain so; and because, when \( \Sigma_0 \) and \( \Sigma_1 \) are strictly compatible, both the projectivity and smoothness of \( \Sigma_1 \) are automatically inherited by \( \Sigma_0 \), and hence it suffices to refine \( \Sigma_1 \). (Note that such an inheritance is not true in general, when \( \Sigma_0 \) is merely induced by \( \Sigma_1 \).) \( \square \)

The proof of Theorem 2.2 is now complete.

5. Morphisms between Satake compactifications

Although Proposition 4.8 does not imply the existence of the compatible collections \( \Sigma_0 \) and \( \Sigma_1 \) needed in the proof of Theorem 2.2, it does allow us to verify directly that certain given compatible collections are already good enough for our purpose, without having to be refined. For this observation to be practical, with the goal of proving Theorem 2.3 in mind, it is desirable to have the following:

Proposition 5.1. Up to replacing \( H_1 \) with a finite index subgroup (without changing \( H_0 \)), we may assume that, for each stratum \( Z_1 \) of \( X_1^{\min} \), there is at most one stratum \( Z_0 \) of \( X_0^{\min} \) that is mapped to \( Z_1 \) under \( f^{\min} : X_0^{\min} \to X_1^{\min} \), so that we do not have to be concerned with possible nontrivial intersections among the images of \( P_1^{\tau_0} \subset P_1^{\tau_1} \). (This statement is only about the cusp labels and cone decompositions, but not about the geometry of minimal compactifications.)

Moreover, in Cases (1) and (2), we may assume in the above that the finite morphism \( f^{\min} : X_0^{\min} \to X_1^{\min} \) is universally injective, which induces an injective map on geometric points. More precisely, up to replacing \( H_1 \) with a finite index subgroup (without changing \( H_0 \)), we may assume that all the finite morphisms \( Z_0 \to Z_1 \) induced by \( f^{\min} \) are closed immersions. (The combination of the previous paragraph and this last assertion imply the first assertion in this paragraph.)

Remark 5.2. The last assertion of Proposition 5.1 provides a slightly more detailed explanation of the corresponding assertion in [30, paragraph preceding Cor. 10.2.3] (which addressed a subtlety in the statement of [29, Thm. 4.1.1]), to which we alluded in the introduction and in Remark 1.2.

We begin with the following:

Lemma 5.3. Suppose that \( H_0 \to H_1 \) is an injective homomorphism of algebraic groups over \( \mathbb{Q} \), and that \( \mathcal{P} \) is a set of primes. Suppose that \( H_0 \subset H_0(\mathbb{A}^{\infty, \mathcal{P}}) \) and \( H_1 \subset H_1(\mathbb{A}^{\infty, \mathcal{P}}) \) are open compact subgroups such that \( H_0 \) is the preimage of \( H_1 \) under \( H_0(\mathbb{A}^{\infty, \mathcal{P}}) \to H_1(\mathbb{A}^{\infty, \mathcal{P}}) \). Suppose that \( Y_0 \) and \( Y_1 \) are sets with actions of \( H_0 \)
and \( H_1 \), respectively, such that \( H_0 \backslash Y_0 \) is a finite set and such that the canonical map \( Y_1 \to \lim_{\to, H_1} (H'_1 \backslash Y_1) \) is injective, where \( H'_1 \) runs over all open compact subgroups of \( H_1 \). Suppose that \( Y_0 \to Y_1 \) is an injective map equivariant with the homomorphism \( H_0 \to H_1 \). Then, up to replacing \( H_1 \) with a finite index subgroup (without changing \( H_0 \)), the induced map \( H_0 \backslash Y_0 \to H_1 \backslash Y_1 \) is also injective.

**Proof.** For simplicity, let us denote the image of \( H_0 \) in \( H(\mathbb{A}_\infty) \) by the same symbols. Then \( H_0 = \cap_{H_0 \subset H_1 \subset H_1} H'_1 \), where we can take each \( H'_1 \) to be generated by \( H_0 \) and some normal subgroup of \( H_1 \) of finite index; and we have injective maps \( H_0 \backslash Y_0 \to H_0 \backslash Y_1 \to \lim_{\to, H_1 \subset H_1} (H'_1 \backslash Y_1) \cong \lim_{\to, H_0 \subset H_1 \subset H_1} (H'_1 \backslash Y_1) \), by our assumptions. Consequently, if any \( y, y' \in H_0 \backslash Y_0 \) have the same image in \( H'_1 \backslash Y_1 \) for all \( H'_1 \) as above, then they have the same image in \( H_0 \backslash Y_1 \), forcing \( y = y' \). Since \( H_0 \backslash Y_0 \) is a finite set, up to replacing \( H_1 \) with a finite index subgroup \( H'_1 \) as above, we see that the induced map \( H_0 \backslash Y_0 \to H_1 \backslash Y_1 \) is also injective, as desired. \( \square \)

**Proof of Proposition 5.1.** In Cases \( 1 \) and \( 4 \) (resp. Case \( 2 \) and Case \( 3 \)) in Assumption 2.1 by the explanations in [28 Section 6.3 and onwards] and [20 Section 2.1.28] (resp. [18 Section 5.4], and [20 Section 2.1.2] or [19 Section 5]), for each \( i \), the strata \( Z_i \) of \( X^{\text{min}}_0 \) are indexed by a finite set of the form \( H_i \backslash Y_i \), for some set \( Y_i \) such that \( Y_i \cong \lim_{\to, H'_1 \subset H_1} (H'_1 \backslash Y_i) \), where \( H'_1 \) runs over the open compact subgroups of \( H_i \), and we have an injective map \( Y_0 \to Y_1 \) equivariant with \( H_0 \to H_1 \). Then the first paragraph of Proposition 5.1 follows from Lemma 5.3.

As for the second paragraph, we would like to show that, up to replacing \( H_1 \) with a finite index subgroup (still containing the image of \( H_0 \)), each finite morphism \( Z_0 \to Z_1 \) as in the first paragraph of Proposition 5.1 is a closed immersion. In Case \( 1 \), \( Z_0 \) and \( Z_1 \) are some finite quotients of smaller Shimura varieties, and we can follow the same limit argument as in the proof of [33 1.15]. In Case \( 2 \), we may reduce to the case where both \( H_0 \) and \( H_1 \) are principal congruence subgroups of \( G_0(\mathbb{Z}^\infty) \) and \( G_1(\mathbb{Z}^\infty) \), respectively, of some level \( n_0 \geq 3 \) prime to \( \varnothing \). (Then we can recover what we wanted by taking quotient by the original \( H_0 \), by [33 Cor. 7.2.5.2]: As for \( X^{\text{min}}_0 \), this recovers the minimal compactification at the original level \( H_0 \). As for \( X^{\text{min}}_1 \), the quotient by the image of \( H_0 \) gives the potentially higher level corresponding to a finite index subgroup of the original \( H_1 \) generated by the principal congruence subgroup of level \( n_0 \) and the image of \( H_0 \).) At principal level \( n_0 \), the closed immersion \( C_{Z_0} \to C_{Z_1} \), as in Lemma 4.7 (where the subscript \( j \) there is omitted because it is unique under our assumption here) is between abelian schemes over \( Z_0 \) and \( Z_1 \) (see [33 Section 6.2.3]), whose identity sections can be matched by choosing representatives of cusp labels compatibly (by restricting endomorphism structures). Therefore, we have a composition \( Z_0 \to C_{Z_0} \to C_{Z_1} \) of the identity section and a closed immersion, which factors through the identity section \( Z_1 \to C_{Z_1} \), both of which are closed immersions because abelian schemes are separated. Thus, \( Z_0 \to Z_1 \) must be a closed immersion as well, as desired. \( \square \)

However, minimal (Satake–Baily–Borel) compactifications of Shimura varieties are not the only *Satake compactifications* (see [31 (3.9)] and [6 Section III.3]) that are relevant in our theory.

**Proposition 5.4.** For \( i = 0, 1 \), and for \( ? = + \) and \( \emptyset \), let \( \overline{P}^i_{Z_i} := (P^i_{Z_i} - \{0\})/\mathbb{R}^{>0}_x \). For \( ? = + \) and \( \emptyset \), the map \( \overline{P}^i_{Z_0} \to \overline{P}^i_{Z_1} \) maps \( \{0\} \) to \( \{0\} \) and is compatible
with the actions of $\mathbb{R}_{>0}^\times$ by scaling, and therefore induces topological embeddings $\mathbf{P}_{Z_0}^\pm := (\mathbf{P}_{Z_0}^\pm - \{0\})/\mathbb{R}_{>0}^\times \rightarrow \mathbf{P}_{Z_1}^\pm := (\mathbf{P}_{Z_1}^\pm - \{0\})/\mathbb{R}_{>0}^\times$ between certain Riemannian symmetric spaces and their Satake partial compactifications, respectively, which in turn induce continuous maps $\mathbf{P}_{Z_0}/\Gamma_{Z_0} \rightarrow \mathbf{P}_{Z_1}/\Gamma_{Z_1}$ between arithmetic quotients of the above-mentioned Riemannian symmetric spaces and their Satake compactifications. When we have compatible collections $\Sigma_0$ and $\Sigma_1$ of smooth cone decompositions (both satisfying the condition that, for $i = 0, 1$, and for each $\sigma \in \Sigma_{Z_i}^\pm$, the stabilizer $\Gamma_{Z_i, \sigma}$ in $\Gamma_{Z_i}$ is trivial), we have triangularizations of $\mathbf{P}_{Z_0}$ and $\mathbf{P}_{Z_1}$, for $? = +$ and $\emptyset$. All such maps are closed maps with finite fibers, and can be assumed to be closed embeddings up to replacing $\mathcal{H}_1$ with a finite index subgroup (still containing the image of $\mathcal{H}_0$).

**Proof.** The identification of $\mathbf{P}_{Z_0}^\pm$ and $\mathbf{P}_{Z_1}^\pm$ (when they are nonempty) with Riemannian symmetric spaces follows from [2, 3, Chapter II] (see also [11, Chapters I—III]). The identification of $\mathbf{P}_{Z_0}$ and $\mathbf{P}_{Z_1}$ (when they are nonempty) with certain Satake partial compactifications of Riemannian symmetric spaces, and the assertion concerning arithmetic quotients and triangularizations in terms of cone decompositions, are explained in more detail in the proof of [13 Prop. 2.2.1]. Then the map $\mathbf{P}_{Z_0}/\Gamma_{Z_0} \rightarrow \mathbf{P}_{Z_1}/\Gamma_{Z_1}$ is proper and hence closed, and has finite fibers by [12, Section 3.3]. The proof of the assertion that $\mathbf{P}_{Z_0}/\Gamma_{Z_0} \rightarrow \mathbf{P}_{Z_1}/\Gamma_{Z_1}$ and hence $\mathbf{P}_{Z_0}/\Gamma_{Z_0} \rightarrow \mathbf{P}_{Z_1}/\Gamma_{Z_1}$ is injective up to replacing $\mathcal{H}_1$ with a finite index subgroup is similar to the proof of the analogous assertion in Proposition 5.1 (in Case (i)), based on Lemma 5.3 and the same limit argument as in the proof of [8, 1.15]. □

Now we are ready for the following:

**Proof of Theorem 2.3.** By Proposition 4.8 it suffices to show that, up to replacing $\mathcal{H}_1$ with a finite index subgroup, there exist some collection $\Sigma_0$ that is strictly compatible with the prescribed $\Gamma_0$. By Propositions 5.1 and 5.4 up to replacing $\mathcal{H}_1$ with a finite index subgroup, we may assume that there is at most one $Z_0$ mapped into any given $Z_1$, so that the image of $\mathbf{P}_{Z_0}/\Gamma_{Z_0} \rightarrow \mathbf{P}_{Z_1}/\Gamma_{Z_1}$ does not meet the image of any other similarly defined map. By induction on dimensions, we may assume that we have already chosen the desired smooth cone decompositions for the boundary components of $\mathbf{P}_{Z_1}$ in $\mathbf{P}_{Z_1} - \mathbf{P}_{Z_1}^\pm$. Then we may ($\Gamma_{Z_1}$-equivariantly) refine the cone decomposition of $\mathbf{P}_{Z_1}$ and obtain some $\Gamma_{Z_1}$-admissible cone decomposition $\Sigma_1$ of $\mathbf{P}_{Z_1}$ that is not necessarily smooth, without further refining the cone decomposition $\Sigma_{Z_0}$ of $\mathbf{P}_{Z_0}$ and the cone decompositions of the boundary components of $\mathbf{P}_{Z_1}$ in $\mathbf{P}_{Z_1} - \mathbf{P}_{Z_1}^\pm$. Since these latter cone decompositions are already smooth, we can inductively construct smooth refinements of each nonsmooth cone in $\mathbf{P}_{Z_1}$ with smooth faces, by the same arguments as in [14] Chapter I, Section 2, proof of Thm. 11 on pages 33–35, and obtain the desired $\Sigma_1$ in finitely many steps.

Moreover, if $\Sigma_0$ is projective smooth and if $f : X_0 \rightarrow X_1$ already extends to some finite morphism $f_{\text{tor}, \Sigma_1} : X_{\text{tor}, \Sigma_0} \rightarrow X_{\text{tor}, \Sigma_1}$, for some projective smooth $\Sigma_1$, then $\Sigma_0$ is induced by $\Sigma_1$ (see Proposition 4.4(5)), in which case the compatible collection of polarization functions for $\Sigma_0$ (i.e., the compatible collection of convex piecewise linear functions $\phi_{Z_0} : \mathbf{P}_{Z_0} \rightarrow \mathbb{R}_{\geq 0}$ as in [2 Chapter IV, Section 2.1, Def.] or [3 Chapter IV, Def. 2.1]; [10] Chapter V, Def. 5.1; and [18] Def. 7.3.1.1) can be taken to be the pullbacks of some polarization functions $\phi'_{Z_1} : \mathbf{P}_{Z_1} \rightarrow \mathbb{R}_{\geq 0}$ for $\Sigma'_1$ under
Here the (perhaps confusing) traditional convention is that $\phi'_Z$ is convex in the sense that the subset of $\mathbb{P}_{Z_1} \times \mathbb{R}_{\geq 0}$ above the graph of $\phi'_Z$ is convex (i.e., $\phi'_Z$ is actually a concave function in the usual sense). In order to justify the last assertion in Theorem 2.3, it suffices to show that, up to replacing $\mathcal{H}_1$ with a finite index subgroup such that Propositions 5.1 and 5.4 hold, we can modify the polarization functions $\phi'_Z : \mathbb{P}_{Z_1} \to \mathbb{R}_{\geq 0}$ for $\Sigma'_1$ such that they are linear exactly on the cones in some (projective) smooth refinement $\Sigma_1$ of $\Sigma'_1$, including all the images of the cones in $\Sigma_0$ (without refining these images).

For this purpose, let us temporarily replace the requirement that the polarization functions $\phi_Z : \mathbb{P}_Z \to \mathbb{R}_{\geq 0}$ take integral values on $\mathbb{Z}_{\Sigma_1} \cap \mathbb{P}_Z$ with the weaker requirement that they take rational values. Let us begin with $\phi'_Z = \phi'_Z$. Then we proceed by induction on the dimensions of the cones in $\Sigma'_1$, and (by $\Gamma_{Z_1}$-equivariance) adjust the polarization functions by the same arguments as in [14], Chapter I, Section 2, proof of Thm. 11 on pages 33–35, which we already used above, such that $\phi_Z : \mathbb{P}_Z \to \mathbb{R}_{\geq 0}$ are (by $\Gamma_{Z_1}$-invariant) convex piecewise linear functions which are linear exactly on some smooth refinement $\Sigma_1$ of $\Sigma'_1$ still including all the images of the cones in $\Sigma_0$, and agree with $\phi'_Z$ on the images of $\mathbb{P}_{Z_0} \to \mathbb{P}_{Z_1}$.

More precisely, for each $Z_1$, on each cone $\sigma$ representing an $\Gamma_{Z_1}$-orbit in $\Sigma'_1$, assume by induction that the adjustments have been done on its finitely many proper faces (which are of smaller dimensions). Let us first take the function on $\sigma$ such that the subset of $\sigma \times \mathbb{R}_{\geq 0}$ above its graph is the $\mathbb{R}_{\geq 0}$-span of the corresponding subsets for the adjusted functions on the proper faces of $\sigma$, which is a convex piecewise linear function on $\sigma$ by construction, and we further adjust this “convex hull” by adding some sufficiently small convex piecewise linear function such that the resulted function is still convex and is linear exactly on a smooth refinement of $\sigma$. In order to see that we can arrange such a refinement to include any image $\tau$ of some cone of $\Sigma_0$, first note that, by the assumption that $\Sigma_0$ is induced by $\Sigma'_1$, and by the assumption that Propositions 5.1 and 5.4 hold, there is at most one such $\sigma \subset \tau$, which is necessarily the intersection of $\tau$ with the image of the linear map $\mathbb{Z}_{\Sigma_0, \mathbb{R}} \to \mathbb{Z}_{\Sigma_1, \mathbb{R}}$, and we just need the adjusted piecewise linear function on $\sigma$ to be discontinuous at $\tau$ but restricts to a linear function on $\tau$. If $\tau$ is closed in $\sigma$, then the proper faces of $\sigma$ are irrelevant, and we can find some adjustment of $\phi_Z |_{\sigma}$ with the desired properties. If $\tau$ is not closed in $\sigma$, then it is $\mathbb{R}_{\geq 0}$-spanned by the intersections of its closure $\tau$ in $\mathbb{P}_{Z_1}$ with the proper faces of $\sigma$, and hence the above “convex hull” is already discontinuous at $\tau$, but still restricts to a linear function on $\tau$, and we can retain this property when further adjusting the function.

Since there are only finitely many $Z_1$, and since (by $\Gamma_{Z_1}$-admissibility) there are only finitely many $\Gamma_{Z_1}$-orbits of cones in $\Sigma'_1$ for each $Z_1$, up to multiplication by a common positive integer that clears the denominators, we may assume that all the adjusted functions $\phi_Z : \mathbb{P}_Z \to \mathbb{R}_{\geq 0}$ take integral values on $\mathbb{Z}_{\Sigma_1} \cap \mathbb{P}_Z$, which then provide the desired modifications of $\phi'_Z : \mathbb{P}_{Z_1} \to \mathbb{R}_{\geq 0}$, as desired. \(\square \)

6. Examples

Example 6.1. In Case 2, suppose that we have the following:

1. $\mathcal{O}_0 = \mathbb{Z} \times \mathbb{Z}$ and $\mathcal{O}_1 = \mathbb{Z}$ is diagonally embedded in $\mathcal{O}_0$, and $\ast_0$ and $\ast_1$ are trivial;
(2) $L_1 = \mathbb{Z}^{\oplus 4}$, with the first (resp. second) factor of $\mathcal{O}_0 = \mathbb{Z} \times \mathbb{Z}$ acting naturally on the first and third (resp. second and fourth) factors of $L_1 = \mathbb{Z}^{\oplus 4}$ and trivially on the remaining factors;

(3) Let $(\cdot, \cdot)_1 : L_1 \times L_1 \to \mathbb{Z}(1)$ be the self-dual pairing defined by composing the standard symplectic pairing $((x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4)) \mapsto x_1y_3 + x_2y_4 - x_3y_1 - x_4y_2$ with a fixed choice of isomorphism $2\pi\sqrt{-1} : \mathbb{Z} \to \mathbb{Z}(1)$, and let $h_1(a+b\sqrt{-1})$ act on $L_{1,\mathbb{R}} \cong \mathbb{R}^{\oplus 4}$ via the left multiplication by $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ on the first and third factors, and similarly on the second and fourth factors.

Then $G_0 \otimes \mathbb{Q}$ is the subgroup of $GL_{2,\mathbb{Q}} \times GL_{2,\mathbb{Q}}$ consisting of elements $(g_1, g_2)$ with $\det(g_1) = \det(g_2)$, and $G_0(\mathbb{R}) \cdot h_0$ is the disjoint union of $\mathcal{H}_+ \times \mathcal{H}_+$ and $\mathcal{H}_- \times \mathcal{H}_-$, where $\mathcal{H}_+$ and $\mathcal{H}_-$ denote the Poincaré upper and lower half-planes, respectively. Moreover, $(G_1 \otimes \mathbb{Q}, G_1(\mathbb{R}) \cdot h_0) = (GSp(4, \mathbb{Q}), \mathcal{H}_{2,\pm})$, where $\mathcal{H}_{2,\pm}$ is the union of the Siegel upper and lower half-planes of genus two. In both cases, the reflex field is $\mathbb{Q}$, so that we can take $F = \mathbb{Q}$, and there are no bad primes for the integral PEL data.

Let $H_0 \subset G_0(\mathbb{Z}^\mathbb{Q})$ be any neat open compact subgroup, which can be assumed to be of the form $H_1 \cap G_0(\mathbb{Z}^\mathbb{Q})$ for some neat open compact subgroup of $G_1(\mathbb{Z}^\mathbb{Q})$ (cf. the proofs of [20, Lem. 2.1.1.18] or [19, Lem. 4.16]). Then the moduli problem defined by $(O_1, \ast_1, L_1, (\cdot, \cdot)_1, h_1)$ and $H_1$ is a smooth integral model $X_1$ of the Siegel threefold over $S_0 = \text{Spec}(\mathbb{Z}(\mathbb{Q}))$, parameterizing principally polarized abelian surfaces with symplectic level-$H_1$ structures; and the moduli problem defined by $(O_0, \ast_0, L_0, (\cdot, \cdot)_0, h_0)$ and $H_0$ is the closed moduli subscheme $X_0$ of $X_1$ parameterizing principally polarized abelian surfaces of the form $(E_1 \times E_2, \lambda_1 \times \lambda_2)$, where $(E_1, \lambda_1)$ and $(E_2, \lambda_2)$ are canonically principally polarized elliptic curves, with level-$H_1$ structures satisfying some conditions. At the level of connected components, up to replacing $H_0$ with a finite index subgroup, $X_0$ can be viewed as the product of two smooth integral models of modular curves. In this case, we have a closed immersion $f : X_0 \hookrightarrow X_1$, and Theorem 2.3 guarantees the existence of some closed immersion of toroidal compactifications $f_{\text{tor}}^{\text{int}} : X_0^{\text{tor}}_{\Sigma_0} \hookrightarrow X_1^{\text{tor}}_{\Sigma_1}$ extending $f$, defined by some strictly compatible collections $\Sigma_0$ and $\Sigma_1$ of cone decompositions.

The map $\overline{P}_{Z_0} = (\overline{P}_{Z_0}^+ - \{0\})/\mathbb{R}^{\times} \to \overline{P}_{Z_1} = (\overline{P}_{Z_1}^+ - \{0\})/\mathbb{R}^{\times}$ can be from the empty set to the empty set; from a single point to a single point; or from the vertical half-line $i\mathbb{R}_{>0}$ to $\mathcal{H}_+$ (up to some identifications), where $i$ denotes the square root of $-1$ in $\mathcal{H}_+$. In the last case, $\Gamma_{Z_0}$ acts trivially on $i\mathbb{R}_{>0}$ because of neatness, while $\Gamma_{Z_1}$ acts via a neat congruence subgroup of $SL_2(\mathbb{Z})$ on $\mathcal{H}_+$ (with no nontrivial stabilizers). Then $\Sigma_{Z_0}$ gives a possibly nontrivial subdivision of $i\mathbb{R}_{>0}$, while $\Sigma_{Z_1}$ gives a triangularization of $\mathcal{H}_+$ that is compatible with $\Gamma_{Z_1}$ and descends to a triangularization of $\mathcal{H}_+/\Gamma_{Z_1}$. Note that any nontrivial subdivision of $i\mathbb{R}_{>0}$ means, when we view the connected components of $X_0$ as products of those of two smooth integral models of modular curves, we have blowups at some products of cusps.

Nevertheless, by Theorem 2.3, for each neat $H_0$, such blowups at products of cusps can be avoided up to replacing $H_1$ with a finite index subgroup. Concretely, according to the proof of Theorem 2.3 in Section 5 (see, in particular, the proof of the second paragraph of Proposition 5.1 in Case (2)), if $H_0 \subset G_0(\mathbb{Z}^\mathbb{Q})$ and $H_1 \subset G_1(\mathbb{Z}^\mathbb{Q})$ are principal congruence subgroups of the same level $n_0 \geq 3$, then there is at most one $Z_0$ mapped to each given $Z_1$, and the map $\overline{P}_{Z_0} \to \overline{P}_{Z_1}$ that
is not from either the empty set or a single point can be identified (up to \(SL_2(\mathbb{Z})\)-action on the target) with the map from the vertical half-line \(i\mathbb{R}_{>0}\) to \(\mathcal{H}_+\), in which case we can arrange a geodesic triangularization on \(\mathcal{H}_+\) without subdividing \(i\mathbb{R}_{>0}\), which results in compatible collections \(\Sigma_0\) and \(\Sigma_1\) that are strictly compatible, without blowups at products of cusps as above (cf. the discussions with \(n = 2\) in [2,3, Chapter II, Section 6]). (This is the end of Example 6.1.)

Example 6.2. In Case [2], suppose that we have the following:

1. \(n \geq 2\) is any integer;
2. \(K\) is an imaginary quadratic extension of \(\mathbb{Q}\), with maximal order \(\mathcal{O}_K\);
3. \(\mathcal{O}_0 = \mathcal{O}_K \times \mathcal{O}_K\) and \(\mathcal{O}_1 = \mathcal{O}_K\) is diagonally embedded in \(\mathcal{O}_0\), and \(\mathcal{O}_0\) and \(\mathcal{O}_1\) are the complex conjugations (simultaneously on both factors of \(\mathcal{O}_0\));
4. \(L_1 = \mathcal{O}_K^{n+1}\), with the first (resp. second) factor of \(\mathcal{O}_0 = \mathcal{O}_K \times \mathcal{O}_K\) acting naturally on the first \(n\) factors (resp. last factor) of \(L_1 = \mathcal{O}_K^{n+1}\) and trivially on the remaining factors;
5. Let \(\varepsilon \in \text{Diff}_{\mathcal{O}_K/\mathbb{Z}}^{-1}\) be any element in the inverse different that is invariant under the complex conjugation, and let \(\langle \cdot, \cdot \rangle_1 : L_1 \times L_1 \to \mathbb{Z}(1)\) be the pairing defined by composing the pairing \(\langle (x_1, x_2, \ldots, x_{n+1}), (y_1, y_2, \ldots, y_2, y_3, y_4) \rangle \mapsto (2\pi \sqrt{-1}) \text{Tr}_{\mathcal{O}_K/\mathbb{Z}}(\varepsilon \cdot (x_1 y_1 + x_2 y_2 + \cdots + x_{n+1} y_{n+1}))\) with a fixed choice of isomorphism \(2\pi \sqrt{-1} : \mathbb{Z} \to \mathbb{Z}(1)\), and let \(h_1(z)\) act on \(L_{1,\mathbb{R}} \cong \mathbb{C}^{n+1}\) via the left multiplication by the complex conjugate \(\overline{\varepsilon}\) on the first factor, and by \(z\) itself on the remaining factors.

Then \(G_0 \otimes \mathbb{R} \cong G(U_{n-1,1} \times U_1)\), which is the subgroup of \(GU_{n-1,1} \times GU_1\) consisting of elements \((g_1, g_2)\) with equal similitudes \(\nu(g_1) = \nu(g_2)\); and \(G_1 \otimes \mathbb{R} \cong GU_{n,1}\). (Note that \(G_0 \otimes \mathbb{R}\) is very close to \(GU_{n-1,1}\).) In both cases, the reflex field is \(K\) because \(n \geq 2\), so that we can take \(F = K\); and the bad primes are those ramified in \(K\) and divides \(\text{Tr}_{\mathcal{O}_K/\mathbb{Z}}(\varepsilon)\), and we can take \(\mathcal{O}_0\) to be any set of rational primes other than these. Let us choose \(\mathcal{H}_0\) and \(\mathcal{H}_1\) suitably, so that we have smooth integral models \(X_0\) and \(X_1\) over \(\mathcal{S}_0\), with a closed immersion \(f : X_0 \to X_1\), which can be interpreted as mapping a smooth integral model of a \(GU_{n-1,1}\) Shimura variety to a smooth integral model of a \(GU_{n,1}\) Shimura variety defined by taking fiber products of the universal abelian scheme with some CM elliptic curves (which explains the need of the \(U_1\) part). (It is perhaps better to work with abelian type Shimura varieties and arrange \(G_0 \otimes \mathbb{R} \to G_1 \otimes \mathbb{R}\) to be \(U_{n-1,1} \to U_{n,1}\), but the difference is on the centers and hence unimportant for our results on the cone decompositions.)

By Theorem 2.2 there exists some closed immersion of toroidal compactifications \(f_{\Sigma_0, \Sigma_1}^{\text{tor}} : X_{0, \Sigma_0}^{\text{tor}} \to X_{1, \Sigma_1}^{\text{tor}}\) extending \(f\), defined by some strictly compatible collections \(\Sigma_0\) and \(\Sigma_1\) of cone decompositions. But note that we have no choice to make for \(\Sigma_0\) and \(\Sigma_1\). All possible maps \(P_{\mathbb{Z}_0} \to P_{\mathbb{Z}_1}\) can be identified with either \(\{0\} \to \{0\}\) or \(\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}\), and in all cases the cone decompositions are uniquely determined and trivial (and satisfy all the usual conditions we impose). Hence, Theorem 2.2 just says that the canonical morphism \(f^{\text{tor}} : X_0^{\text{tor}} \to X_1^{\text{tor}}\) between smooth integral models of toroidal compactifications over \(\mathcal{S}_0\), where all the collections of cone decompositions are now justifiably omitted from the notation, is a closed immersion; and Theorem 2.3 is trivially true without the need to replace \(\mathcal{H}_1\).
Nevertheless, such a discussion is not completely meaningless. The fact that smooth toroidal compactifications of $X_0$ and $X_1$ uniquely exist is well known, but the fact that closed immersions $f: X_0 \to X_1$ extend to closed immersions $f^{\text{tor}}: X_0^{\text{tor}} \to X_1^{\text{tor}}$ is probably less so. Also, as soon as we have such a $f^{\text{tor}}$, we can consider the closed immersion $(\text{Id}_{X_0}, f): X_0 \to X_0 \times X_1$, which then extends to the closed immersion $(\text{Id}_{X_0^{\text{tor}}}, f^{\text{tor}}): X_0^{\text{tor}} \to X_1^{\text{tor}}$, which provides the justification for some usual geometric considerations related to the Gan–Gross–Prasad conjecture.

We have similar assertions in Case (3). (This is the end of Example 6.3.)

**Example 6.3.** In Case (1), suppose $G_0$ is the special orthogonal group over $\mathbb{Q}$ defined by a quadratic space $V_0$ of signature $(n-1,2)$ at $\infty$, for some integer $n \geq 2$, and let $G_1$ be the special orthogonal group over $\mathbb{Q}$ defined by $V_1 := V_0 \otimes (\mathbb{Q}: e_{n+2})$, where the quadratic form is defined by setting $e_{n+2}$ to have norm $+1$. Then $G_0 \otimes \mathbb{R} \cong \text{SO}_{n-1,2}$ and $G_1 \otimes \mathbb{R} \cong \text{SO}_{n,2}$. Let $h_0$ and $h_1$ be defined by mapping $\text{Gm}_m, \mathbb{C} \hookrightarrow \text{SO}_{2,\mathbb{R}}$ so $r(\cos \theta + i \sin \theta) \mapsto (\cos 2\theta - \sin 2\theta \cos 2\theta \sin 2\theta)$ into the second factor of the diagonally embedded compact subgroups $\text{SO}_{n-1,2} \times \text{SO}_{2,\mathbb{R}}$ and $\text{SO}_{n,1} \times \text{SO}_{2,\mathbb{R}}$ of $\text{SO}_{n-1,2}$ and $\text{SO}_{n,2}$, respectively. Then the reflex field of both Shimura data $(G_0, G_0(\mathbb{R}) \cdot h_0)$ and $(G_1, G_1(\mathbb{R}) \cdot h_1)$ is $\mathbb{Q}$, and we can take $F$ to be $\mathbb{Q}$ (or any field extension in $\mathbb{C}$). Let $\mathcal{H}_0$ and $\mathcal{H}_1$ be chosen such that $f: X_0 \to X_1$ is a closed immersion over $S_0 = \text{Spec}(F)$. Then $\mathcal{P}_{Z_0}^+ \to \mathcal{P}_{Z_1}^+$ can be either from the empty set to the empty set; from a single point to a single point; or from the hyperbolic $(n-1)$-space to the hyperbolic $n$-space (equivariant with $\text{SO}_{n-1,1} \to \text{SO}_{n,1}$, up to some identifications). (The map $i\mathbb{R}_{>0} \to \mathcal{H}_+$ in Example 6.1 can be viewed as a special case of the last possibility, with $n = 2$.) By Theorem 2.2 there exists some closed immersion of toroidal compactifications $f^{\text{tor}}_{\Sigma_0, \Sigma_1}: X^{\text{tor}}_{0, \Sigma_0} \hookrightarrow X^{\text{tor}}_{1, \Sigma_1}$ extending $f$, for some compatible collections $\Sigma_0$ and $\Sigma_1$ of cone decompositions that are strictly compatible.

Moreover, by Theorem 2.3 up to replacing $\mathcal{H}_1$ with a finite index subgroup, we may extend $f$ to a closed immersion $f^{\text{tor}}_{\Sigma_0, \Sigma_1}$ with any prescribed smooth compatible collection $\Sigma_0$, which gives triangularizations of some hyperbolic $n$-spaces extending some prescribed geodesic triangularizations on some hyperbolic $(n-1)$-subspaces.

We have similar assertions in Case (1) if we replace special orthogonal groups above with the corresponding general spin groups, with suitable associated Shimura data and Siegel embeddings. (This is the end of Example 6.3.)

**References**


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