DUAL BGG COMPLEXES FOR AUTOMORPHIC BUNDLES

KAI-WEN LAN AND PATRICK POLO

Abstract. We generalize the construction of dual BGG complexes in Faltings–Chai and Mokrane–Tilouine–Polo (from the case of Siegel modular varieties) to all smooth integral models of PEL-type Shimura varieties.

1. INTRODUCTION

Shimura varieties are generalizations of modular curves, whose cohomology groups with coefficients in the so-called automorphic bundles set up natural stages for relating automorphic representations to Galois representations. In order to understand the Hodge structures on the de Rham version of such cohomology groups, Faltings introduced the dual BGG spectral sequence (over \( \mathbb{C} \)) in [9], following older ideas of Bernstein, I. M. Gelfand, and S. I. Gelfand [2], and verified its degeneration in [10, Ch. VI] using toroidal compactifications of fiber products of the universal abelian schemes over the Siegel modular varieties.

The geometric construction of compactifications in [10, Ch. VI] is actually carried out over \( \mathbb{Z} \), and (parabolic) BGG complexes have been constructed over \( \mathbb{Z}(p) \) (under a \( p \)-smallness assumption on the highest weights) by Tilouine and the second author in [33]. Based on these inputs, Mokrane and Tilouine studied the de Rham cohomology of Siegel modular varieties with coefficients in vector bundles over \( \mathbb{Z}(p) \) in [31] by constructing analogues of Faltings’s dual BGG complexes, and obtained several interesting applications to the cohomology of Siegel modular varieties. (In [8], Dimitrov applied similar ideas to the cohomology of Hilbert modular varieties.)

The aim of this article is to explain that the constructions of dual BGG complexes in [9, Sec. 3], [10, Ch. VI], and [31, Sec. 5] have analogues over all (smooth integral models of) PEL-type Shimura varieties, under a \( p \)-smallness assumption.

The main geometric input, generalizing the constructions of toroidal compactifications in [10], has been carried out by the first author in [22] and [25]. (We will refer to the published revision [26] instead of the original thesis [22]. When the Shimura variety we consider is compact, the shorter article [23] would suffice, because it explains that no compactification is needed.)


Key words and phrases. Shimura varieties; automorphic bundles and differential operators; BGG and dual BGG complexes.

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In [33], Theorems 2.8 and 4.3 were stated (and proved) for a connected, split reductive group \( G \) with a simply-connected derived group. In fact, these hypotheses can be relaxed, and we will show that a similar result still holds when \( G \) has a factor isomorphic to some orthogonal group \( O_2 \), (whose derived group is not simply-connected). (For readers familiar with the classification of PEL-type Shimura varieties, the point is that we allow all possibilities, including those with factors of type D.)

We will review the geometric setup in Section 2, review the representation theory we need in Section 3, explain the construction of differential operators in Section 4, and prove our main results in Section 5.

We shall follow [26, Notations and Conventions] unless otherwise specified. By symplectic isomorphisms between modules with symplectic pairings, we always mean isomorphisms between the modules matching the pairings up to an invertible scalar multiple. (These are often called symplectic similitudes, but our understanding is that the codomains of pairings are modules rather than rings, which ought to be matched as well.) Sheaves on schemes, algebraic spaces, or algebraic stacks are étale sheaves by default, although for coherent sheaves on schemes it would suffice to work in the Zariski topology.

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2. Geometric setup

2.1. Linear algebraic data. Let $\mathcal{O}$ be an order in a finite-dimensional semisimple $\mathbb{Q}$-algebra with a positive involution $\ast$. Here, an involution means an anti-automorphism of order 2, and positivity of $\ast$ means that for every $x \neq 0$ in $\mathcal{O} \otimes \mathbb{R}$, one has $\text{Tr}_{\mathcal{O} \otimes \mathbb{R}/\mathbb{R}}(xx^\ast) > 0$. We assume that $\mathcal{O}$ is stable under $\ast$. We shall denote the center of $\mathcal{O} \otimes \mathbb{Q}$ by $F$. (Then $F$ is a product of number fields.)

Let $\mathbb{Z}(1) := \ker(\exp : \mathbb{C} \to \mathbb{C}^\times) = (2\pi \sqrt{-1})\mathbb{Z}$, which is a free $\mathbb{Z}$-module of rank one. For any $\mathbb{Z}$-module $M$, we denote by $M(1)$ the module $M \otimes \mathbb{Z}(1)$, called the Tate twist of $M$, which is noncanonically isomorphic to $M$ as $\mathbb{Z}$-modules.

By a PEL-type $\mathcal{O}$-lattice $(L, \langle \cdot, \cdot \rangle, h_0)$, we mean the following data:

1. An $\mathcal{O}$-lattice $L$, namely, a finite free $\mathbb{Z}$-module $L$ with the structure of an $\mathcal{O}$-module.
2. An alternating pairing $\langle \cdot, \cdot \rangle : L \times L \to \mathbb{Z}(1)$ satisfying $\langle bx, y \rangle = \langle x, b^\ast y \rangle$ for all $x, y \in L$ and $b \in \mathcal{O}$, together with an $\mathbb{R}$-algebra homomorphism $h_0 : \mathbb{C} \to \text{End}_{\mathcal{O} \otimes \mathbb{R}}(L \otimes \mathbb{R})$,

satisfying:

(a) For any $z \in \mathbb{C}$ and $x, y \in L \otimes \mathbb{R}$, we have $\langle h_0(z)x, y \rangle = \langle x, h_0(z^c)y \rangle$, where $z \mapsto z^c$ is complex conjugation.
(b) The $\mathbb{R}$-bilinear pairing $(2\pi \sqrt{-1})^{-1}\langle \cdot, h_0(\sqrt{-1})\cdot \rangle$ on $L \otimes \mathbb{R}$ is (symmetric and) positive definite. (See [26] Def. 1.2.1.3, where $h_0$ was denoted by $h$.)

The tuple $(\mathcal{O}, \ast, L, \langle \cdot, \cdot \rangle, h_0)$ will be called an integral PEL datum. It is an integral version of the data $(B, \ast, V, \langle \cdot, \cdot \rangle, h_0)$ in [21] and related works.

**Definition 2.1 (cf. [26] Def. 1.2.1.6).** Let $\mathcal{O}$ and $(L, \langle \cdot, \cdot \rangle)$ be given as above. We define for each $\mathbb{Z}$-algebra $R$

$$G(R) := \left\{ (g, r) \in \text{Aut}_{\mathcal{O} \otimes \mathbb{R}}(L \otimes R) \times \text{G}_m(R) : \langle gx, gy \rangle = r \langle x, y \rangle, \forall x, y \in L \otimes R \right\}.$$ 

The assignment is functorial in $R$ and defines a group functor $G$ over $\mathbb{Z}$. The projection to the second factor $(g, r) \mapsto r$ defines a homomorphism $v : G \to \text{G}_m$, which we call the similitude character. For simplicity, we shall often denote elements $(g, r)$ in $G$ by simply $g$, and denote by $v(g)$ the value of $r$ when we need it.

The homomorphism $h_0 : \mathbb{C} \to \text{End}_{\mathcal{O} \otimes \mathbb{R}}(L \otimes \mathbb{R})$ defines a Hodge structure of weight $-1$ on $L$, with Hodge decomposition

$$(2.2) \quad L \otimes \mathbb{C} = V_0 \oplus V_0^c,$$

such that $h_0(z)$ acts by $1 \otimes z$ on $V_0$, and by $1 \otimes z^c$ on $V_0^c$. One can easily check that $V_0$ is (maximal) totally isotropic under the non-degenerate pairing $\langle \cdot, \cdot \rangle$, and hence [22] induces canonically an isomorphism

$$(2.3) \quad V_0^c \cong V_0^c(1) := \text{Hom}_{\mathbb{C}}(V_0, \mathbb{C})(1).$$
Let $F_0$ be the reflex field of the $\mathcal{O} \otimes \mathbb{C}$-module $V_0$. Recall (see [21, p. 389] or [20, Def. 1.2.5.4]) that $F_0$ is the subfield of $\mathbb{C}$ generated over $\mathbb{Q}$ by $\{\text{Tr}_C(b|V_0)\}_{b \in \mathcal{O}}$.

By abuse of notation, we shall denote the ring of integers in $F$ (resp. $F_0$) by $\mathcal{O}_F$ (resp. $\mathcal{O}_{F_0}$). This is in conflict with the notation of the order $\mathcal{O}$ in the integral PEL datum, but the precise interpretation will be clear from the context.

We say that a rational prime number $p > 0$ is good if it satisfies the following conditions (cf. [21, Sec. 5] or [20, Def. 1.4.1.1]):

1. $p$ is unramified in $\mathcal{O}$ (as in [20, Def. 1.1.1.8]).
2. $p \neq 2$ if $\mathcal{O} \otimes \mathbb{Q}$ involves simple factors of type D (as in [20, Def. 1.2.1.15]).
3. If we consider $L^{\#} := \{x \in L \otimes \mathbb{Q} : \langle x, y \rangle \in \mathbb{Z}(1), \forall y \in L\}$, the dual lattice of $L$ under the $\mathbb{Z}(1)$-valued pairing $\langle \cdot, \cdot \rangle$, then $p \mid [L^{\#} : L]$. Equivalently, after base change to $\mathbb{Z}((p))$, the pairing $\langle \cdot, \cdot \rangle$ is perfect in the sense that it induces an isomorphism $L \otimes \mathbb{Z}((p)) \cong L^\vee \otimes \mathbb{Z}((p))(1)$.

Let us fix any choice of a good prime $p$.

By [20, Lem. 1.2.5.9], there exists a finite field extension $F'_0$ of $F_0$ in $\mathbb{C}$, unramified at $p$, and an $\mathcal{O}_{F'_0}$-torsion-free $\mathcal{O} \otimes \mathcal{O}_{F'_0,(p)}$-module $L_0$, such that $L_0 \otimes_{\mathcal{O}_{F'_0,(p)}} \mathbb{C} \cong V_0$.

Let us fix the choices of $F'_0$ and $L_0$ from now on. (In practice, there might be optimal choices in each special case.)

Let us denote (cf. [20, Lem. 1.1.4.13]) by

$$\langle \cdot, \cdot \rangle_{\text{can}} : (L_0 \oplus L_0^\vee(1)) \times (L_0 \oplus L_0^\vee(1)) \to \mathcal{O}_{F'_0,(p)}(1)$$

the alternating pairing defined by

$$\langle (x_1, f_1), (x_2, f_2) \rangle_{\text{can}} := f_2(x_1) - f_1(x_2).$$

The natural right action of $\mathcal{O}$ on $L_0^\vee(1)$ defines a natural left action of $\mathcal{O}$ by composition with $*: \mathcal{O} \to \mathcal{O}^{\text{op}}$. Then (2.3) induces canonically an isomorphism $L_0^\vee(1) \otimes \mathbb{C} \cong V_0^\vee(1) \cong V_0^\vee \otimes \mathbb{C}$ of $\mathcal{O} \otimes \mathbb{C}$-modules.

**Definition 2.4.** For any $\mathcal{O}_{F'_0,(p)}$-algebra $R$, set

$$G_0(R) := \left\{ (g, r) \in \text{Aut}_{\mathcal{O} \otimes R}(L_0 \oplus L_0^\vee(1)) \otimes_{\mathcal{O}_{F'_0,(p)}} R \times \mathbb{G}_m(R) : (gx, gy)_{\text{can}} = r(x, y)_{\text{can}}, \forall x, y \in (L_0 \oplus L_0^\vee(1)) \otimes_{\mathcal{O}_{F'_0,(p)}} R \right\},$$

$$P_0(R) := \left\{ (g, r) \in G_0(R) : g(L_0^\vee(1) \otimes_{\mathcal{O}_{F'_0,(p)}} R) = L_0^\vee(1) \otimes_{\mathcal{O}_{F'_0,(p)}} R \right\},$$

$$M_0(R) := \text{Aut}_{\mathcal{O} \otimes R}(L_0^\vee(1) \otimes_{\mathcal{O}_{F'_0,(p)}} R) \times \mathbb{G}_m(R),$$

We shall view $M_0(R)$ canonically as a quotient of $P_0(R)$ by

$$P_0(R) \to M_0(R) : (g, r) \mapsto (g|_{L_0^\vee(1)} \otimes_{\mathcal{O}_{F'_0,(p)}} R, r).$$

The assignments are functorial in $R$, and define group functors $G_0$, $P_0$, and $M_0$ over $\mathcal{O}_{F'_0,(p)}$. 
By [26] Prop. 1.1.1.21, Cor. 1.2.5.7, and Cor. 1.2.3.10, and by our choice that \( F'_0 \) is unramified at \( p \), there exists a discrete valuation ring \( R_1 \) over \( \mathcal{O}_{F'_0,(p)} \) satisfying the following:

**Condition 2.5.**

1. The maximal ideal of \( R_1 \) is generated by \( p \), and the residue field \( \kappa_1 \) of \( R_1 \) is a finite field of characteristic \( p \). In this case, the \( p \)-adic completion of \( R_1 \) is isomorphic to the Witt vectors \( W(\kappa_1) \) over \( \kappa_1 \).
2. The \( \mathbb{Z} \)-algebra \( \mathcal{O}_F \) is split over \( R_1 \), in the sense that \( \Upsilon := \text{Hom}_{\mathbb{Z}\text{-alg.}}(\mathcal{O}_F, R_1) \) has cardinality \( |F : \mathbb{Q}| \). Then there is a canonical isomorphism

\[
\mathcal{O}_F \otimes \mathcal{O}_F_{R_1} \cong \prod_{\tau \in \Upsilon} \mathcal{O}_F,\tau,
\]

where each \( \mathcal{O}_F,\tau \) can be identified as the \( \mathcal{O}_F \)-algebra \( R_1 \) via \( \tau \).
3. There exists an isomorphism

\[
(L \otimes \mathcal{O}_F_{R_1}, \langle \cdot, \cdot \rangle) \cong (L_0 \otimes L_0^{\vee}(1), \langle \cdot, \cdot \rangle_{\text{can.}}) \otimes \mathcal{O}_{F'_0,(p)}
\]

inducing an isomorphism

\[
\mathcal{O}_F \otimes \mathcal{O}_{F'_0,(p)} \cong \mathcal{O}_F \otimes \mathcal{O}_F_{R_1} \otimes \mathcal{O}_{F'_0,(p)}
\]

realizing \( \mathcal{O}_0 \otimes \mathcal{O}_F \) as a subgroup of \( \mathcal{O}_F \otimes \mathcal{O}_F^{\vee}_{R_1} \). (The existence of the isomorphism \([2,7]\) follows from [26, Cor. 1.2.3.10].)

From now on, let us fix the choice of \( R_1 \) and the isomorphism \([2,7]\), and set \( \mathcal{O}_F,1 := \mathcal{O}_F \otimes \mathcal{O}_F_{R_1}, \mathcal{O}_1 := \mathcal{O}_F \otimes \mathcal{O}_F_{R_1}, L_1 := L \otimes \mathcal{O}_F_{R_1}, L_{0,1} := L_0 \otimes \mathcal{O}_F_{R_1}, G_1 := G_0 \otimes \mathcal{O}_F_{R_1}, P_1 := P_0 \otimes \mathcal{O}_F_{R_1} \), and \( M_1 := M_0 \otimes \mathcal{O}_F_{R_1} \).

**Remark 2.9.** The group functors in Definitions \([2.7]\) and \([2.4]\) are representable because they are defined by closed conditions in general linear group schemes. By the same explicit classification as in the proof of [26, Prop. 1.2.3.11] (which works verbatim over the \( R_1 \) here instead of the \( R' \) there), \( G_1 = G_0 \otimes \mathcal{O}_F_{R_1} \) is a split reductive group scheme over \( R_1 \), the group scheme \( P_1 \) is a parabolic subgroup scheme of \( G_1 \), and \( M_1 \) is canonically isomorphic to the Levi quotient of \( P_1 \).

### 2.2. PEL-Type Shimura varieties and automorphic bundles

Let \( \mathcal{H} \) be a neat open compact subgroup of \( \text{G}(\widehat{\mathbb{Z}}^p) \). (See [32] 0.6 or [26] Def. 1.4.1.8 for the definition of neatness.) By [26] Def. 1.4.1.4 (with \( \square = \{ p \} \) there), the data of \( (L, \langle \cdot, \cdot \rangle, h_0) \) and \( \mathcal{H} \) define a moduli problem \( \mathcal{M}_\mathcal{H} \) over \( S_0 = \text{Spec}(\mathcal{O}_{F'_0,(p)}) \), parameterizing tuples \( (A, \lambda, i, \alpha_\mathcal{H}) \) over schemes \( S \) over \( S_0 \) of the following form:

1. \( A \to S \) is an abelian scheme.
2. \( \lambda : A \to A^\vee \) is a polarization of degree prime to \( p \).
3. \( i : \mathcal{O} \to \text{End}_{\mathcal{O}}(A) \) is an \( \mathcal{O} \)-endomorphism structure as in [26] Def. 1.3.3.1.
4. \( \text{Lie}_{A/S} \) with its \( \mathcal{O} \otimes \mathbb{Z}(p) \)-module structure given naturally by \( i \) satisfies the determinantal condition in [26] Def. 1.3.4.1] given by \( (L \otimes \mathcal{O}_F, \langle \cdot, \cdot \rangle, h_0) \).
5. \( \alpha_\mathcal{H} \) is an (integral) level-\( \mathcal{H} \) structure of \( (A, \lambda, i) \) of type \( (L \otimes \mathcal{O}_F, \langle \cdot, \cdot \rangle) \) as in [26] Def. 1.3.7.6.
(The definition can be identified with the one in [21 Sec. 5] by [26 Prop. 1.4.3.4].) By [26 Thm. 1.4.11 and Cor. 7.2.3.10], \( \mathcal{M}_H \) is representable by a (smooth) quasi-projective scheme over \( S_0 \) (under the assumption that \( \mathcal{H} \) is neat).

Let \( (A, \lambda, i, \alpha_H) \to \mathcal{M}_H \) be the universal tuple over \( \mathcal{M}_H \). Consider the relative de Rham cohomology \( H^1_{\text{dR}}(A/\mathcal{M}_H) \), with the dual

\[
H^1_{\text{dR}}(A/\mathcal{M}_H) := \text{Hom}_{\mathcal{O}_{\mathcal{M}_H}}(H^1_{\text{dR}}(A/\mathcal{M}_H), \mathcal{O}_{\mathcal{M}_H})
\]
defined to be the relative de Rham homology. Consider the canonical pairing

\[
\langle \cdot, \cdot \rangle_{\lambda} : H^1_{\text{dR}}(A/\mathcal{M}_H) \times H^1_{\text{dR}}(A/\mathcal{M}_H) \to \mathcal{O}_{\mathcal{M}_H}(1)
\]
behind (2.10) defined by the pullback under \( \text{Id} \times \lambda_* \) of the canonical perfect pairing

\[
H^1_{\text{dR}}(A/\mathcal{H}) \times H^1_{\text{dR}}(A^\vee/\mathcal{M}_H) \to \mathcal{O}_{\mathcal{M}_H}(1)
\]
defined by the first Chern class of the Poincaré line bundle over \( A \times A^\vee \). (See for example [7 1.5].) Under the assumption that \( \lambda \) has degree prime to \( p_1 \), we know that \( \lambda_* \) is separable, that \( \lambda_* \) is an isomorphism, and hence that the pairing \( \langle \cdot, \cdot \rangle_{\lambda} \) above is perfect. Let \( \langle \cdot, \cdot \rangle_{\lambda} \) also denote the induced pairing on \( H^1_{\text{dR}}(A/\mathcal{M}_H) \times H^1_{\text{dR}}(A/\mathcal{M}_H) \) by duality. By [3 Lem. 2.5.3], we have canonical short exact sequences

\[
0 \to \text{Lie}_{\mathcal{M}_H}(1) \to H^1_{\text{dR}}(A/\mathcal{M}_H) \to \text{Lie}_{A/\mathcal{M}_H} \to 0
\]

and

\[
0 \to \text{Lie}_{A/\mathcal{M}_H}(1) \to H^1_{\text{dR}}(A/\mathcal{M}_H) \to \text{Lie}_{A^\vee/\mathcal{M}_H} \to 0.
\]
The submodules \( \text{Lie}_{\mathcal{M}_H}(1) \) and \( \text{Lie}_{A/\mathcal{M}_H}(1) \) are maximal totally isotropic with respect to \( \langle \cdot, \cdot \rangle_{\lambda} \). (The Tate twists on \( \text{Lie}_{\mathcal{M}_H}(1) \) and \( \text{Lie}_{A/\mathcal{M}_H}(1) \) were omitted in some of the first author’s earlier writings, which we have reinstated for the sake of clarity.)

Let \( \tilde{M}^{(m)}_H \) be the \( m \)-th infinitesimal neighborhood of the diagonal image of \( \mathcal{M}_H \) in \( \mathcal{M}_H \times \mathcal{M}_H \), and let \( \text{pr}_1, \text{pr}_2 : \tilde{M}^{(m)}_H \to \mathcal{M}_H \) be the two projections. Then we have by definition the canonical morphism \( \mathcal{O}_{\mathcal{M}_H} \to \mathcal{O}_{\mathcal{M}_H/S_0} := \text{pr}_1^* \circ \text{pr}_2^*(\mathcal{O}_{\mathcal{M}_H}) \). The isomorphism \( s : \tilde{M}^{(m)}_H \to \tilde{M}^{(m)}_H \) over \( \mathcal{M}_H \) swapping two components of the fiber product then defines an automorphism \( s^* \) of \( \mathcal{O}_{\mathcal{M}_H/S_0} \). When \( m = 1 \), the kernel of the structural morphism \( \text{str}^* : \mathcal{O}_{\mathcal{M}_H/S_0} \to \mathcal{O}_{\mathcal{M}_H} \) canonically isomorphic to \( \Omega^1_{\mathcal{M}_H/S_0} \) by definition, is spanned by the image of \( s^* - \text{Id}^* \) (induced by \( \text{pr}_1^* - \text{pr}_2^* \)).

An important property of the relative de Rham cohomology of any smooth morphism like \( A \to \mathcal{M}_H \) is that, for any two smooth lifts \( \tilde{A}_1 \to \tilde{M}^{(1)}_H \) and \( \tilde{A}_2 \to \tilde{M}^{(1)}_H \) of \( A \to \mathcal{M}_H \), there is a canonical isomorphism \( H^1_{\text{dR}}(\tilde{A}_2/\tilde{M}^{(1)}_H) \cong H^1_{\text{dR}}(\tilde{A}_1/\tilde{M}^{(1)}_H) \) lifting the identity morphism on \( H^1_{\text{dR}}(A/\mathcal{M}_H) \). (See for example [26 Prop. 2.1.6.4].) If we consider \( \tilde{A}_1 := \text{pr}_1^* A \) and \( \tilde{A}_2 := \text{pr}_2^* A \), then we obtain a canonical isomorphism

\[
\text{pr}_2^* H^1_{\text{dR}}(A/\mathcal{M}_H) \cong H^1_{\text{dR}}(\text{pr}_2^* A/\tilde{M}^{(1)}_H) \cong H^1_{\text{dR}}(\text{pr}_1^* A/\tilde{M}^{(1)}_H) \cong \text{pr}_1^* H^1_{\text{dR}}(A/\mathcal{M}_H),
\]

which we denote by \( \text{Id}^* \) by abuse of notation. On the other hand, pullback by the swapping automorphism \( s : \tilde{M}^{(1)}_H \to \tilde{M}^{(1)}_H \) defines another canonical isomorphism

\[
s^* : \text{pr}_2^* H^1_{\text{dR}}(A/\mathcal{M}_H) \cong H^1_{\text{dR}}(\text{pr}_2^* A/\tilde{M}^{(1)}_H) \cong H^1_{\text{dR}}(\text{pr}_1^* A/\tilde{M}^{(1)}_H) \cong \text{pr}_1^* H^1_{\text{dR}}(A/\mathcal{M}_H).
\]

Hence, we can define the Gauss–Manin connection as follows (cf. [26 Rem. 2.1.7.4]):
Definition 2.11. The Gauss–Manin connection
\begin{equation}
\nabla : H^1_{\text{dR}}(A/M_H) \to H^1_{\text{dR}}(A/M_H) \otimes \Omega^1_{M_H/S_0}
\end{equation}
on 

on $H^1_{\text{dR}}(A/M_H)$ is the composition
\[
H^1_{\text{dR}}(A/M_H) \xrightarrow{pr^*_2} H^1_{\text{dR}}(pr^*_2 A/\hat{M}_H) \xrightarrow{\pi^* - \text{Id}^*} H^1_{\text{dR}}(A/M_H) \otimes \Omega^1_{M_H/S_0}.
\]

Definition 2.12. The composition
\[
\text{Lie}^\vee_{A/M_H}(1) \hookrightarrow H^1_{\text{dR}}(A/M_H) \xrightarrow{\nabla} H^1_{\text{dR}}(A/M_H) \otimes \Omega^1_{M_H/S_0} \xrightarrow{\text{Lie}^\vee_{A/M_H} \otimes \text{Lie}^\vee_{A^\vee/M_H}(1)} \Omega^1_{M_H/S_0},
\]
defines by duality a morphism
\begin{equation}
\KS_{A/M_H/S_0} : \text{Lie}^\vee_{A/M_H} \otimes \text{Lie}^\vee_{A^\vee/M_H}(1) \to \Omega^1_{M_H/S_0},
\end{equation}
which we call the Kodaira–Spencer morphism.

Definition 2.13. The Kodaira–Spencer isomorphism
\[
KS_{A/M_H/S_0} \circ \nabla = \Omega^1_{M_H/S_0}.
\]

Proposition 2.16 (see \cite{23} Prop. 2.3.5.2). The Kodaira–Spencer morphism \eqref{2.14} factors through the canonical quotient $\text{Lie}^\vee_{A/M_H} \otimes \text{Lie}^\vee_{A^\vee/M_H}(1) \to \Omega^1_{M_H/S_0}$ and induces an isomorphism
\begin{equation}
\KS_{A/M_H/S_0} \circ \nabla \cong \Omega^1_{M_H/S_0},
\end{equation}
which we call the Kodaira–Spencer isomorphism, and denote again (by abuse of notation) by $\KS_{A/M_H/S_0}$.
As in [25 Sec. 6A] and [27 Sec. 1.3], let us define the principal bundles

\begin{equation}
\mathcal{E}_G := \text{Isom}_{\mathcal{M}_{H,1}} \left( (H^1_{\text{ad}}(A/M_{H,1}), \langle \cdot, \cdot \rangle), \mathcal{O}_{\mathcal{M}_{H,1}}(1) \right),
\end{equation}

\begin{equation}
\mathcal{E}_{P_1} := \text{Isom}_{\mathcal{M}_{H,1}} \left( (H^1_{\text{ad}}(A/M_{H,1}), \langle \cdot, \cdot \rangle), \text{Lie}_{A/\mathcal{M}_{H,1}}(1) \right),
\end{equation}

and

\begin{equation}
\mathcal{E}_{M_1} := \text{Isom}_{\mathcal{O}} \circ \mathcal{O}_{\mathcal{M}_{H,1}} \left( (\text{Lie}_{\mathcal{M}_{H,1}}(1), \mathcal{O}_{\mathcal{M}_{H,1}}(1)), (L^0_{0,1}(1) \otimes \mathcal{O}_{\mathcal{M}_{H,1}}(1), \mathcal{O}_{\mathcal{M}_{H,1}}(1)) \right),
\end{equation}

which are étale torsors for $G_1$, $P_1$, and $M_1$, respectively, over $\mathcal{M}_{H,1}$. (The entries $\mathcal{O}_{\mathcal{M}_{H,1}}(1)$ in the tuples represent the values of the pairings, which are matched up to unit by the isomorphisms, by our convention.)

**Definition 2.21.** For any $R_1$-algebra $R$, we denote by $\text{Rep}_R(G_1)$ (resp. $\text{Rep}_R(P_1)$, resp. $\text{Rep}_R(M_1)$) the category of $R$-modules of finite presentation with algebraic actions of $G_1 \otimes R$ (resp. $P_1 \otimes R$, resp. $M_1 \otimes R$).

**Definition 2.22.** Let $R$ be any $R_1$-algebra. For any $W \in \text{Rep}_R(G_1)$, we define

\begin{equation}
\mathcal{E}_{G_1, R}(W) := (\mathcal{E}_{G_1} \otimes R) \times W,
\end{equation}

called the automorphic sheaf over $\mathcal{M}_{H,1} \otimes R$ associated with $W$. It is called an automorphic bundle if $W$ is locally free as an $R$-module. We define similarly for $W \in \text{Rep}_R(P_1)$ (resp. $W \in \text{Rep}_R(M_1)$) by replacing $G_1$ with $P_1$ (resp. with $M_1$) in (2.23). (These are coherent sheaves by fpqc descent. See [15 VIII, 1.1 and 1.10].)

By [27 Lem. 1.18, 1.19, and 1.20, and Cor. 1.21], we have the following:

**Lemma 2.24.** Let $R$ be any $R_1$-algebra.

1. The assignment $\mathcal{E}_{G_1, R}(\cdot)$ (resp. $\mathcal{E}_{P_1, R}(\cdot)$, resp. $\mathcal{E}_{M_1, R}(\cdot)$) defines an exact functor from $\text{Rep}_R(G_1)$ (resp. $\text{Rep}_R(P_1)$, resp. $\text{Rep}_R(M_1)$) to the category of coherent sheaves on $\mathcal{M}_{H,1}$.

2. If we consider an object $W \in \text{Rep}_R(G_1)$ as an object in $\text{Rep}_R(P_1)$ by restriction to $P_1$, then we have a canonical isomorphism $\mathcal{E}_{G_1, R}(W) \cong \mathcal{E}_{P_1, R}(W)$.

3. If we view an object $W \in \text{Rep}_R(M_1)$ as an object in $\text{Rep}_R(P_1)$ in the canonical way (under the canonical surjection $P_1 \to M_1$), then we have a canonical isomorphism $\mathcal{E}_{P_1, R}(W) \cong \mathcal{E}_{M_1, R}(W)$.

4. Suppose $W \in \text{Rep}_R(P_1)$ has a decreasing filtration by subobjects $F^n(W) \subset W$ in $\text{Rep}_R(P_1)$ such that each graded piece $\text{Gr}_F^n(W) := F^n(W)/F^{n+1}(W)$ can be identified with an object of $\text{Rep}_R(M_1)$. Then $\mathcal{E}_{P_1, R}(W)$ has a filtration $\mathcal{E}_{P_1, R}(F^n(W))$ with graded pieces $\mathcal{E}_{M_1, R}(\text{Gr}_F^n(W))$.  

2.3. Toroidal compactifications and canonical extensions. Under the assumption that \( \mathcal{H} \) is neat, by [26] Thm. 6.4.1.1 and 7.3.3.4, \( M_\mathcal{H} \) admits a toroidal compactification \( M_\mathcal{H}^{tor} = M_\mathcal{H}^{tor}_\Sigma \), a scheme projective and smooth over \( S_0 \), depending on a compatible collection \( \Sigma \) of cone decompositions that is projective and smooth in the sense of [26] Def. 6.3.3.4 and 7.3.1.3, with the following properties:

1. The universal abelian scheme \( A \to M_\mathcal{H} \) extends to a semi-abelian scheme \( A^{ext} \to M_\mathcal{H}^{tor} \), the polarization \( \lambda : A \to A^{\vee} \) extends to a prime-to-\( p \) isogeny \( \lambda^{ext} : A^{ext} \to (A^{ext})^{\vee} \) between semi-abelian schemes, and the endomorphism structure \( i : \mathcal{O} \hookrightarrow \text{End}_{M_\mathcal{H}}(A) \) extends to an endomorphism structure \( i^{ext} : \mathcal{O} \hookrightarrow \text{End}_{M_\mathcal{H}^{tor}}(A^{ext}) \). (These extensions are unique because the base is noetherian and normal. See [10] Ch. I, Prop. 2.7.)

2. The complement of \( M_\mathcal{H} \) in \( M_\mathcal{H}^{tor} \) (with its reduced structure) is a relative Cartier divisor \( D = D_{\infty, \mathcal{H}} \) with simple normal crossings. Here simpleness of the normal crossings uses [26] Cond. 6.2.5.25 and Lem. 6.2.5.27 (cf. [10] Ch. IV, Rem. 5.8(a)) and the assumption that \( \mathcal{H} \) is neat.

3. Let

\[
\text{KS}_{A^{ext}/M_\mathcal{H}^{tor}} := \text{KS}_{(A^{ext}, \lambda^{ext}, i^{ext})/M_\mathcal{H}^{tor}}
\]

be the quotient of \( \text{Lie}_{A^{ext}/M_\mathcal{H}^{tor}}^{\vee} \otimes \text{Lie}_{(A^{ext})^{\vee}/M_\mathcal{H}^{tor}}^{\vee} \) by the relations as in Definition 2.15. Let

\[
\Omega^1_{M_\mathcal{H}^{tor}/S_0} := \Omega^1_{M_\mathcal{H}^{tor}/S_0}(\log D) := \Omega^1_{M_\mathcal{H}^{tor}/S_0}[d \log D]
\]

be the sheaf of modules of log \( 1 \)-differentials on \( M_\mathcal{H}^{tor} \) over \( S_0 \), with respect to the relative Cartier divisor \( D \) with normal crossings. Then the Kodaira–Spencer morphism (2.14) extends to a morphism

\[
\text{KS}_{A^{ext}/M_\mathcal{H}^{tor}}/S_0 : \text{Lie}_{A^{ext}/M_\mathcal{H}^{tor}}^{\vee} \otimes \text{Lie}_{(A^{ext})^{\vee}/M_\mathcal{H}^{tor}}^{\vee}(1) \to \Omega^1_{M_\mathcal{H}^{tor}/S_0},
\]

called the extended Kodaira–Spencer morphism, which factors through the canonical quotient \( \text{Lie}_{A^{ext}/M_\mathcal{H}^{tor}}^{\vee} \otimes \text{Lie}_{(A^{ext})^{\vee}/M_\mathcal{H}^{tor}}^{\vee}(1) \to \text{KS}_{A^{ext}/M_\mathcal{H}^{tor}}(1) \) and induces a canonical isomorphism

\[
\text{KS}_{A^{ext}/M_\mathcal{H}^{tor}}/S_0 : \text{KS}_{A^{ext}/M_\mathcal{H}^{tor}}(1) \cong \Omega^1_{M_\mathcal{H}^{tor}/S_0}
\]

extending (2.17), called the extended Kodaira–Spencer isomorphism.

In what follows, we shall fix the choice of a (projective and smooth) \( \Sigma \), and suppress \( \Sigma \) from the notation.

Let \( M_{\mathcal{H},0}^{tor} \) denote the schematic closure of \( \text{Sh}_{\mathcal{H}} \) in \( M_{\mathcal{H}}^{tor} \), and let \( M_{\mathcal{H},1}^{tor} \) denote the pullback of \( M_{\mathcal{H},0}^{tor} \) under \( S_1 \to S_0 \). Then \( M_{\mathcal{H},1}^{tor} \) is smooth over \( S_1 \), and \( M_{\mathcal{H},1}^{tor} \to S_1 \) is proper smooth with properties analogous to those of \( M_{\mathcal{H}}^{tor} \to S_0 \). By abuse of notation, let us denote the pullback of \( D \) to \( M_{\mathcal{H},1}^{tor} \) by the same notation \( D \).

Proposition 2.27 (see [25] Prop. 6.9). The locally free sheaf \( H_1^{dR}(A/M_{\mathcal{H},1}^{tor}) \) extends to a locally free sheaf \( H_1^{dR}(A/M_{\mathcal{H},1})^{can} \) over \( M_{\mathcal{H},1}^{tor} \), which can be characterized by the following properties:

1. The sheaf \( H_1^{dR}(A/M_{\mathcal{H},1})^{can} \), canonically identified as a subsheaf of the quasi-coherent sheaf \( (M_{\mathcal{H},1} \to M_{\mathcal{H},1})_* (H_1^{dR}(A/M_{\mathcal{H},1})) \), is self-dual under
the pairing \((M_{H,1} \to M_{\text{tor}}{\text{ext}}_{H,1})_\lambda((\cdot, \cdot)_\lambda)\). We shall denote the induced pairing by \((\cdot, \cdot)_\lambda^{\text{can}}\).

(2) \(H^1_{\text{dr}}(A/M_{H,1})^{\text{can}}\) contains \(\text{Lie}_{A^{\text{ext}}/M_{\text{tor}}{\text{ext}}_{H,1}}^\vee\) as a subsheaf totally isotropic under \((\cdot, \cdot)_\lambda^{\text{can}}\).

(3) The quotient sheaf \(H^1_{\text{dr}}(A/M_{H,1})^{\text{can}}/\text{Lie}_{A^{\text{ext}}/M_{\text{tor}}{\text{ext}}_{H,1}}^\vee\) can be canonically identified with the subquotient \(\text{Lie}_{(A^{\text{ext}}/M_{\text{tor}}{\text{ext}}_{H,1})}^\vee\) of \((M_{H,1} \to M_{\text{tor}}{\text{ext}}_{H,1}), \text{Lie}_{A^{\text{ext}}/M_{H,1}}^\vee\).

(4) The pairing \((\cdot, \cdot)_\lambda^{\text{can}}\) induces an isomorphism \(\text{Lie}_{A^{\text{ext}}/M_{\text{tor}}{\text{ext}}_{H,1}}^\vee \rightarrow \text{Lie}_{(A^{\text{ext}}/M_{\text{tor}}{\text{ext}}_{H,1})}^\vee\) which coincides with \(d\lambda^{\text{ext}}\).

(5) Let \(H^1_{\text{dr}}(A/M_{H,1})^{\text{can}} := \text{Hom}_{M_{H,1}^{\text{can}}}(H^1_{\text{dr}}(A/M_{H,1})^{\text{can}}, \mathcal{O}_{M_{H,1}^{\text{can}}}).\) The Gauss–Manin connection \((2.12)\) extends to an integrable connection

\[
(2.28) \quad \nabla : H^1_{\text{dr}}(A/M_{H,1})^{\text{can}} \rightarrow H^1_{\text{dr}}(A/M_{H,1})^{\text{can}} \otimes \Omega^1_{H^{\text{can}}_{H,1}}/S_1
\]

with log poles along \(D\), called the extended Gauss–Manin connection, such that the composition

\[
\text{Lie}_{A^{\text{ext}}/M_{\text{tor}}{\text{ext}}_{H,1}}^\vee(1) \rightarrow H^1_{\text{dr}}(A/M_{H,1})^{\text{can}} \rightarrow H^1_{\text{dr}}(A/M_{H,1})^{\text{can}} \otimes \Omega^1_{M_{\text{tor}}{\text{ext}}_{H,1}}/S_1
\]

induces by duality the extended Kodaira–Spencer morphism \((2.25)\) (and hence the extended Kodaira–Spencer isomorphism \((2.26)\)).

Remark 2.29. Any construction achieving the properties in Proposition 2.27 will serve the same purpose in what follows. Therefore, one can refer to \([10\text{ Ch. VI}]\) and related works in special cases, without having to explain the consistency with \([25]\). (This is desirable because the methods in \([10\text{ Ch. VI}]\) and \([25]\) are different.)

As in \([23\text{ Sec. 6B}]\) and \([28\text{ Sec. 4.2}]\), by replacing

\[
(H^1_{\text{dr}}(A/M_{H,1}), (\cdot, \cdot)_\lambda, O_{M_{H,1}}(1), \text{Lie}_{A^{\text{ext}}/M_{H,1}}^\vee(1))
\]

(and its subtuples) with

\[
(H^1_{\text{dr}}(A/M_{H,1})^{\text{can}}, (\cdot, \cdot)_\lambda^{\text{can}}, O_{M_{H,1}^{\text{can}}}(1), \text{Lie}_{(A^{\text{ext}}/M_{\text{tor}}{\text{ext}}_{H,1})}^\vee(1))
\]

(and the corresponding subtuples) in the definitions \((2.18), (2.19),\) and \((2.20),\) the principal bundles \(E_{G_1}, E_{P_1},\) and \(E_{M_{H,1}}\) over \(M_{H,1}\) extend canonically to the principal bundles \(E_{G_1}^{\text{can}}, E_{P_1}^{\text{can}},\) and \(E_{M_{H,1}^{\text{can}}}\) over \(M_{\text{tor}}{\text{ext}}_{H,1}\), respectively.

Definition 2.30. Let \(R\) be any \(R_1\)-algebra. For any \(W \in \text{Rep}_{R}(G_1)\), we define

\[
(2.31) \quad E_{G_1,R}^{\text{can}}(W) := (E_{G_1}^{\text{can}} \otimes_R R_{G_1}) \otimes W,
\]

called the canonical extension of \(E_{G_1,R}(W)\), and define

\[
E_{G_1,R}^{\text{sub}}(W) := E_{G_1,R}^{\text{can}}(W) \otimes \mathcal{J}_D,
\]

called the subcanonical extension of \(E_{G_1,R}(W)\), where \(\mathcal{J}_D\) is the \(O_{M_{H,1}^{\text{can}}}\)-ideal defining the relative Cartier divisor \(D\). Also, we define similarly \(E_{P_1,R}^{\text{can}}(W)\),

\[
E_{P_1,R}^{\text{can}}(W) := (E_{P_1}^{\text{can}} \otimes_R R_{P_1}) \otimes W, \quad E_{P_1,R}^{\text{sub}}(W) := E_{P_1,R}^{\text{can}}(W) \otimes \mathcal{J}_D,
\]

called the canonical and subcanonical extensions of \(E_{P_1,R}(W)\), respectively.
Lemma 2.32 (cf. [28, Lem. 4.14]). Lemma 2.24 remains true if we replace the automorphic sheaves with their canonical or subcanonical extensions.

2.4. De Rham complexes. Let $R$ be any $R_1$-algebra. For simplicity, we shall denote pullbacks of objects from $R_1$ to $R$ by replacing the subscript “1” with “$R$”, although we shall use the same notation $D$ for its pullback.

First, let us explain how the Gauss–Manin connection (2.12) induces integrable connections on automorphic sheaves.

In Definition 2.11 the Gauss–Manin connection (2.12) is defined by the difference between the two isomorphisms $\text{Id}^*, s^* : \text{pr}_2^* \mathcal{H}^1_{\text{dR}}(A/M_H) \overset{\sim}{\to} \text{pr}_1^* \mathcal{H}^1_{\text{dR}}(A/M_H)$ lifting the identity morphism on $\mathcal{H}^1_{\text{dR}}(A/M_H)$. Since $s^*$ has a simple definition, we can interpret $\text{Id}^*$ (whose definition as in [26, Prop. 2.1.6.4] is far from simple) as induced by the Gauss–Manin connection (2.12) (and $s^*$).

By construction of $\mathcal{E}_{G_1, R}(-)$ (cf. (2.23)), for each $W \in \text{Rep}_R(G_1)$, the two isomorphisms above induce two isomorphisms $\text{Id}^*, s^* : \text{pr}_2^*(\mathcal{E}_{G_1, R}(W)) \overset{\sim}{\to} \text{pr}_1^*(\mathcal{E}_{G_1, R}(W))$ lifting the identity morphism on $\mathcal{E}_{G_1, R}(W)$. Hence, the difference $s^* - \text{Id}^*$ induces an integrable connection

\[\nabla : \mathcal{E}_{G_1, R}(W) \to \mathcal{E}_{G_1, R}(W) \otimes \Omega^1_{\text{M}_{N,R}/S}.
\]

Definition 2.34. The integrable connection $\nabla$ in (2.33) above is called the Gauss–Manin connection for $\mathcal{E}_{G_1, R}(W)$.

Next, let us explain how the extended Gauss–Manin connection (2.28) induces integrable connections on canonical and subcanonical extensions (extending the integrable connections induced by the Gauss–Manin connection (2.12)). Set

\[\Omega^{*,*}_{\text{M}_{N,R}^\text{can},1} := \Omega^{*,*}_{\text{M}_{N,R}^\text{can},1}(\log D) \cong \Omega^{1}_{\text{M}_{N,R}^\text{can},1}/S_1.[d\log D].\]

Let $\mathcal{F}^1_{\text{M}_{N,R}/S_1}$ be the subsheaf of $(\text{M}_H \hookrightarrow \text{M}_{M_H}^\text{can})_*(\text{M}_{N} \hookrightarrow \text{M}_{N}^\text{can}(\Omega^{1}_{\text{M}_{N,R}^\text{can},1}/S_1)$ corresponding to the subsheaf $\text{M}_{N/R}^\text{can} \oplus \Omega^{1}_{\text{M}_{N,R}^\text{can},1}/S_0$ of $(\text{M}_H \hookrightarrow \text{M}_{M_H}^\text{can})_*(\text{M}_{N} \hookrightarrow \text{M}_{N}^\text{can}(\Omega^{1}_{\text{M}_{N,R}^\text{can},1}/S_1)$ under the canonical splitting $\mathcal{F}^1_{\text{M}_{N/R}/S_1} \cong \Omega^{1}_{\text{M}_{N,R}^\text{can},1}/S_0 \oplus \Omega^{1}_{\text{M}_{N,R}^\text{can},1}/S_0$ with the summand $\text{M}_{N/R}^\text{can} \oplus \Omega^{1}_{\text{M}_{N,R}^\text{can},1}/S_1$ spanned by the image of $\text{pr}_1^* - \text{pr}_2^*$.

Let $\mathcal{F}^1_{\text{M}_{N,R}/S_0}$ be the subsheaf of $(\text{M}_H \hookrightarrow \text{M}_{M_H}^\text{can})_*(\text{M}_{N} \hookrightarrow \text{M}_{N}^\text{can}(\Omega^{1}_{\text{M}_{N,R}^\text{can},1}/S_0)$ corresponding to the subsheaf $\text{M}_{N/R}^\text{can} \oplus \Omega^{1}_{\text{M}_{N,R}^\text{can},1}/S_0$ of $(\text{M}_H \hookrightarrow \text{M}_{M_H}^\text{can})_*(\text{M}_{N} \hookrightarrow \text{M}_{N}^\text{can}(\Omega^{1}_{\text{M}_{N,R}^\text{can},1}/S_1)$ under the canonical splitting $\mathcal{F}^1_{\text{M}_{N/R}/S_0} \cong \Omega^{1}_{\text{M}_{N,R}^\text{can},1}/S_0 \oplus \Omega^{1}_{\text{M}_{N,R}^\text{can},1}/S_0$ with the summand $\text{M}_{N/R}^\text{can} \oplus \Omega^{1}_{\text{M}_{N,R}^\text{can},1}/S_1$ spanned by the image of $\text{pr}_1^* - \text{pr}_2^*$. Then the morphisms $\text{pr}_1^*, \text{pr}_2^*, \text{M}_R^\text{can} \otimes \Omega^{1}_{\text{M}_{N,R}^\text{can},1}/S_1$ induce respectively morphisms $\text{pr}_1^*, \text{pr}_2^* : \text{M}_R^\text{can} \to \mathcal{F}^1_{\text{M}_{N,R}/S_0}$ and $\mathcal{F}^1_{\text{M}_{N,R}/S_0} \cong \mathcal{F}^1_{\text{M}_{N,R}/S_0}$ such that $s^* \cdot \nabla$ induces the universal log derivation $d : \mathcal{F}^1_{\text{M}_{N,R}/S_0} \to \mathcal{F}^1_{\text{M}_{N,R}/S_0}$.

Since $\mathcal{F}^1_{\text{dR}}(\text{M}_{N,H})$ is only axiomatic, it is convenient that the above objects are uniquely determined by their pullbacks to $\text{M}_{N,H}$, and that we can define them as induced objects, without having to resort to their interpretations in log geometry. (Certainly, any reasonable theory should be compatible with such extensions.)

The property (4) in Proposition 2.27 states that the Gauss–Manin connection (2.12) induces the extended Gauss–Manin connection (2.28), which is equivalent to the statement that the extended Gauss–Manin connection (2.28) is defined by the difference between the two isomorphisms $\text{Id}^*, s^* : \text{pr}_2^*(\mathcal{F}^1_{\text{dR}}(\text{M}_{N,H})^\text{can}) \overset{\sim}{\to} \text{pr}_1^*(\mathcal{F}^1_{\text{dR}}(\text{M}_{N,H})^\text{can})$ lifting the identity morphism on $\mathcal{F}^1_{\text{dR}}(\text{M}_{N,H})^\text{can}$. (Again, we
can interpret $\text{Id}^*$ as induced by the extended Gauss–Manin connection (2.28) and $\pi^*$). Note that here $\text{pr}_1$ and $\text{pr}_2$ are morphisms with their targets tensored with $\mathcal{F}_{\text{M}_{\text{tor}}}/S_0$, but not $\mathcal{F}_{\text{M}_{\text{tor}}}/S_0$ (which can be identified with the structural sheaf of the first infinitesimal neighborhood of $\mathcal{M}_{\text{tor}}$ in $\mathcal{M}_{\text{tor}} \times \mathcal{M}_{\text{tor}}$). By construction of $\mathcal{E}_{\text{G}_1,\text{R}}^\text{can}(\cdot)$ (cf. (2.31)), for each $W \in \text{Rep}_R(G_1)$, the two isomorphisms above induce two isomorphisms $\text{Id}^*, \pi^* : \text{pr}_2^*(\mathcal{E}_{\text{G}_1,\text{R}}^\text{can}(W)) \sim \text{pr}_1^*(\mathcal{E}_{\text{G}_1,\text{R}}^\text{can}(W))$ lifting the identity morphism on $\mathcal{E}_{\text{G}_1,\text{R}}^\text{can}(W)$. Hence, the difference $\pi^* - \text{Id}^*$ induces a morphism

$$
\nabla : \mathcal{E}_{\text{G}_1,\text{R}}^\text{can}(W) \to \mathcal{E}_{\text{G}_1,\text{R}}^\text{can}(W) \otimes \mathcal{F}_{\text{M}_{\text{tor}}}/S_R
$$

of sheaves of $R$-modules. Since the connection $\nabla$ in (2.35) is induced by the connection $\nabla$ in (2.33), the conditions for being an integrable connection with log poles are tautologically verified. By applying $\otimes \mathcal{I}_D$, we obtain an integrable connection

$$
\nabla : \mathcal{E}_{\text{G}_1,\text{R}}^\text{can}(W) \to \mathcal{E}_{\text{G}_1,\text{R}}^\text{can}(W) \otimes \mathcal{F}_{\text{M}_{\text{tor}}}/S_R
$$

with log poles.

**Definition 2.37.** The integrable connection $\nabla$ (with log poles) in (2.33) (resp. (2.36)) is called the **extended Gauss–Manin connection** for $\mathcal{E}_{\text{G}_1,\text{R}}^\text{can}(W)$ (resp. $\mathcal{E}_{\text{G}_1,\text{R}}^\text{can}(W)$).

**Definition 2.38.** The connections (2.33), (2.35), and (2.36) define respectively the de Rham complex

$$(\mathcal{E}_{\text{G}_1,\text{R}}(W) \otimes \Omega^*_\mathcal{M}_{\text{tor}}/S_R, \nabla)$$

and the log de Rham complexes

$$(\mathcal{E}_{\text{G}_1,\text{R}}^\text{can}(W) \otimes \mathcal{F}_{\text{M}_{\text{tor}}}/S_R, \nabla)$$

and

$$(\mathcal{E}_{\text{G}_1,\text{R}}^\text{can}(W) \otimes \mathcal{F}_{\text{M}_{\text{tor}}}/S_R, \nabla).$$

### 3. Representation theory

**3.1. Decomposition of reductive groups.** By [26 Prop. 1.1.1.21], and by the decomposition of the center $\mathcal{O}_{F,1} = \mathcal{O}_F \otimes R_1$ given in (2.6), $\mathcal{O}_1 = \mathcal{O} \otimes R_1$ is canonically isomorphic to a direct product $\prod_{\tau \in \mathcal{Y}} \mathcal{O}_\tau$, where for each $\tau \in \mathcal{Y} = \text{Hom}_{\mathbb{Z}-\text{alg}}(\mathcal{O}_F, R_1)$ we have $\mathcal{O}_\tau \cong \mathcal{M}_{t_\tau}(\mathcal{O}_{F,\tau})$ for some $t_\tau$, whose center is $\mathcal{O}_{F,\tau} = R_1$, on which $\mathcal{O}_F$ acts via the homomorphism $\tau : \mathcal{O}_F \to R_1$.

By [26 Lem. 1.1.3.4], for each $\tau \in \mathcal{Y}$, there is a unique (up to isomorphism) indecomposable projective $\mathcal{O}_\tau$-module, which we shall denote by $V_\tau$. Concretely, since $\mathcal{O}_\tau \cong \mathcal{M}_{t_\tau}(\mathcal{O}_{F,\tau})$, we can take $V_\tau$ to be $\mathcal{O}_{F,\tau}^{\oplus t_\tau}$, in which case $\text{End}_{\mathcal{O}_\tau}(V_\tau) \cong \mathcal{O}_{F,\tau} \cong R_1$. Moreover, every finitely generated projective $\mathcal{O}_\tau$-module decomposes (up to isomorphism) into a direct sum $\bigoplus_{\tau \in \mathcal{Y}} V_\tau^{\oplus m_\tau}$ for some integers $m_\tau$. We call the tuple $(m_\tau)_{\tau \in \mathcal{Y}}$ of integers the **multi-rank** of such an $\mathcal{O} \otimes R_1$-module. (See [26]...
Then \( q_{\tau} = p_{\tau_{oc}} \), where \( c : \mathcal{O}_F \to \mathcal{O} \) is the restriction of \( * : \mathcal{O} \to \mathcal{O} \). The multi-rank of \( L_1 \) is \((p_\tau + q_\tau)_{\tau \in \mathcal{Y}}\), because we have the isomorphism \([2.7] \) over \( R_1 \).

Fix any choice of an isomorphism \( L_{0,1} \cong \bigoplus_{\tau \in \mathcal{Y}} V_{\tau}^{\oplus p_\tau} \), and fix any choices of the (non-canonical) isomorphisms \( V_{\tau_{oc}}(1) := \text{Hom}_{R_1}(V_{\tau_{oc}}, R_1(1)) \cong V_{\tau} \) (for \( \tau \in \mathcal{Y} \)). Then these choices induce canonically an isomorphism

\[
L_1 \cong \left( \bigoplus_{\tau \in \mathcal{Y}} V_{\tau}^{\oplus p_\tau} \right) \oplus \left( \bigoplus_{\tau \in \mathcal{Y}} (V_{\tau_{oc}}(1))^{\oplus q_\tau} \right) \cong \bigoplus_{\tau \in \mathcal{Y}} V_{\tau}^{\oplus (p_\tau + q_\tau)}
\]

by \([2.7] \), matching the pairing \( \langle \cdot, \cdot \rangle \) with the pairing

\[
(\langle x_1, \tau \rangle, f_{1, \tau_{oc}})(\langle x_2, \tau \rangle, f_{2, \tau_{oc}})) \mapsto \sum_{\tau \in \mathcal{Y}} (f_{2, \tau}(x_1, \tau) - f_{1, \tau}(x_2, \tau)).
\]

**Lemma 3.3** (see \([27] \) Lem. 2.4]). There exists a cocharacter \( G_m \otimes R_1 \to G_1 \) splitting the similitude character \( v : G_1 \to G_m \otimes R_1 \), which acts trivially on \( L_{0,1}^{\vee} \) (under the identification \([2.7] \)).

For each \( \tau \in \mathcal{Y} \), set \( L_\tau := V_{\tau}^{\oplus p_\tau} \oplus (V_{\tau_{oc}}(1))^{\oplus q_\tau} \), and define the canonical pairing

\[
\langle \cdot, \cdot \rangle_\tau : L_\tau \times L_{\tau_{oc}} \to R_1(1)
\]

by

\[
((x_1, \tau), f_{1, \tau_{oc}}), (x_2, \tau_{oc}, f_{2, \tau}) \mapsto f_{2, \tau}(x_1, \tau) - f_{1, \tau_{oc}}(x_2, \tau_{oc}).
\]

(The two factors \( L_\tau \) and \( L_{\tau_{oc}} \) of the domain of \( \langle \cdot, \cdot \rangle_\tau \) are not the same when \( \tau \neq \tau \circ c \).) Then we see that the pairing \((3.2) \) is simply the sum of \( \langle \cdot, \cdot \rangle_\tau \) over \( \tau \in \mathcal{Y} \). Note that \( \text{Aut}_{\mathcal{C}}(1) \) for every \( R_1 \)-algebra \( R \). If we define for each \( R_1 \)-algebra \( R \)

\[
G_\tau(R) := \left\{ g \in \text{Aut}_{\mathcal{O}}(L_\tau \otimes R) \mid (gx, gy)_\tau = (x, y)_\tau, \forall x \in L_\tau \otimes R, \forall y \in L_{\tau_{oc}} \otimes R \right\}
\]

then we obtain a group functor \( G_\tau \) over \( R_1 \) which falls into three possibilities (by the same explicit classification as in the proof of \([26] \) Prop. 1.2.3.11] again, as in Remark \([2.9] \).

1. \( G_\tau \cong \text{Sp}_{2r_\tau} \otimes R_1 \), where \( r_\tau = p_\tau = q_\tau \) and \( \text{Sp}_{2r_\tau} \) is the (split) symplectic group of rank \( r_\tau \) over \( \mathbb{Z} \).
2. \( G_\tau \cong \text{O}_{2r_\tau} \otimes R_1 \), where \( r_\tau = p_\tau = q_\tau \) and \( \text{O}_{2r_\tau} \) is the (split) even orthogonal group of rank \( r_\tau \) over \( \mathbb{Z} \).
3. \( G_\tau \cong \text{GL}_{r_\tau} \otimes R_1 \), where \( r_\tau = p_\tau + q_\tau \) and \( \text{GL}_{r_\tau} \) is the general linear group of rank \( r_\tau \) over \( \mathbb{Z} \).

Since \( \langle \cdot, \cdot \rangle_\tau = -\langle \cdot, \cdot \rangle_{\tau_{oc}} \) as pairings between \( L_\tau \) and \( L_{\tau_{oc}} \), the two group functors \( G_\tau \) and \( G_{\tau_{oc}} \) are canonically isomorphic.

Thus, we obtain a decomposition

\[
G_1 \cong \left( \prod_{\tau \in \mathcal{Y} / c} G_\tau \right) \times (G_m \otimes R_1)
\]
over $\mathcal{S}_1 = \text{Spec}(R_1)$, where $\tau \in \mathcal{Y}/c$ means (by abuse of language) we pick exactly one representative $\tau$ in its $c$-orbit in $\mathcal{Y}$, and where the last factor $G_m \otimes R_1$ is given by the cocharacter given by Lemma 3.3 splitting the similitude character.

3.2. Decomposition of parabolic subgroups. Under the identification (2.7), the submodule $L^\mathcal{Y}_{0,1}(1)$ of $L_1$ is matched with the submodule $0 \oplus \bigoplus_{\tau \in \mathcal{Y}} (V^\mathcal{Y}_{\text{roc}}(1))^{\oplus q_\tau}$ of the second member in (3.1). For each $\tau \in \mathcal{Y}$, define group functors $P_\tau$ and $M_\tau$ over $R_1$ by setting for each $R_1$-algebra $R$

\[ P_\tau(R) := \left\{ g \in G_\tau(R) : g\left(0 \oplus (V^\mathcal{Y}_{\text{roc}}(1))^{\oplus q_\tau} \otimes R) = (0 \oplus (V^\mathcal{Y}_{\text{roc}}(1))^{\oplus q_\tau} \otimes R) \right\} \]

and

\[ M_\tau(R) := \left\{ g \in P_\tau(R) : g((V^\mathcal{Y}_{\text{roc}}(1))^{\oplus q_\tau} \otimes R) \oplus 0 \right\} \]

Then the subgroup $P_1$ of $G_1$ can be identified with the subgroup

\[ \left( \prod_{\tau \in \mathcal{Y}/c} P_\tau \right) \times (G_m \otimes R_1) \subset \left( \prod_{\tau \in \mathcal{Y}/c} G_\tau \right) \times (G_m \otimes R_1), \]

and the canonical surjection $P_1 \to M_1$ has a splitting $M_1 \subset P_1$ given by

\[ \left( \prod_{\tau \in \mathcal{Y}/c} M_\tau \right) \times (G_m \otimes R_1) \subset \left( \prod_{\tau \in \mathcal{Y}/c} P_\tau \right) \times (G_m \otimes R_1). \]

For each $\tau \in \mathcal{Y}$, since $\text{End}_{G_\tau}(V_{\tau}) \cong \text{End}_{G_\tau}(V^\mathcal{Y}_{\text{roc}}(1)) \cong \mathcal{O}_{F,\tau} \cong R_1$, we have a diagonal action of $(G^p_m \times G^p_m)(R)$ on $(V^\mathcal{Y}_{\text{roc}}(1))^{\oplus q_\tau} \otimes R$, which is functorial in $R$ and hence defines a homomorphism $(G^p_m \times G^p_m) \otimes R_1 \to M_\tau$.

3.3. Hodge filtrations. Let $R$ be any $R_1$-algebra. Fix any choice of a cocharacter as in Lemma 3.3 and consider its reciprocal $H : G_m \otimes R_1 \to G_1$. (By definition, $H$ factors through $P_1$.)

**Definition 3.7.** Given any object $W \in \text{Rep}_R(P_1)$, the induced action of $G_m \otimes R_1$ decomposes $W$ into weight spaces $W^{(a)}$ for $G_m \otimes R_1$, indexed by integers. Then the Hodge filtration $F$ on $W$ is the decreasing filtration $F(W) = \{ F^a(W) \}_{a \in \mathbb{Z}}$ defined by $F^a(W) := \bigoplus_{b \geq a} W^{(b)}$. (Note that the choice of $H$ is not unique in general, but the resulting filtration is independent of this choice.)

**Example 3.8.** Since the cocharacter $H$ acts with weight 0 on $L^\mathcal{Y}_{0,1}(1)$ (as a submodule of $L_1$) and with weight $-1$ on $L_{0,1}$ (as a quotient module of $L_1$), the Hodge filtration $F$ on $L_1$ is given by $F^{-1}(L_1) = L_1$, $F^0(L_1) = L^\mathcal{Y}_{0,1}(1)$, and $F^1(L_1) = \{0\}$. Then the only possibly nonzero graded pieces are $G^p_{F^{-1}}(L_1) = L_{0,1}$ and $G^p_{F^0}(L_1) = L^\mathcal{Y}_{0,1}(1)$.

**Lemma 3.9** (see [27, Lem. 2.11]). Let $W \in \text{Rep}_R(P_1)$ and let $\{ F^a(W) \}_{a \in \mathbb{Z}}$ be its Hodge filtration defined in Definition 3.7. Then the unipotent radical $U_1$ of $P_1$ acts trivially on $G^p_F(W)$ for every $a \in \mathbb{Z}$. In other words, each graded piece $G^p_F(W)$ can be identified as an object in $\text{Rep}_R(M_1)$. 
By Lemmas 2.24 and 2.32, the Hodge filtration on $W$ defines similar filtrations on $E_{P_1}(W)$, $E_{P_1,R}(W)$, and $E_{\text{sub}}(W)$, which we shall denote by $F^a(E_{P_1}(W))$, $F^a(E_{P_1,R}(W))$, and $F^a(E_{\text{sub}}(W))$, for $a \in \mathbb{Z}$. We have the canonical isomorphisms $\text{Gr}_p^a(E_{P_1,R}(W)) \cong \text{Gr}_p^a(E_{\text{sub}}(W))$, $\text{Gr}_p^a(E_{P_1}(W)) \cong \text{Gr}_p^a(E_{\text{sub}}(W))$, and $\text{Gr}_p^a(E_{P_1}(W)) \cong \text{Gr}_p^a(E_{\text{sub}}(W))$ between the graded pieces.

**Definition 3.10.** The filtration $F(E_{P_1,R}(W)) = \{F^a(E_{P_1,R}(W))\}_{a \in \mathbb{Z}}$, $F(E_{P_1,R}(W)) = \{F^a(E_{P_1,R}(W))\}_{a \in \mathbb{Z}}$, and $F(E_{\text{sub}}(W)) = \{F^a(E_{\text{sub}}(W))\}_{a \in \mathbb{Z}}$ are called the Hodge filtration on $E_{P_1,R}(W)$, $E_{P_1,R}(W)$, and $E_{\text{sub}}(W)$, respectively.

**Definition 3.11.** Let $W \in \text{Rep}_R(G_1)$. By considering $W$ as an object of $\text{Rep}_R(P_1)$ by restriction from $G_1$ to $P_1$, we can define the Hodge filtration on $E_{G_1,R}(W) \cong E_{P_1,R}(W)$ (resp. $E_{G_1,R}(W) \cong E_{P_1,R}(W)$, resp. $E_{\text{sub}}(W) \cong E_{\text{sub}}(W)$) (see Lemmas 2.24 and 2.32 as in Definition 3.10). The Hodge filtration on the de Rham complex $E_{G_1,R}(W)$ is defined by

$$F^a(E_{G_1,R}(W)) \otimes \Omega^*_{M_{N,R}/S_R}$$

The Hodge filtrations on the log de Rham complexes $E_{G_1,R}^\text{can}(W)$ and $E_{G_1,R}^\text{sub}(W)$ are defined similarly.

These are respectively subcomplexes of the full de Rham complexes for the Gauss–Manin connections, thanks to the Griffiths transversality. We shall postpone the explanation for the Griffiths transversality to the end of Section 4.3 (This is not ideal for the exposition, but we will not need Griffiths transversality before then.)

**Lemma 3.12** (see [28 Lem. 4.21]). Suppose $W_1$ and $W_2$ are two objects in $\text{Rep}_R(G_1)$ such that the induced actions of $P_1$ and $\text{Lie}(G_1)$ on them satisfy $W_1|_{P_1} \cong W_2|_{P_1}$ and $W_1|_{\text{Lie}(G_1)} \cong W_2|_{\text{Lie}(G_1)}$. Then we have a canonical isomorphism

$$E_{G_1,R}(W_1) \otimes \Omega^*_{M_{N,R}/S_R} \cong E_{G_1,R}(W_2) \otimes \Omega^*_{M_{N,R}/S_R} \cong E_{G_1,R}(W_2) \otimes \Omega^*_{M_{N,R}/S_R}$$

respecting the Hodge filtrations on both sides. (Consequently, the same is true with $E_{G_1,R}(\cdot)$ replaced with $E_{G_1,R}(\cdot)$ and $E_{G_1,R}(\cdot)$.)

**Remark 3.14.** Lemma 3.12 will be needed only when $G_1$ is not connected, i.e., when $O \otimes \mathbb{Q}$ involves simple factors of type D (as in [26 Def. 1.2.1.15]). (This happens exactly when $G_\tau \cong O_{2r_\tau} \otimes R_1$ for some $\tau \in \mathcal{T}$.)

### 3.4. Roots and weights

We shall choose a maximal torus $T_\tau$ of $M_\tau$ by choosing a subgroup of $(G_m^\tau \times G_m^\tau) \otimes \mathbb{Z} \otimes R_1$ that embeds into $M_\tau$ under the natural homomorphism $(G_m^\tau \times G_m^\tau) \otimes \mathbb{Z} \otimes R_1 \rightarrow M_\tau$ defined at the end of Section 3.2. There are two cases:

1. If $\tau = \tau_{oc}$, then $p_\tau = q_\tau$ and we take $T_\tau = \{t_\tau = (t_{\tau,i})_{1 \leq i \leq r_\tau}\}$, embedded in $(G_m^\tau \times G_m^\tau) \otimes \mathbb{Z} \otimes R_1$ by $t_\tau \mapsto (t_\tau^{-1}, t_\tau).$
(2) If \( \tau \neq \tau \circ c \), then we take \( T_\tau = \{ t_\tau = (t_{\tau,i})_{1 \leq i \leq r} \} \), identified with 
\((\mathbb{G}_m^r \times \mathbb{G}_m^s) \otimes R_1\) by \((t_{\tau,i})_{1 \leq i \leq \tau} \mapsto ((t_{\tau,q,i}^{-1})_{1 \leq i \leq \rho}, (t_{\tau,i})_{1 \leq i \leq \tau})\).

We take \( T_1 \subset M_1 \) to be the subgroup corresponding to

\[
(3.15) \quad \left( \prod_{\tau \in \mathcal{T}/c} T_\tau \right) \times (\mathbb{G}_m \otimes R_1) \subset \left( \prod_{\tau \in \mathcal{T}/c} M_\tau \right) \times (\mathbb{G}_m \otimes R_1)
\]

(where the products are over \( S_0 \)). Then the split torus \( T_1 \) is a maximal torus in both \( M_1 \) and \( G_1 \) (by comparing the ranks).

Elements in \( T_1 \) can be written as \( t = ((t_{\tau,i})_{\tau \in \mathcal{T}/c}; t_{0}) = (((t_{\tau,i})_{1 \leq i \leq \tau})_{\tau \in \mathcal{T}/c}; t_{0}), \) and therefore elements in the character group \( X := \text{Hom}_R(T_1, \mathbb{G}_m \otimes R_1) \) of \( T_1 \) are of the form \( \mu = ((\mu_\tau)_{\tau \in \mathcal{T}/c}; \mu_0) = (((\mu_{\tau,i})_{1 \leq i \leq \tau})_{\tau \in \mathcal{T}/c}; \mu_0) \), given concretely by

\[
t \mapsto \left( \prod_{\tau \in \mathcal{T}/c} \mu_\tau(t_\tau) \right) \mu_0(t_0) = \left( \prod_{\tau \in \mathcal{T}/c} \prod_{1 \leq i \leq \tau} t_{\tau,i}^{\mu_\tau,i} \right) \mu_0.
\]

Let \( X^\vee := \text{Hom}_R(\mathbb{G}_m \otimes R_1, T_1) \) be the cocharacter group of \( T_1 \), and let \((\cdot, \cdot) : X \times X^\vee \to \mathbb{Z}\) be the canonical pairing between \( X \) and \( X^\vee \) defined by sending \((\mu, \nu^\vee) \in X \times X^\vee \) to \( \mu \circ \nu^\vee \in \text{End}_R(\mathbb{G}_m \otimes R_1) \cong \mathbb{Z} \). Let \( \Phi_{G_1} \subset X \) (resp. \( \Phi_{G_1}^+ \subset X^\vee \)) be the roots (resp. coroots) of the split reductive group scheme \( G_1 \) over \( R_1 \). The choice of \( \Phi_{G_1}^+ \) corresponds to the choice of a Borel subgroup \( B_1 \) in \( G_1 \). Using the explicit identifications in (3.4), (3.5), (3.6), and (3.15), we can choose \( B_1 \) to contain the unipotent radical \( U_1 \) of \( P_1 \), and accordingly the positive roots \( \Phi_{G_1}^+ \) in \( \Phi_{G_1} \), such that the set \( X_{G_1}^+ \) of dominant weights of \( G_1 \) consists of those \( \mu \in X \) as above with \( \mu_{\tau,i} \geq \mu_{\tau,i+1} \) for every \( \tau \in \mathcal{T}/c \) and for every \( 1 \leq i < r, \) satisfying in addition:

1. If \( G_\tau \cong \text{Sp}_{2r}, \) then \( \mu_{\tau,r} \geq 0. \)
2. If \( G_\tau \cong \text{O}_{2r}, \) then \( \mu_{\tau,r-1} \geq \mu_{\tau,r}. \)

(If \( G_\tau \cong \text{GL}_{2r}, \) then there is no other condition on \( \mu_{\tau,r} \).)

**Remark 3.16.** When \( G_1 \) is not connected (i.e., \( G_\tau \cong \text{O}_{2r}, \) \( \otimes R_1 \) for some \( \tau \in \mathcal{T} \)), it is isomorphic to a semi-direct product \( G_1^0 \times \Gamma \), where \( G_1^0 \) is the identity component of \( G_1 \), and where \( \Gamma \) is an elementary abelian 2-group normalizing \( B_1 \) and \( T_1 \); and the irreducible representations \( V \) of \( G_1 \) over \( \mathbb{R} \otimes \mathbb{Q} \) are parameterized not exactly by a single dominant weight \( \mu \in X_{G_1}^+ \), but instead by the \( \Gamma \)-orbit \([\mu]\) of \( \mu \) in \( X_{G_1}^+ \) plus the action on \( V \) of the stabilizer of \( \mu \) in \( \Gamma \). Suppose any \( V \) as above has \( \mu \in X_{G_1}^+ \) as a highest weight. Then \( V_{|G_1|} \cong \bigoplus_{\mu' \in [\mu]} V_{\mu'} \), where each \( V_{\mu'} \) is an irreducible representation of \( G_1^0 \) over \( \mathbb{R} \otimes \mathbb{Q} \) of highest weight \( \mu' \), and where \([\mu]\) is the \( \Gamma \)-orbit of \( \mu \) in \( X_{G_1}^+ \). (Since there is a central isogeny \( \left( \prod_{\tau \in \mathcal{T}/c} G_\tau \right) \times (\mathbb{G}_m \otimes R_1) \to G_1 \), it suffices to verify the analogous statements for the factors \( G_\tau \), or rather just those such that \( G_\tau \cong \text{O}_{2r}, \otimes R_1 \), by following the same argument as in [14, Sec. 5.5.5].)

By Lemma 3.12, two representations of \( G_1 \) will serve the same purpose for us if their restrictions to \( G_1^0 \) are isomorphic. Hence, we shall abusively denote by \( V_{[\mu]} \) any irreducible representation \( V \) (as above) having \( \mu \) as a highest weight.
Let $\Phi_{M_1}$ be the roots of the split reductive group scheme $M_1$ over $R_1$. Then intersection of the above-chosen $B_i$ with $M_1$ (realized as a subgroup in $P_1$ as above) determines the choice of positive roots $\Phi_{M_1}^+$ in $\Phi_{M_1}$, so that $\Phi_{M_1}^+ = \Phi_{M_1} \cap \Phi_{G_1}^+$. Then the set $X_{M_1}^+$ of dominant weights of $M_1$ consists of those $\mu \in X$ as above with $\mu_{\tau, i} \geq \mu_{\tau, i+1}$ for every $\tau \in Y / \tau$ and for every $1 \leq i < q_{\tau}$ or $q_{\tau} < i < r_{\tau}$. (When $G_{\tau} \cong Sp_{2r_\tau} \otimes R_1$ or $G_{\tau} \cong O_{2r_\tau} \otimes R_1$, this means we drop the conditions 1 and 2 above. When $G_{\tau} \cong GL_{r_\tau} \otimes R_1$, this means we drop the condition $\mu_{\tau, q_{\tau}} \geq \mu_{\tau, q_{\tau}+1}$.)

It is conventional to say that a root $\alpha \in \Phi_{G_1}$ is compact if it is an element of $\Phi_{M_1}$, and that $\alpha$ is non-compact if otherwise. We denote the non-compact roots of $\Phi_{G_1}$ by $\Phi_{M_1}^+$, and denote the collection of positive non-compact roots by $\Phi_{M_1, \mu}$. For negative roots, we replace $+$ with $-$ in the above notation.

Let $W_{G_1}$ (resp. $W_{M_1}$) be the Weyl group of $G_1$ (resp. $M_1$). The realization of $M_1$ as a subgroup of $G_1$ containing $T_1$ identifies $W_{M_1}$ as a subgroup of $W_{G_1}$. We define

$$W_{M_1} := \{w \in W_{G_1} : w(X_{G_1}^+) \subset X_{M_1}^+\}.$$ 

Then any element in $W_{G_1}$ has a unique expression as $w = w_1 w_2$ with $w_1 \in W_{M_1}$ and $w_2 \in W_{M_1}$. (The elements of $W_{M_1}$ are the minimal length representatives of $W_{M_1} \setminus W_{G_1}$.)

For any root $\alpha \in \Phi_{G_1}$, we shall denote by $\alpha^\vee \in \Phi_{G_1}^+$ the associated coroot. Let $\rho := \frac{1}{2} \sum_{\alpha \in \Phi_{G_1}^+} \alpha$ be the half-sum of positive roots in $\Phi_{G_1}$. The dot action of $W_{G_1}$ (and its subset $W_{M_1}$) is defined by setting $w \cdot \mu := w(\mu + \rho) - \rho$ for each $w \in W_{G_1}$.

### 3.5. p-small weights and Weyl modules.

**Definition 3.17.** Let $\mu \in X$. We say $\mu$ is $p$-small for $G_1$ (resp. for $M_1$) if $(\mu + \rho, \alpha^\vee) \leq p$ for every $\alpha \in \Phi_{G_1}$ (resp. $\alpha \in \Phi_{M_1}$). We denote by $X_{G_1}^{<p}$ (resp. $X_{M_1}^{<p}$) the subset of $X$ consisting of $\mu \in X$ that are $p$-small for $G_1$ (resp. $M_1$), and we set $X_{G_1}^{+<p} := X_{G_1}^+ \cap X_{G_1}^{<p}$ (resp. $X_{M_1}^{+<p} := X_{M_1}^+ \cap X_{M_1}^{<p}$).

**Remark 3.18.** Note that $X_{G_1}^{<p}$ is stable under the dot action of $W_{G_1}$, and that $w \cdot \mu$ belongs to $X_{M_1}^{+<p}$ for any $w \in W_{M_1}$ and $\mu \in X_{G_1}^{+<p}$.

**Remark 3.19 (cf. [33, 1.9]).** Since $\rho_{M_1} := \frac{1}{2} \sum_{\alpha \in \Phi_{M_1}^+} \alpha$ satisfies $(\rho, \alpha^\vee) = (\rho_{M_1}, \alpha^\vee)$ for every $\alpha \in \Phi_{M_1}$, the given definition of $p$-smallness for the Levi subgroup $M_1$ is the same as the one when $M_1$ is regarded as a reductive group on its own.

Since $G_1$ (resp. $M_1$) is split reductive over $R_1$, there exists a split reductive group scheme $G_{split}$ (resp. $M_{split}$) over $\mathbb{Z}(p)$ such that $G_1 \cong G_{split, R_1}$ (resp. $M_1 \cong M_{split, R_1}$). Note that $G_{split}$ (resp. $M_{split}$) has the same roots and weights as $G_1$ (resp. $M_1$), and is a semi-direct product of $G_m$ with the split symplectic, (even) orthogonal, and general linear groups over $\mathbb{Z}(p)$. For $\mu \in X_{G_1}^+$ (resp. $\mu \in X_{M_1}^+$), let $V_{[\mu], Q}$ (resp. $W_{\mu, Q}$) be any irreducible $Q$-representation of $G_{split}$ (resp. $M_{split}$) having $\mu$ as a highest weight (see Remark 3.16). As in [33, 1.5], a $\mathbb{Z}(p)$-lattice in a $Q$-representation of a group scheme over $\mathbb{Z}(p)$ is called admissible if it is stable under the group scheme action. Let $V_{[\mu], Z(p)} \subset V_{[\mu], Q}$ (resp. $W_{\mu, Z(p)} \subset W_{\mu, Q}$) be the span of a highest weight vector under the action of the group scheme over $\mathbb{Z}(p)$, which is (by construction) minimal among admissible lattices in $V_{[\mu], Q}$ (resp. $W_{\mu, Q}$).
that contain the same highest weight vector. If we denote by $G_{\text{split}}^\circ$ the identity component of $G_{\text{split}}$, then $G_{\text{split}, R_1}^\circ \cong G_1^\circ$ and $V_{[\mu], Z(p)}|_{G_{\text{split}}^\circ} \cong \bigoplus_{\mu' \in [\mu]} V_{\mu', Z(p)}$ (see Remark 3.16), where each $V_{\mu', Z(p)}$ is the span of some highest weight vector under the action of $G_{\text{split}}^\circ$ in an irreducible $\mathbb{Q}$-representation of highest $\mu'$. (Then $V_{\mu', Z(p)}$ and $W_{\mu, Z(p)}$ are Weyl modules of $G_{\text{split}}^\circ$ and $M_{\text{split}}$, respectively; cf. [23, 1.3].)

According to [23, Cor. 1.9] (cf. [23, Cor. 5]), if $\mu \in X_{G_1}^{+,-p}$ (resp. $\mu \in X_{M_1}^{+,-p}$), then all admissible $\mathbb{Z}(p)$-lattices in $V_{[\mu], Q}$ (resp. $W_{\mu, Q}$), including ones constructed by plethysm as in [11] or [14], are isomorphic to $V_{[\mu], Z(p)}$ (resp. $W_{\mu, Z(p)}$). Then we set $V_{[\mu]} := V_{[\mu], Z(p)} \otimes R_1$ (resp. $W_{\mu} := W_{\mu, Z(p)} \otimes R_1$), and set $V_{[\mu], R} := V_{[\mu]} \otimes R$ (resp. $W_{\mu, R} := W_{\mu} \otimes R$) for each $R_1$-algebra $R$.

4. Differential operators

4.1. Verma modules. Let $U_1$ be the unipotent radical of the parabolic subgroup $P_1$ of $G_1$. Then $u_1 := \text{Lie}(U_1)$ is the unipotent radical of the parabolic subalgebra $p_1 := \text{Lie}(P_1)$ of $g_1 := \text{Lie}(G_1)$. Let $p_1^-$ be the parabolic subalgebra of $g_1$ opposite to $p_1$, and let $u_1^-$ be the unipotent radical of $p_1^-$. Our convention in Section 3.4 is that the weights of $u_1$ are in $\Phi^{M_1,+}$, and so that the weights of $u_1^-$ are in $\Phi^{M_1,-}$. Let $U(g_1)$ (resp. $U(p_1)$, resp. $U(u_1^-)$) denote the universal enveloping algebra of $g_1$ (resp. $p_1$, resp. $u_1^-$.). As always, for each $R_1$-algebra $R$, we denote the pullbacks of objects from $R_1$ to $R$ by replacing the subscript “1” with “R”.

Now let us fix the choice of an $R_1$-algebra $R$. We view $g_R$, $p_R$, and $u_R$ as objects in $\text{Rep}_R(P_1)$ canonically, and we view $u_R$ as an object in $\text{Rep}_R(P_1)$ by $u_R \cong g_R/p_R$.

We also view $u_R$ and $u_R$ as objects in $\text{Rep}_R(M_1)$ because $U_1$ acts trivially on them.

**Definition 4.1.** By a $U(g_R)$-$P_1$-module, we mean a module with actions of $U(g_R)$ and $P_1$ that induce the same action of $p_R$. By a morphism between $U(g_R)$-$P_1$-modules, we mean a morphism of $U(g_R)$-modules that induces a morphism of $U(p_R)$-modules coming from an algebraic morphism between $P_1$-modules. We shall use the notation $\text{Hom}_{U(g_R)}(\cdot, \cdot)$ to mean the group of morphisms between $U(g_R)$-$P_1$-modules.

**Lemma 4.2.** Let $W \in \text{Rep}_R(P_1)$. Then the module

$$(4.3) \quad \text{Verm}(W) := U(g_R) \otimes_{U(p_R)} W$$

with canonical action of $U(g_R)$ on the first component, and with canonical diagonal action of $P_1$ on both components, is a $U(g_R)$-$P_1$-module.

**Proof.** We need to show that the two induced actions of $p_R$ agree. Let “ad” denote the adjoint action of $p_R$ on $g_R$, induced by the canonical adjoint action of $P_1$ on $g_1$. Then the lemma follows from the identity

$$(pu) \otimes v = (pu - up) \otimes v + (up) \otimes v = (\text{ad}(p)(u)) \otimes v + u \otimes (pv),$$

for all $p \in p_R$, $u \in U(g_R)$, and $v \in W$. \qed

**Definition 4.4.** Let $W \in \text{Rep}_R(P_1)$. We define the (generalized) Verma module for $W$ to be the $U(g_R)$-$P_1$-module $\text{Verm}(W)$ defined as in (4.3). (Elements in $U(g_R)$
but not in $U(p_R)$ do not act on the second component even when $W$ comes from an object in $\text{Rep}_R(G_1)$."

**Remark 4.5.** Such modules are more often called *generalized Verma modules* because $p_1$ is seldom the Borel subalgebra of $g_1$. Since the choice of the parabolic subalgebra $p_1$ is fixed in what follows, we shall drop the modifier *generalized* from all terminologies for simplicity.

According to the Poincaré–Birkhoff–Witt theorem over $\mathbb{Z}$ (and hence over $\mathbb{Z}(p)$) for the split forms of the Lie algebras, and hence over $R_1$ and over $R$ by base change, we have a canonical isomorphism

$$\text{Verm}(W) = U(g_R) \otimes_{U(p_R)} W \cong U(u_R^-) \otimes_R W$$

of $P_1$-modules. Since $u_R^-$ is abelian, we have a canonical isomorphism $U(u_R^-) \cong \text{Sym}(u_R^-)$, in which $\text{Sym}(u_R^-)$ can be identified with a polynomial algebra over $R$ with variables given by any free $R_1$-basis of $u_R^-$. For any integer $m \geq 0$, we denote by $U(u_R^-)^m$ (resp. $U(u_R^-)^{\leq m}$, resp. $U(u_R^-)^{< m}$) the elements of degree $m$ (resp. at most $m$, resp. strictly less than $m$) in $U(u_R^-)$. We use similar notation for any algebra with a natural grading.

Note that $U(u_R^-)^{\leq m}$ is naturally a filtered $P_1$-module with $U(u_R^-)^m$ as the top graded piece. In general the canonical morphism $U(u_R^-)^{\leq m} \to U(u_R^-)^m$ does not split as a morphism of $P_1$-modules.

**Definition 4.6.** We say that a $P_1$-submodule of $\text{Verm}(W)$ is of bounded degree if it is contained in $U(u_R^-)^{\leq m} \otimes_R W$ for some $m \geq 0$. We say it is of degree $m$ if it is contained in $U(u_R^-)^{\leq m} \otimes_R W$ but not in $U(u_R^-)^{< m} \otimes_R W$.

Let $W_1, W_2 \in \text{Rep}_R(P_1)$. By finiteness of $W_1$ as an $R$-module, we know that any morphism in $\text{Hom}_{U(g_R)-P_1}(\text{Verm}(W_1), \text{Verm}(W_2)) \cong \text{Hom}_{P_1}(W_1, \text{Verm}(W_2))$ sends $W_1$ to a $P_1$-submodule of $\text{Verm}(W_2)$ of bounded degree.

**Definition 4.7.** We say that a morphism in $\text{Hom}_{U(g_R)-P_1}(\text{Verm}(W_1), \text{Verm}(W_2))$ is of degree $m$ if the image of the induced morphism in $\text{Hom}_{P_1}(W_1, \text{Verm}(W_2))$ has the same property.

4.2. **Construction of differential operators.** Let $W_1, W_2 \in \text{Rep}_R(P_1)$, and let $\phi$ be a morphism in $\text{Hom}_{U(g_R)-P_1}(\text{Verm}(W_1), \text{Verm}(W_2))$ of degree $m$, induced by a morphism in $\text{Hom}_{P_1}(W_1, U(u_R^-)^{\leq m} \otimes_R W_2)$ which we denote again by $\phi$. Suppose $W_1$ and $W_2$ are locally free as $R$-modules.

By local freeness of $u_R^-$ over $R_1$, we have a canonical perfect pairing

$$\text{Sym}(u_R^-) \times \Gamma((u_R^-)^\vee) \to R,$$

compatible with the canonical $P_1$-actions, matching elements of the same degree, where $\Gamma(\cdot)$ is the divided power analogue of $\text{Sym}(\cdot)$. (See [4, Appendix A, especially Prop. A.10] for the precise definition of $\Gamma(\cdot)$ and for the perfection of $U(r)$.) For simplicity, we have omitted the subscript "$R$" of $\text{Sym}(\cdot)$ and $\Gamma(\cdot)$ indicating that the constructions are over $R$.)

Let us abusively denote $(u_R^-)^\vee$ as $u_R^\#$, which can be identified with $u_R$ when $p > 2$, or when $G_{\tau} \cong \text{Sp}_{2r, R_1}$ for some $\tau \in Y$. Then (4.8) induces an isomorphism

$$(U(u_R^-)^{\leq m})^\vee \cong \Gamma(u_R^\#)/\Gamma^{> m}(u_R^\#) =: \Gamma_{\leq m}(u_R^\#),$$
and the morphism $\phi : W_1 \to U(u_R^\#) \otimes R W_2$ is canonically dual to a morphism

\[(4.9) \quad \phi^\vee : W_2^\vee \otimes R \Gamma_{\leq m}(u_R^\#) \to W_1^\vee,\]

all considered as morphisms in $\text{Rep}_R(P_1)$.

There is a degree-preserving canonical morphism

$$\text{Sym}(u_R^\#) \to \Gamma(u_R^\#)$$

(of $P_1$-modules), which induces a canonical morphism

\[(4.10) \quad \text{Sym} \leq m(u_R^\#) := \text{Sym}(u_R^\#) / \text{Sym} > m(u_R^\#) \to \Gamma_{\leq m}(u_R^\#)\]

in $\text{Rep}_R(P_1)$. This morphism is an isomorphism either when the residue characteristics of $R$ are all zero, or when $\phi$ is $p$-small, namely when $m < p$, because $m!$ is invertible in $R_1$, and hence in $R$.

The above morphisms (4.9) and (4.10) induce another morphism

\[(4.11) \quad \phi^\vee : W_2^\vee \otimes \text{Sym} \leq m(u_R^\#) \to W_1^\vee\]

in $\text{Rep}_R(P_1)$.

**Lemma 4.12.** For any isomorphism $\iota : R_1(1) \to R_1$ inducing an isomorphism $L_{0,1}^\vee(1) \to L_{0,1}^\vee$ which we also denote by $\iota$, the $R_1$-module $u_1^\# = u_{R_1}^\#$ is isomorphic to

\[
(L_{0,1}^\vee \otimes R_1 L_{0,1}^\vee(1)) / \left( \iota(y) \otimes z - \iota(z) \otimes y + (b^\ast x) \otimes y - x \otimes (by) \right)_{x \in L_{0,1}^\vee(1), y \in L_{0,1}^\vee, b \in \O_1}.
\]

**Proof.** This follows from the definition of $P_1$ in Definition 2.4 \qed

**Corollary 4.13.** There are canonical isomorphisms $\mathcal{E}_{P_1,R}(u_R^\#) \cong \text{KS}_{\text{Act}^\ast/M_{\text{th},R}^\ast}(1)$ and $\mathcal{E}_{\text{can}}(u_R^\#) \cong \text{KS}_{\text{Act}^\ast/M_{\text{th},R}^\ast}(1)$ (cf. Definition 2.15 and the definition of $\text{KS}_{\text{Act}^\ast/M_{\text{th},R}^\ast}$ in Section 2.3).

**Proof.** By definition (cf. 2.10, 2.19, and 2.23), we can identify $\mathcal{E}_{P_1,R}(L_{0,1}^\vee)$ with $\mathcal{E}_{P_1,R}(L_{0,1}^\vee)$ under the isomorphism $\lambda^\ast : \text{Lie}_{A^\ast/M_{\text{th},R}^\ast} \to \text{Lie}_{A^\ast/M_{\text{th},R}^\ast}$. Hence, by Lemma 4.12 and by functoriality, we obtain $\mathcal{E}_{P_1,R}(u_R^\#) \cong \text{KS}_{\text{Act}^\ast/M_{\text{th},R}^\ast}(1)$. The case for $\mathcal{E}_{\text{can}}(u_R^\#) \cong \text{KS}_{\text{Act}^\ast/M_{\text{th},R}^\ast}(1)$ is similar. \qed

We shall always identify $\mathcal{E}_{P_1,R}(u_R^\#)$ with $\Omega^1_{M_{\text{th},R}/S_R}$ using Corollary 4.13 and the Kodaira–Spencer isomorphism (2.17).

**Lemma 4.14.** Under the above identification, consider the morphism

\[(4.15) \quad (\mathcal{E}_{P_1,R} \otimes R) \times \text{Sym}_{\leq 1}(u_R^\#) \to \O_{M_{\text{th},R}/S_R} \oplus \Omega^1_{M_{\text{th},R}/S_R} : (\xi, (r, u)) \to (r, [\xi^{-1} u \xi]),\]

where $\xi$ is any section of $\mathcal{E}_{P_1,R} \otimes R$, where $(r, u) \in \text{Sym}_{\leq 1}(u_R^\#)$, with $r$ in degree zero and $u$ in degree one, and where $[\xi^{-1} u \xi]$ is defined as follows: (For simplicity of notation, let us treat only sections defined globally, although the argument works also locally.) Any section $\xi$ of $\mathcal{E}_{P_1,R} \otimes R$ induces by definition (cf. 2.19) an isomorphism

$$\mathcal{H}_{\text{dR}}^{1}(A/M_{\text{th},R}) \to (L_{0,1} \oplus L_{0,1}^\vee(1)) \otimes \O_{M_{\text{th},R}}$$
(which we again denote by $\xi$) matching the natural filtrations, and hence also induces a splitting

$$H^1_{\text{dR}}(A_R/M_{H,R}) \cong \text{Lie}_{A_R/M_{H,R}} \oplus \text{Lie}_{A_R^{\vee}/M_{H,R}}(1)$$

(corresponding to the canonical splitting of $L_{0,1} \oplus L_{0,1}^{\vee}(1)$). Then $\xi^{-1}u\xi$ induces a morphism $\text{Lie}_{A_R/M_{H,R}} \to \text{Lie}_{A_R^{\vee}/M_{H,R}}(1)$, which in turn induces a section of $\Omega^1_{M_{H,R}/S_R}$ under the Kodaira–Spencer morphism $\text{Sym}^n_{\text{dR}}(\cdot)$, which we denote by $[\xi^{-1}u\xi]$.

For any section $\eta$ of $P_1 \otimes R$, both $\eta(\xi, (r, u)) = (\eta\xi, (r, \eta u^{-1}))$ and $(\xi, (r, u))$

have the same image $(r, [\eta]^{-1}(\eta u^{-1})(\eta\xi)) = (r, [\xi^{-1}u\xi])$, and hence the morphism (4.15) induces a morphism (see Definition 2.22)

$$\mathcal{E}_{P_1,R}(\text{Sym}_{\leq 1}(u_{R})) \to \mathcal{O}_{M_{H,R}} \oplus \Omega^1_{M_{H,R}/S_R}$$

This morphism is an isomorphism of $\mathcal{O}_{M_{H,R}}$-modules.

Similarly, we have an isomorphism

$$\mathcal{E}_{P_1,R}^\text{can}(\text{Sym}_{\leq 1}(u_{R})) \cong \mathcal{O}_{M_{H,R}} \oplus \Omega^1_{M_{H,R}/S_R}.$$}

**Proof.** By trivializing $\mathcal{E}_{P_1,R}$ étale locally, we see that the morphism (4.16) is indeed a morphism of $\mathcal{O}_{M_{H,R}}$-modules. By definition, it sends the submodule $\mathcal{E}_{P_1,R}(u_{R})$ of $\mathcal{E}_{P_1,R}(\text{Sym}_{\leq 1}(u_{R}))$ (induced by the canonical submodule $u_{R}$ of $\text{Sym}_{\leq 1}(u_{R})$ embedded in degree one) to the submodule $\Omega^1_{M_{H,R}/S_R}$ of $\mathcal{O}_{M_{H,R}} \oplus \Omega^1_{M_{H,R}/S_R}$; and the induced morphism $\mathcal{E}_{P_1,R}(u_{R}) \to \Omega^1_{M_{H,R}/S_R}$ is an isomorphism by Corollary 4.13 and by the extended Kodaira–Spencer isomorphism (2.26). On the other hand, the induced morphism $\mathcal{E}_{P_1,R}(R) \to \mathcal{O}_{M_{H,R}}$ between quotient modules is clearly an isomorphism. Therefore, (4.16) is an isomorphism, as desired.

The proof for (4.17) is similar. 

**Lemma 4.18.** For any integer $m \geq 0$, we have a canonical filtered isomorphism

$$\mathcal{E}_{P_1,R}(\text{Sym}_{\leq m}(u_{R})) \cong \mathcal{P}^m_{M_{H,R}/S_R},$$

where $\mathcal{P}^m_{M_{H,R}/S_R}$ is the sheaf of principal parts of order $m$ over $M_{H,R}$ (see [16] IV-4, 16.3)).

**Proof.** Let $\mathcal{M}^{(m)}_{H,R}$ be defined as in the paragraph preceding Definition 2.11 with the canonical splitting $\mathcal{P}^m_{M_{H,R}/S_R} \cong \mathcal{O}_{M_{H,R}} \oplus \Omega^1_{M_{H,R}/S_R}$ when $m = 1$. Then (4.16) can be rewritten as a filtered isomorphism $\mathcal{E}_{P_1,R}(\text{Sym}_{\leq 1}(u_{R})) \cong \mathcal{P}^1_{M_{H,R}/S_R}$. Since the functor $\mathcal{E}_{P_1,R}(\cdot)$ is functorial and exact, and since $\text{Sym}_{\leq m}(u_{R}) \cong \text{Sym}_m(\text{Sym}_{\leq 1}(u_{R}))$ as $P_1,R$-modules, we obtain a canonical filtered isomorphism

$$\mathcal{E}_{P_1,R}(\text{Sym}_{\leq m}(u_{R})) \cong \text{Sym}_{\leq m}(\mathcal{E}_{P_1,R}(u_{R})) \cong \text{Sym}_{\leq m}(\mathcal{P}^1_{M_{H,R}/S_R}).$$

Since $M_{H,R} \to S_R$ is smooth (and hence differential smooth), the canonical filtered morphism $\text{Sym}_{\leq m}(\mathcal{P}^1_{M_{H,R}/S_R}) \to \mathcal{P}^m_{M_{H,R}/S_R}$ is an isomorphism. (It suffices to compare the graded pieces. See [16] IV-4, 17.12.4.) Then the lemma follows by composing all these filtered isomorphisms.

□
Lemma 4.20. For any integer \( m \geq 0 \), the morphism \([4.11]\) corresponds under the functor \( \mathcal{E}_{P,1,R}(-) \) to a morphism

\[
\mathcal{E}_{P,1,R}(W_2^\vee) \otimes_{\mathcal{O}_{M^\tor_{H,R}/S_R}} \mathcal{O}_{M_{H,R}/S_R} \to \mathcal{E}_{P,1,R}(W_1^\vee)
\]

between locally free \( \mathcal{O}_{M_{H,R}} \)-modules. The pre-composition of this morphism with the canonical morphism \( \mathcal{E}_{P,1,R}(W_2^\vee) \to \mathcal{O}_{M_{H,R}/S_R} \otimes \mathcal{E}_{P,1,R}(W_2^\vee) \) gives a differential operator

\[
d_\phi : \mathcal{E}_{P,1,R}(W_2^\vee) \to \mathcal{E}_{P,1,R}(W_1^\vee)
\]

of order \( m \). (See [16] IV-4, 16.8.1.) Moreover, this construction is compatible with composition of morphisms.

Proof. The first statement follows immediately from Lemma \([4.18]\). The construction is compatible with composition of morphisms because the identification

\[
\mathcal{E}_{P,1,R}(\text{Sym}^{-m}(u_R^\#)) \cong \mathcal{O}_{M_{H,R}/S_R}^m
\]

in Lemma \([4.18]\) is compatible with the canonical morphism

\[
\text{Sym}^{-m}(u_R^\#) \otimes_{R} \text{Sym}^{-m'}(u_R^\#) \to \text{Sym}^{-m''}(u_R^\#)
\]

for all \( 0 \leq m, m', m'' \) with \( m'' \leq m + m' \).

For each \( m \geq 0 \), let us define \( \mathcal{O}_{M^\tor_{H,R}/S_R}^m \) to be the canonical extension of \( \mathcal{O}_{M^\tor_{H,R}/S_R} \), namely \( \mathcal{E}_{P,1,R}(\text{Sym}^{-m}(u_R^\#)) \). (This is consistent with the construction of \( \mathcal{O}_{M^\tor_{H,R}/S_R}^m \) in Section 2.4. For our purpose, we do not need to know any interpretation of \( \mathcal{O}_{M^\tor_{H,R}/S_R}^m \) and \( \mathcal{O}_{M^\tor_{H,R}/S_R}^m \) in log geometry.)

Lemma 4.21. The canonical morphism \( \mathcal{O}_{M_{H,R}} \to \mathcal{O}_{M^\tor_{H,R}/S_R} \) over \( M_{H,R} \) admits a unique extension \( \mathcal{O}_{M^\tor_{H,R}} \to \mathcal{O}_{M^\tor_{H,R}/S_R} \) over \( M^\tor_{H,R} \).

Proof. Note that \( \mathcal{O}_{M^\tor_{H,R}/S_R}^m = \mathcal{E}_{P,1,R}^m(\text{Sym}^{-m}(u_R^\#)) \) is locally free, and there is a canonical decomposition \( \mathcal{O}_{M^\tor_{H,R}/S_R}^m \cong \oplus_{0 \leq a \leq m} \text{Sym}^a(\Omega^1_{M^\tor_{H,R}/S_R}) \) as \( \mathcal{O}_{M^\tor_{H,R}} \)-modules. (This decomposition is compatible with restriction to \( M_{H,R} \).) For any \( 0 \leq a \leq m \), since the composition of the canonical morphism \( \mathcal{O}_{M_{H,R}} \to \mathcal{O}_{M^\tor_{H,R}/S_R}^m \) with the canonical projection \( \mathcal{O}_{M^\tor_{H,R}/S_R}^m \to \text{Sym}^a(\Omega^1_{M^\tor_{H,R}/S_R}) \) is nothing but the \( a \)-th symmetric power of the universal derivation \( d : \mathcal{O}_{M_{H,R}} \to \Omega^1_{M_{H,R}/S_R} \), the lemma follows from the fact that, by the very definition of \( \Omega^1_{M^\tor_{H,R}/S_R} \), the morphism \( (M_{H,R} \to M^\tor_{H,R})_* (d) \) sends the subsheaf \( \mathcal{O}_{M^\tor_{H,R}} \) of \( (M_{H,R} \to M^\tor_{H,R})_* (\mathcal{O}_{M_{H,R}}) \) to the subsheaf \( \Omega^1_{M^\tor_{H,R}/S_R} \) of \( (M_{H,R} \to M^\tor_{H,R})_* (\Omega^1_{M_{H,R}/S_R}) \).

Proposition 4.21. For any integer \( m \geq 0 \), the morphism \([4.11]\) corresponds under the functor \( \mathcal{E}_{P,1,R}^m(-) \) to a morphism

\[
\mathcal{E}_{P,1,R}(W_2^\vee) \otimes_{\mathcal{O}_{M^\tor_{H,R}/S_R}} \mathcal{O}_{M^\tor_{H,R}/S_R} \to \mathcal{E}_{P,1,R}(W_1^\vee)
\]
between locally free $\mathcal{O}_{M_{W_1}^\mathrm{tor}}$-modules. The pre-composition of this morphism with the canonical morphism $E_{P_1,R}(W_2^\vee) \to \mathcal{F}^m_{M_{W_1}^\mathrm{tor}/S_R} \otimes E_{P_1,R}^\mathrm{can}(W_2^\vee)$ (induced by the extended canonical morphism in Lemma 4.20) gives a log differential operator

$$d_\phi : E_{P_1,R}^\mathrm{can}(W_2^\vee) \to E_{P_1,R}^\mathrm{can}(W_1^\vee)$$

of order $m$. Moreover, this construction is compatible with composition of morphisms.

The analogous statements for the functor $E_{G_1,R}^\mathrm{sub}(-)$ are also true.

**Proof.** The case of $E_{G_1,R}^\mathrm{can}(-)$ is similar to the case of $E_{G_1,R}(-)$. (See the proof of Proposition 4.19.) The case of $E_{G_1,R}^\mathrm{sub}(-)$ then follows by applying $\otimes \mathcal{F}_D$ to all $\mathcal{O}_{M_{W_1}^\mathrm{tor}}$-modules.

**Remark 4.22.** While the restriction gives rise to (log) differential operators, the original morphism gives rise to (log) HPD differential operators. The attachment of the restriction to then corresponds to the attachment of a (log) differential operator to a (log) HPD (or rather PD) differential operator, as in paragraph following Def. 4.4.

For later reference, let us record the following observation:

**Lemma 4.23.** If there exists integers $a_0$ and $m_0$ such that $\text{Gr}^{a_1}_{\mathcal{F}}(W_2^\vee) \neq 0$ only when $a_1 \leq a_0$, but $\text{Gr}^{a_2}_{\mathcal{F}}(W_1^\vee) = 0$ for all $a_2 > a_0 + m_0$, then there is no nonzero morphism as in (4.9) with $m > m_0$. Therefore, the construction of Proposition 4.19 (resp. Proposition 4.21) gives no nonzero differential operator (resp. log differential operator) of order greater than $m_0$ from $W_2^\vee$ to $W_1^\vee$.

**Proof.** This is because all elements in $u_R$ have $H$-weight $-1$. \qed

### 4.3. Standard complexes and de Rham complexes.

Let $W \in \text{Rep}_R(G_1)$ be locally free as an $R$-module, which we also consider as an element of $\text{Rep}_R(P_1)$ by restriction to $P_1$.

Let us identify $u_R$ with $\mathfrak{g}_R/\mathfrak{p}_R$ as algebraic representations of $P_1$ as usual. Let $n$ denote the relative dimension of $M_{W_1}^\mathrm{tor}$ over $S_0$, which is also the rank of $u_R$ as a free $R$-module. Consider the complex of $U(\mathfrak{g}_R)$-P$_1$-modules

$$0 \to \text{Verm}(\wedge^n(u_R^{-} \otimes W)) \xrightarrow{d_n} \text{Verm}(\wedge^{n-1}(u_R^{-} \otimes W))$$

$$\xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} \text{Verm}(u_R^{-} \otimes W) \xrightarrow{d_1} \text{Verm}(W)$$

with differentials given by morphisms

$$d_a : \text{Verm}(\wedge^a(u_R^{-} \otimes W)) = U(\mathfrak{g}_R) \otimes (\wedge^a(u_R^{-} \otimes W))$$

$$\to \text{Verm}(\wedge^{a-1}(u_R^{-} \otimes W)) = U(\mathfrak{g}_R) \otimes (\wedge^{a-1}(u_R^{-} \otimes W))$$

$$\to \cdots \to \text{Verm}(\wedge^1(u_R^{-} \otimes W)) = U(\mathfrak{g}_R) \otimes (\wedge^1(u_R^{-} \otimes W))$$

$$\to \text{Verm}(\wedge^0(u_R^{-} \otimes W)) = U(\mathfrak{g}_R) \otimes (\wedge^0(u_R^{-} \otimes W)) = \text{Verm}(W)$$
of $U(g_R)$-$P_1$-modules, for $1 \leq a \leq n$, defined by
\[
d_a(u \otimes ((x_1 \wedge x_2 \wedge \ldots \wedge x_a) \otimes v)) := \sum_{1 \leq i \leq a} (-1)^{i-1}(ux_i) \otimes ((x_1 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge x_a) \otimes v) \]
\[
+ \sum_{1 \leq i \leq a} (-1)^iu \otimes ((x_1 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge x_a) \otimes (x_i v))
\]
for all $u \in U(g_R)$, $x_1, \ldots, x_a \in u_R^\#$, and $v \in W$. We omitted the usual terms involving $[x_i, x_j]$ (see [33, 2.2]) because $u_R^\#$ is abelian. (When $W = R$ is the trivial representation, this is induced by the standard Koszul complex resolving the trivial $U(g_R)$-module $R$.) One can easily check (with any free $R$-basis of $u_R^\#$) that this complex is exact.

**Definition 4.26.** The complex $\begin{equation} \tag{4.24} \end{equation}$ with differentials given by $\begin{equation} \tag{4.25} \end{equation}$ is called the standard complex (of $U(g_R)$-$P_1$-modules) for the module $W$ in $\text{Rep}_R(G_1)$, which we shall denote as $\text{Std}_*(W)$.

**Proposition 4.27.** Under the functor $\mathcal{E}_{P_1,R}(\cdot)$ (as in Proposition 4.19), the canonical morphism
\[
W^\vee \otimes \text{Sym}_{\leq 1}(u_R^\#) \rightarrow W^\vee \otimes u_R^\# : w \otimes (c + e) \mapsto w \otimes e + \sum_{1 \leq j \leq n} (y_jw) \otimes (cf_j),
\]
for all $c \in R_1$, $e \in u_R^\#$, and $w \in W^\vee$, and for any free $R$-basis $y_1, \ldots, y_n$ of $u_R^\#$ with dual free $R$-basis $f_1, \ldots, f_n$ of $u_R^\#$, is associated with the canonical morphism
\[
\mathcal{E}_{P_1,R}(W^\vee) \otimes \mathcal{D}^1_{M_{H,R}/S_R} \rightarrow \mathcal{E}_{P_1,R}(W^\vee) \otimes \Omega^1_{M_{H,R}/S_R}
\]
inducing the Gauss–Manin connection defined as in Definition 2.34.

**Proof.** As explained in the proof of Lemma 4.18, the canonical morphism $\mathcal{D}^1_{M_{H,R}/S_R} \rightarrow \Omega^1_{M_{H,R}/S_R}$ corresponds to $\text{Sym}_{\leq 1}(u_R^\#) \rightarrow u_R^\#$, $c + e \mapsto e$, for all $c \in R$ and $e \in u_R^\#$. The canonical morphism (4.29) inducing the Gauss–Manin connection is defined by (the restriction to $\text{pr}_2(\mathcal{E}_{P_1,R}(W^\vee))$ of) $s^* - \text{Id}^*$ on $\mathcal{E}_{P_1,R}(W^\vee) \otimes \mathcal{D}^1_{M_{H,R}/S_R}$, satisfying
\[
(s^* - \text{Id}^*)(z \otimes x) = ((s^* - \text{Id}^*)(z \otimes 1))x + z \otimes ((s^* - \text{Id}^*)x)
\]
for all sections $z$ of $\mathcal{E}_{P_1,R}(W^\vee)$ and sections $x$ of $\mathcal{D}^1_{M_{H,R}/S_R}$, because
\[
((s^* - \text{Id}^*)(z \otimes 1)) \otimes ((s^* - \text{Id}^*)x) = 0.
\]
Since $(s^* - \text{Id}^*)x$ is known to agree with the image of the canonical morphism $\mathcal{D}^1_{M_{H,R}/S_R} \rightarrow \Omega^1_{M_{H,R}/S_R}$ when restricted to sections $x$ in $\text{pr}_2^*(\mathcal{O}_{M_{H,R}})$, it remains to study $(s^* - \text{Id}^*)(z \otimes 1)$.

Let us adopt the notation in the proof of Lemma 4.18 with two projections $\text{pr}_1, \text{pr}_2 : \hat{M}^{(1)}_{H,R} \rightarrow M_{H,R}$. Then $\text{pr}_1^* \mathcal{H}^{\text{dR}}_1(A/M_{H,R}) \cong \mathcal{H}^{\text{dR}}_1(\text{pr}_2^* A/\hat{M}^{(1)}_{H,R})$, and we obtain a morphism $(s^* - \text{Id}^*) : \mathcal{H}^{\text{dR}}_1(A/M_{H,R}) \rightarrow \mathcal{H}^{\text{dR}}_1(\text{pr}_2^* A/\hat{M}^{(1)}_{H,R}) \otimes \Omega^1_{M_{H,R}/S_R}$.

For any section $v$ of $\text{Der}^1_{M_{H,R}/S_R}$, we obtain a morphism $\mathcal{H}^{\text{dR}}_1(A/M_{H,R}) \rightarrow \mathcal{H}^{\text{dR}}_1(A/M_{H,R})$ respecting $\langle \cdot, \cdot \rangle_\lambda$, and inducing a trivial morphism on the top
Hodge graded piece (after taking quotient by the bottom Hodge graded pieces). If we identify
\[(H^1 \text{dR}(A/M_{H,R}), \langle \cdot, \cdot \rangle, \mathcal{O}_{M_{H,R}}(1), \text{Lin}_{A^\vee/M_{H,R}}(1))\]
with
\[(\langle L_{0,1} \otimes L^\vee_{0,1}(1) \rangle \otimes \mathcal{O}_{M_{H,R}}, \langle \cdot, \cdot \rangle_{\text{can}}, \mathcal{O}_{M_{H,R}}(1), L^\vee_R(1) \otimes \mathcal{O}_{M_{H,R}})\]
by any section of \(E \otimes_R R\), this morphism induced by \(v\) defines a section \(u_v\) of the pullback of \(u^*_R\) to \(M_{H,R}\). (This is compatible with the identification \(E_{P_1,R}(u^*_R) \cong \Omega^1_{M_{H,R}}/S_R\) based on Corollary 4.13 and the Kodaira–Spencer isomorphism.)

Hence,
\[v(s^* - \text{Id}^*)(z \otimes 1) = u_v z,\]
and so
\[(s^* - \text{Id}^*)(z \otimes 1) = \sum_{1 \leq j \leq n} (y_j z) \otimes f_j\]
by duality, as desired.

\[\square\]

**Corollary 4.30.** The complex associated with \(\text{Std}_*(W)\) under the functor \(E_{P_1,R}(\cdot)\) (as in Proposition 4.19) is canonically isomorphic to the de Rham complex
\[(E_{G_1,R}(W^\vee) \otimes \Omega^*_M_{M_{H,R}/S_R}, \nabla).\]

**Proof.** For each \(1 \leq a \leq n\), the morphism \(d_a\) corresponds to the morphism
\[d_a^\vee : W^\vee \otimes \wedge^{a-1}(u^*_R) \otimes \text{Sym}^1(u^*_R) \to W^\vee \otimes \wedge^a(u^*_R)\]
defined by
\[d_a^\vee (w \otimes ((e_1 \wedge e_2 \wedge \cdots \wedge e_{a-1}) \otimes (e + e_a))) := w \otimes (e_1 \wedge e_2 \wedge \cdots \wedge e_{a-1} \wedge e_a) + \sum_{1 \leq j \leq n} (y_j w) \otimes (e_1 \wedge e_2 \wedge \cdots \wedge e_{a-1} \wedge (e f_j)),\]
for all \(w \in W^\vee, e_1, \ldots, e_{a-1}, e_a \in u^*_R\), and \(c \in R\), and for any free \(R\)-basis \(y_1, \ldots, y_n\) of \(u^*_R\) dual to a free \(R\)-basis \(f_1, \ldots, f_n\) of \(u^*_R\). (This can be checked using the explicit bases we have chosen.)

By Proposition 4.27 the morphism associated with \(d_a^\vee\) under the functor \(E_{P_1,R}(\cdot)\) is the composition of the canonical morphisms
\[E_{G_1,R}(W^\vee) \otimes R^\vee_{M_{H,R}/S_R} \otimes \mathcal{O}^a_{M_{H,R}/S_R} \to E_{G_1,R}(W^\vee) \otimes R^\vee_{M_{H,R}/S_R} \otimes \Omega^1_{M_{H,R}/S_R} \to E_{G_1,R}(W^\vee) \otimes \Omega^a_{M_{H,R}/S_R},\]
inducing the Gauss–Manin connection, as desired. \[\square\]

**Proposition 4.31.** The analogues of Proposition 4.27 for the canonical and sub-canonical extensions are true, and the complexes associated with \(\text{Std}_*(W)\) under the functors \(E_{P_1,R}(\cdot)\) and \(E_{P_1,R}(\cdot)\) (as in Proposition 4.21) are canonically isomorphic to the log de Rham complexes
\[(E^\text{can}_{G_1,R}(W^\vee) \otimes \Omega^*_M_{M_{H,R}/S_R}, \nabla).\]
and
\[(\mathcal{E}_{G_1, R}^{\text{can}}(W^\vee) \otimes \Omega^*_{M_{H, R}/S_R^R, \nabla}),\]
respectively.

**Proof.** By functoriality and compatibility between \(\mathcal{E}_{P_1, R}(\cdot)\) and \(\mathcal{E}_{P_1, R}^{\text{can}}(\cdot)\), the identification \(\mathcal{E}_{P_1, R}(u^\#_R) \cong \Omega^1_{M_{H, R}/S_R^R}\) based on Corollary 4.13 and the Kodaira–Spencer isomorphism (2.17) extends to the identification \(\mathcal{E}_{P_1, R}^{\text{can}}(u^\#_R) \cong \Omega^1_{M_{H, R}/S_R^R}\) based on the extended Kodaira–Spencer isomorphism (2.20). Then, by \(\mathcal{E}_{P_1, R}(\cdot)\) in Proposition 2.27, we have the canonical morphism \(\mathcal{E}_{G_1, R}(W^\vee) \otimes \Omega^1_{M_{H, R}/S_R^R, \nabla}\) extending \(\mathcal{E}_{P_1, R}(\cdot)\) in its verify construction in Section 2.4. Then the proofs of Proposition 4.27 and Corollary 4.30 also work for the canonical extensions, and shows that \(\mathcal{E}_{G_1, R}^{\text{can}}(W^\vee) \otimes \Omega^1_{M_{H, R}/S_R^R, \nabla}\) is associated with \(\text{Std}_*(W)\) under the functor \(\mathcal{E}_{P_1, R}(\cdot)\). (The case of \(\mathcal{E}_{G_1, R}^{\text{can}}(\cdot)\) then follows by applying \(\mathcal{E}_{P_1, R}(\cdot)\) to all \(\Omega^1_{M_{H, R}/S_R^R, \nabla}\)-modules, as usual.) \(\square\)

**Corollary 4.32.** The de Rham complex
\[(\mathcal{E}_{G_1, R}(W^\vee) \otimes \Omega^*_{M_{H, R}/S_R^R, \nabla})\]
and the log de Rham complexes
\[(\mathcal{E}_{G_1, R}^{\text{can}}(W^\vee) \otimes \Omega^*_{M_{H, R}/S_R^R, \nabla})\]
and
\[(\mathcal{E}_{G_1, R}^{\text{can}}(W^\vee) \otimes \Omega^*_{M_{H, R}/S_R^R, \nabla})\]
all satisfy the Griffiths transversality. (The remark following Definition 3.11 is now justified.)

**Proof.** By Corollary 4.30 and Proposition 4.31 it suffices to note that in (4.25) the action of \(u^\#_R\) on \(W\) increases the \(R\)-weights by 1 (cf. the proof of Lemma 3.9). \(\square\)

5. MAIN RESULTS

5.1. **Notation.** Let \(R\) be any \(R_1\)-algebra. Let \(X_{G_1, R}^{+, <p}\) := \(X_{G_1, R}^{+, <p}\) (resp. \(X_{M_{1, R}}^{+, <p}\) := \(X_{M_{1, R}}^{+, <p}\)) if the residue characteristics of \(R\) are all zero, and let \(X_{G_1, R}^{+, <p}\) := \(X_{G_1, R}^{+, <p}\) (resp. \(X_{M_{1, R}}^{+, <p}\) := \(X_{M_{1, R}}^{+, <p}\)) as in Definition 3.11 if otherwise.

For each \(\mu \in X_{G_1, R}^{+, <p}\), let \(V_{[\mu], R} \in \text{Rep}_R(G_1)\) be defined as in the last paragraph of Section 3.5. Since the underlying \(R\)-module of \(V_{[\mu], R}\) is locally free, we can consider the contragredient representation \(V_{[\mu], R}^\vee \in \text{Rep}_R(G_1)\). Then we have the associated automorphic bundle
\[V_{[\mu], R}^\vee := \mathcal{E}_{G_1, R}(V_{[\mu], R}^\vee)\]
over $M_{H,R}$, and its canonical and subcanonical extensions
\[ (V_{\mu,R}^\vee)^{\text{can}} := \mathcal{E}_{G_1,R}^{\text{can}}(V_{\mu,R}^\vee), \]
and
\[ (V_{\mu,R}^\vee)^{\text{sub}} := \mathcal{E}_{G_1,R}^{\text{sub}}(V_{\mu,R}^\vee), \]
respectively, to $M_{H,R}^{\text{tor}}$. Similarly, for each $\nu \in X_{M_1,R}^+$, let $W_{\nu,R}$ be defined as in the last paragraph of Section 3.5. For any $w \in W_{M_1}$, we define
\[ W_{w,\mu,R} := \bigoplus_{\nu \in w[\mu]} W_{\nu,R}. \]

(Although $V_{\mu,R} \otimes Q$ is irreducible by definition, $W_{w,\mu,R} \otimes Q$ is not necessarily irreducible when $G_1$ is not connected.) Then we define $W_{w,\mu,R}^{\vee}$, $W_{w,\mu,R}^{\vee}$, $(W_{w,\mu,R}^{\vee})^{\text{can}}$, and $(W_{w,\mu,R}^{\vee})^{\text{sub}}$ in the obvious way.

The connections (2.33), (2.35), and (2.36) define respectively the de Rham complex
\[ (V_{\mu,R}^\vee)_{\mathcal{E}_{M_{H,R}}^{\text{can}}} \otimes \Omega_{M_{H,R}}^* \mathcal{O}_{M_{H,R}/S_R,R} \nabla \]
and the log de Rham complexes
\[ ((V_{\mu,R}^\vee)^{\text{can}})_{\mathcal{E}_{M_{H,R}}^{\text{can}}} \otimes \Omega_{M_{H,R}}^* \mathcal{O}_{M_{H,R}/S_R,R} \nabla \]
and
\[ ((V_{\mu,R}^\vee)^{\text{sub}})_{\mathcal{E}_{M_{H,R}}^{\text{sub}}} \otimes \Omega_{M_{H,R}}^* \mathcal{O}_{M_{H,R}/S_R,R} \nabla. \]

5.2. BGG complexes.

**Definition 5.1.** A complex of $U(\mathfrak{g}_R)$-$P_1$-modules formed by direct sums of Verma modules (see Definition 4.4) is called a summand of degree zero of another complex of $U(\mathfrak{g}_R)$-$P_1$-modules formed by direct sums of Verma modules if both the embedding and the splitting morphisms defining the summand are defined by direct sums of morphisms of $U(\mathfrak{g}_R)$-$P_1$-modules of degree zero (see Definition 4.7).

For any integer $a \geq 0$, we denote by $W_{M_1}(a)$ the elements $w$ in $W_{M_1}$ with length $l(w) = a$.

**Theorem 5.2.** Let $\mu \in X_{G_1,R}^+$, and let $V_{\mu,R} \in \text{Rep}_R(G_1)$ be defined as in Section 3.5. Then there exists an $F$-filtered complex of $U(\mathfrak{g}_R)$-$P_1$-modules
\[ (5.3) \]
\[ 0 \to \text{BGG}_n(V_{\mu,R}) \to \text{BGG}_{n-1}(V_{\mu,R}) \to \ldots \to \text{BGG}_1(V_{\mu,R}) \to \text{BGG}_0(V_{\mu,R}), \]
which is canonically $F$-filtered quasi-isomorphically embedded as a summand of $\text{Std}_*(V_{\mu,R})$ (see Definition 4.26) in the category of $F$-filtered complexes of $U(\mathfrak{g}_R)$-$P_1$-modules, where
\[ (5.4) \]
\[ \text{BGG}_n(V_{\mu,R}) \cong \bigoplus_{w \in W_{M_1}(a)} \text{Verm}(W_{w[\mu],R}), \]
(as $U(\mathfrak{g}_R)$-$P_1$-submodules) for each $0 \leq a \leq n$. Moreover, the induced complex
\[ \text{Gr}_F(\text{BGG}_*(V_{\mu,R})) \]
is a canonical (quasi-isomorphic) summand of degree zero of $\text{Gr}_F(\text{Std}_*(V_{\mu,R}))$ with trivial differentials.
Proof. By the argument in [33, 4.4] (using [20, Sec. 8.2] and [33, Cor. 1.11 b]), since $U_1$ and hence $U_R$ are abelian, we have $\wedge^a(u_R) \cong \bigoplus_{w \in W^{M_1}(a)} W_{w[0],R}$ as $P_1$-modules, and hence

$$\text{Std}_a(V_{[0],R}) \cong \bigoplus_{w \in W^{M_1}(a)} \text{Verm}(W_{w[0],R})$$

as $U(\mathfrak{g}_R)$-$P_1$-modules. (When some residue characteristic of $R$ is $p > 0$, the argument in [33, 4.4] requires that $p \geq h - 1$, where $h$ is the Coxeter number of $G_1$. As pointed out in [33, Rem. 2.1], this is automatic if there is any $\mu \in \mathfrak{X}_{G_1}^{+,<p}$ to begin with.) As in [33, 4.5], since $\text{Std}_a(V_{[0],R}) \cong \text{Std}_a(V_{[0],R}) \otimes V_{[0],R}$ (as complexes of $U(\mathfrak{g}_R)$-$P_1$-modules), we deduce from (5.5) (and the tensor identity [12, Prop. 1.7]) that

$$\text{Std}_w(V_{[0],R}) \cong \bigoplus_{w \in W^{M_1}} \text{Std}_w(V_{[0],R})$$

where

$$\text{Std}_w(V_{[0],R}) := \text{Verm}(W_{w[0],R} \otimes V_{[0],R})$$

($W_{w[0],R} \otimes V_{[0],R}$ being a tensor product in $\text{Rep}_R(P_1)$) appears in degree $l(w)$, and where the differentials are inherited from those of $\text{Std}_w(V_{[0],R})$. (In particular, (5.6) is not a decomposition into subcomplexes.)

By the same argument as in [33, 2.7] (using also [18]), the complex $\text{Std}_w(V_{[0],R})$ admits a canonical direct sum decomposition

$$\text{Std}_w(V_{[0],R}) \cong \bigoplus_{\mu \in J} \text{Std}_w(V_{[0],R})\chi_j \cong \bigoplus_{\mu \in J} \bigoplus_{w \in W^{M_1}(a)} \text{Std}_w(V_{[0],R})\tilde{\chi}_j,$$

indexed by some finite set $J$, such that the center of $U(\mathfrak{g}_R)$ acts on the reduction mod $p$ of each direct summand $\text{Std}_w(V_{[0],R})\tilde{\chi}_j$ by a distinct character $\tilde{\chi}_j$. The direct sum with respect to $j \in J$ is a decomposition into subcomplexes of $U(\mathfrak{g}_R)$-$P_1$-modules, because the action of the center of $U(\mathfrak{g}_R)$ commutes with the action of $P_1$. Take the unique index $j_0 \in J$ such that $\tilde{\chi}_{j_0} = \bar{\chi}_{[\mu],p}$, where the latter is the unique character of the center of $U(\mathfrak{g}_R)$ that acts nontrivially on the reduction mod $p$ of $V_{[0],R}$. Then we define

$$\text{BGG}_w(V_{[0],R}) := \text{Std}_w(V_{[0],R})\tilde{\chi}_{j_0} = \text{Std}_w(V_{[0],R})\bar{\chi}_{[\mu],p}.$$

Thus, (5.6) has a refinement

$$\text{BGG}_w(V_{[0],R}) \cong \bigoplus_{w \in W^{M_1}} \text{Std}_w(V_{[0],R})\bar{\chi}_{[\mu],p}.$$

Since $\mu \in \mathfrak{X}_{G_1}^{+,<p}$, by [33, Lem. 2.3], all weights of $\wedge^a(u_R) \otimes V_{[0],R}$ are $p$-small. Then, for each $\nu \in W^{M_1}$, there exists a finite filtration on $W_{\nu,[0],R} \otimes V_{[0],R}$ such that the graded pieces are of the form $W_{\nu,R}$ for some $\nu \in \mathfrak{X}_{M_1}^{+,<p}$. Since the functor $\text{Verm}(\cdot)$ (see (4.3)) is exact (because $U(\mathfrak{g}_R)$ is free over $U(\mathfrak{p}_R)$), there is a corresponding finite filtration on $\text{Std}_w(V_{[0],R}) = \text{Verm}(W_{\nu,[0],R} \otimes V_{[0],R})$ by $U(\mathfrak{g}_R)$-$P_1$-modules, whose graded pieces are of the form $\text{Verm}(W_{\nu,R})$ with $W_{\nu,R}$ appearing as a graded piece on the finite filtration on $W_{\nu,[0],R} \otimes V_{[0],R}$. We have a similar finite filtration for the direct summand $\text{Std}_w(V_{[0],R})\bar{\chi}_{[\mu],p}$ of $\text{Std}_w(V_{[0],R})$. As in [33, 2.7, 2.8] (using also [18]), for $\nu \in \mathfrak{X}_{M_1}^{+,<p}$, the $U(\mathfrak{g}_R)$-$P_1$-module $\text{Verm}(W_{\nu,R})$
appears as a graded piece of such a filtration on $\text{Std}_w(V_{[\mu],R})_{\bar{\chi}_{[\mu],p}}$ if and only if the following two conditions hold:

1. $W_{\nu,R}$ appears as a graded piece on the finite filtration on $W_{w,[0],R} \otimes V_{[\mu],R}$.

2. $\bar{\chi}_{[\mu],p} = \bar{\chi}_{[\mu],p}$, or equivalently when $\nu = w', \mu'$ for some $w' \in W^{M_1}$ and some $\mu' \in [\mu]$.

But then, as explained in [33] proof of Lem. 4.5, this happens only when $w' = w$, and the multiplicity is exactly one for each $\mu'$.

(5.8) $\text{Std}_w(V_{[\mu],R})_{\bar{\chi}_{[\mu],p}} \cong \text{Verm}(W_{w,[\mu],R})$.

Thus, (5.4) follows from the combination of (5.7) and (5.8).

As for last statement, first note that the morphisms of $U(g_R)$-P$_1$-modules defining $\text{Gr}_F(\text{BGG}_\ast(V_{[\mu],R}))$ as a summand of $\text{Gr}_F(\text{Std}_\ast(V_{[\mu],R}))$ are of degree zero because $U_R$ and hence $u_R$ act trivially on $F$-graded pieces. Since there is no nonzero $P_1$-morphism from any $P_1$-summand of $W_{w_2,[\mu],R} \otimes W_{w_1,[\mu],R}$ when $w_1, w_2 \in W^{M_1}$ satisfy $w_1 \neq w_2$ (which is the case when $l(w_1)$, $l(w_2)$), the differentials of $\text{Gr}_F(\text{BGG}_\ast(V_{[\mu],R}))$ (which are sums of morphisms between Verma modules) are sums of morphisms that are either zero or of positive degree (in the sense of Definition 4.6). By Lemma 4.23 (with $m_0 = 0$), this shows that the differentials of $\text{Gr}_F(\text{BGG}_\ast(V_{[\mu],R}))$ are all zero, as desired.

5.3. Dual BGG complexes.

Theorem 5.9. For any $\mu \in X_{G_1,R}^+$, there is a canonical $F$-filtered complex

$$\text{BGG}_\ast((V_{[\mu],R})_{\text{can}},$$

with trivial differentials on its $F$-graded pieces, such that

$$\text{Gr}_F(\text{BGG}_\ast((V_{[\mu],R})_{\text{can}})) \cong \bigoplus_{w \in W^{M_1}(m)} (W_{w,[\mu],R})_{\text{can}}$$

as $\mathcal{O}_{M_{\text{tor}},R}$-modules, together with a canonical quasi-isomorphic embedding

(5.10) $\text{Gr}_F(\text{BGG}_\ast((V_{[\mu],R})_{\text{can}})) \hookrightarrow \text{Gr}_F((V_{[\mu],R})_{\text{can}} \otimes \Omega_{M_{\text{tor}},R}^\ast / S_R)$

in the category of complexes of $\mathcal{O}_{M_{\text{tor}},R}$-modules, realizing the left-hand side as a summand of the right-hand side.

The embedding (5.10) is induced by taking $F$-graded pieces of a canonical $F$-filtered morphism

(5.11) $\text{BGG}_\ast((V_{[\mu],R})_{\text{can}}) \rightarrow (V_{[\mu],R})_{\text{can}} \otimes \Omega_{M_{\text{tor}},R}^\ast / S_R$

in the categories of complexes of sheaves of $R$-modules, with morphisms in each degree given by differential operators (rather than morphisms of $\mathcal{O}_{M_{\text{tor}},R}$-modules).

Proof. The existence of the complex $\text{BGG}_\ast((V_{[\mu],R})_{\text{can}})$ (with required properties) follows from Theorem 5.2 and Proposition 4.21 (Implicit in the condition of being $F$-filtered is that its differential and its Hodge filtration satisfy the Griffiths transversality, which is true because the action of $u_R$ on any $P_1$-module increases the $H$-weights by 1, as in the proof of Corollary 4.32).
The existence of the $F$-filtered morphisms \((5.10)\) and \((5.11)\) in the categories of complexes of sheaves of $R$-modules, with morphisms in each degree given by differential operators, also follows from Theorem 5.2 and Proposition 4.21, and from Proposition 4.31. The fact that \((5.10)\) is an embedding in the category of complexes of $\mathcal{O}_{\mathcal{M}^\infty_{\mathcal{H},R}}$-modules follows from the fact that $\text{Gr}_F(\mathcal{BGG}_{\mathcal{M}^\infty_{\mathcal{H},R}}(V_{[\mu],R}))$ is a quasi-isomorphic summand of degree zero of $\text{Gr}_F(\text{Std}_{\mathcal{M}^\infty_{\mathcal{H},R}}(V_{[\mu],R}))$ (see the last statement in Theorem 5.2).

\[ \text{Corollary 5.12. With the setting in Theorem 5.9 by setting} \]
\[ \mathcal{BGG}^*((V^\vee_{[\mu],R})^{\text{sub}}) := \mathcal{BGG}^*((V^\vee_{[\mu],R})^{\text{can}}) \otimes_{\mathcal{O}_{\mathcal{M}^\infty_{\mathcal{H},R}}} \mathcal{S}_D, \]

we obtain an $F$-filtered complex
\[ \mathcal{BGG}^*((V^\vee_{[\mu],R})^{\text{sub}}), \]
with trivial differentials on $F$-graded pieces, such that
\[ \text{Gr}_F(\mathcal{BGG}^a((V^\vee_{[\mu],R})^{\text{sub}})) \cong \bigoplus_{w \in Wk_{1}(a)} (W^\vee_{w-[\mu],R})^{\text{sub}} \]
as $\mathcal{O}_{\mathcal{M}^\infty_{\mathcal{H},R}}$-modules, together with a canonical quasi-isomorphic embedding
\[ \text{(5.13)} \quad \text{Gr}_F(\mathcal{BGG}^*((V^\vee_{[\mu],R})^{\text{sub}})) \hookrightarrow \text{Gr}_F((V^\vee_{[\mu],R})^{\text{sub}}) \otimes_{\mathcal{O}_{\mathcal{M}^\infty_{\mathcal{H},R}}} \mathcal{I}_{\mathcal{M}^\infty_{\mathcal{H},R}/S_R} \]
in the category of complexes of $\mathcal{O}_{\mathcal{M}^\infty_{\mathcal{H},R}}$-modules, realizing the left-hand side as a summand of the right-hand side.

The embedding \((5.13)\) is induced by taking $F$-graded pieces of a canonical $F$-filtered morphism
\[ \text{BGG}^*((V^\vee_{[\mu],R})^{\text{sub}}) \hookrightarrow (V^\vee_{[\mu],R})^{\text{sub}} \otimes_{\mathcal{O}_{\mathcal{M}^\infty_{\mathcal{H},R}}} \mathcal{I}_{\mathcal{M}^\infty_{\mathcal{H},R}/S_R} \]
in the categories of complexes of sheaves of $R$-modules, with morphisms in each degree given by differential operators (rather than morphisms of $\mathcal{O}_{\mathcal{M}^\infty_{\mathcal{H},R}}$-modules).

\[ \text{Proof. Simply apply} \quad \otimes \quad \mathcal{S}_D \text{ to all} \quad \mathcal{O}_{\mathcal{M}^\infty_{\mathcal{H},R}} \text{-modules in Theorem 5.9.} \quad \square \]

\[ \text{Remark 5.15. The canonical objects and morphisms in Theorem 5.9 and Corollary 5.12 are all functorial in } R. \text{ In fact, by the smoothness of } \mathcal{M}^\infty_{\mathcal{H},1} \rightarrow S_1, \text{ in order to prove Theorem 5.9 and Corollary 5.12 it suffices to treat the universal cases } R = R_1 \text{ (for } \mu \in X_{G_1}^-), \text{ and } R = R_1 \otimes \mathbb{Q} \text{ (for } \mu \in X_{G_1}, \text{ but } \mu \not\in X_{G_1}^+), \text{ which can happen only when the residue characteristics of } R \text{ are all zero).} \]

\[ \text{Remark 5.16. If we are in the modular curve case, and if } |\mu| \text{ is chosen such that } V^\vee_{[\mu],R} \cong H^1_{\text{dR}}(A/\mathcal{M}_{\mathcal{H},R}), \text{ then the degree zero term of (5.11) can be identified with a morphism} \]
\[ \text{Lie}_{A^\vee/\mathcal{M}_{\mathcal{H},R}} \rightarrow H^1_{\text{dR}}(A/\mathcal{M}_{\mathcal{H},R}) \]
of sheaves of $R$-modules. If the residue characteristics of $R$ are all zero, then we know that this is an embedding which splits the (relative) Hodge filtration \textit{globally}. This is a differential operator but not a morphism of $\mathcal{O}_{\mathcal{M}^\infty_{\mathcal{H},R}}$-modules.
5.4. Characteristic zero bases.

**Proposition 5.17.** If the residue characteristics of \( R \) are all zero, then the canonical morphisms \( [5.11] \) and \( [5.14] \) are (\( F \)-filtered) quasi-isomorphic embeddings (cf. \([9]\) Sec. 3 and \([10]\) Ch. VI, Sec. 5)).

**Proof.** Since all objects involved can be defined over a field extension of \( F_0 \) that can be embedded into \( \mathbb{C} \), we may assume that \( R = \mathbb{C} \). Then we can verify the statements of quasi-isomorphisms of sheaves after analytification. We shall denote the analytifications of objects over \( \mathbb{C} \) by the subscript “an”.

By the comparisons given in \([24]\) Sec. 4 and 5.2, the algebraic and analytic constructions are compatible with each other for the Shimura varieties and their toroidal compactifications, and for the automorphic bundles and their canonical extensions. Based on these objects, the algebraic and analytic constructions of differential operators from morphisms between Verma modules are also compatible with each other.

Then the proposition is known thanks to the same arguments in \([9]\) Sec. 3 and \([10]\) Ch. VI, Sec. 5:

Over \( M_{H,an} \), which is a finite union of arithmetic quotients of Hermitian symmetric spaces, we know (by \([9]\) Sec. 3 and 7), by verifying the statements over each connected components) that the canonical morphism

\[
\text{BGG}^\bullet(V_{\mu,an}^\vee) \to \text{BGG}^\bullet(\mathcal{O}_{H,an}^\vee) \oplus \Omega_M^\bullet(\mathcal{O}_{M,an}^\vee)
\]

is a quasi-isomorphic embedding realizing the left-hand side as a summand of the right-hand side, and both sides give resolutions of the same local system \( V_{\mu,an}^\vee \). Based on these objects, the algebraic and analytic constructions of resolutions in general.)

According to \([1]\) Ch. III, Sec. 5, Main Thm. I and its proof\], the connected local charts of the toroidal compactifications about a boundary divisor can be given...
by partial toroidal embeddings of punctured polydisk bundles with fundamental groups canonically identified with a discrete subgroup of the unipotent radical of some maximal parabolic subgroup of $G \otimes \mathbb{Z}$. (The unipotent radical depends on the boundary divisor in question.) Since the local system $V^\vee_{[\mu], \text{an}}$ is defined over $M_{H, \text{an}}$ using the action of the same group $G \otimes \mathbb{Z}$, the monodromy transformations along irreducible components of the boundary divisor $D$ are all unipotent. (The automorphic bundles $V^\vee_{[\mu], \text{an}}$ can be realized as summands of the relative de Rham cohomology of the log smooth morphisms from toroidal compactifications of Kuga families to $M_{H, \text{an}}$. By [19, VII], this implies that the eigenvalues of the residue maps of the Gauss–Manin connections along irreducible components of $D$ are non-negative rational numbers strictly less than one. Therefore, by [19, VI], this shows that the condition that the monodromy transformations of $V^\vee_{[\mu], \text{an}}$ are unipotent are equivalent to the condition that the Gauss–Manin connections of $V^\vee_{[\mu], \text{an}}$ are nilpotent.)

Hence, since $V^\vee_{[\mu], \text{an}} \otimes \Omega^\bullet_{M_{H, \text{an}}}$ is a resolution of $V^\vee_{[\mu], \text{Betti}}$, as explained in [10, Ch. VI, Prop. 5.4], by local calculations and by reducing to the one variable case using Künneth, we know that (5.18) and (5.19) are indeed quasi-isomorphisms, as desired. (The unipotence of the monodromy is used in the standard argument reducing these statements to the trivial coefficient case; see for example [6, II, Lem. 6.9].) □

**Remark 5.20.** Suppose any residue characteristic of $R$ is $p > 0$. Then differential operators behave pathologically in general, and the left-hand sides of the morphisms (5.11) and (5.14) might fail to be summands of the right-hand sides for trivial reasons (because the differential operators may annihilate too many elements). This can be salvaged by introducing the language of divided powers, which are quite natural because log crystals (realized as coherent sheaves with log HPD stratifications) can be canonically attached to the generalized Verma modules. This has been studied in the Siegel case in [31, Sec. 4–5]. However, since this has not been needed in the applications we have in mind, we shall not carry this out in this article.

### 5.5. Decomposition of (log) Hodge cohomology.

Variants of the following consequence of the canonical quasi-isomorphic embeddings (5.10) and (5.13) on the $F$-graded pieces (without any reference to (5.11) and (5.14)) suffice in most applications we know (including those in [31] and subsequent works using patterns of Hodge–Tate weights):

**Corollary 5.21.** For any $\mu \in X^\times_{G_{1,R}}$, we have a canonical isomorphism

$$H^{a+b}(M^a_{H,R}, \text{Gr}^0_{H,R}(V^\vee_{[\mu], R})^\text{can} \otimes \Omega^\bullet_{M^a_{H,R}/S_R})$$

(5.22)

$$\cong \bigoplus_{\omega \in \mathcal{W}_{M^1}} H^{a+b-l(\omega)}(M^a_{H,R}, \text{Gr}^0_{H,R}(V^\vee_{[\mu \omega], R})^\text{can}).$$

The same is true if we replace $(V^\vee_{[\mu], R})^\text{can}$ with $(V^\vee_{[\mu], R})^\text{sub}$.

The upshot is that the left-hand side of (5.22) is a hypercohomology of complexes of sheaves, while the right-hand side is a direct sum of cohomology of sheaves. In practice, the right-hand side can be much easier to study.
Proof of Corollary 5.21. This is because of the quasi-isomorphisms (5.10) and (5.13), and because the left-hand sides of them have trivial differentials. □

Remark 5.23. Applications of (5.10) and (5.13) to the study of torsion in the cohomology of PEL-type Shimura varieties can be found in the joint work of the first author and Junecue Suh. (See [27] and [28].)

Remark 5.24. If the residue characteristics of $R$ are all zero, then it is known that the Hodge spectral sequences

$$E_1^{a,b} := H^{a+b}(M_{H,R}^\text{tor}, Gr_F^d((V_{[\mu]}^\psi, R)^\text{can}) \otimes \Omega^{M_{H,R}^\text{tor} / S_R})$$

(5.25)

$$\Rightarrow H^{a+b}(M_{H,R}^\text{tor}, (V_{[\mu]}^\psi, R)^\text{can}) \otimes \Omega^{M_{H,R}^\text{tor} / S_R})$$

and

$$E_1^{a,b} := H^{a+b}(M_{H,R}^\text{tor}, Gr_F^d((V_{[\mu]}^\psi, R)^\text{sub}) \otimes \Omega^{M_{H,R}^\text{tor} / S_R})$$

(5.26)

$$\Rightarrow H^{a+b}(M_{H,R}^\text{tor}, (V_{[\mu]}^\psi, R)^\text{can})$$

degenerate at their $E_1$ terms. As a result, being defined by $F$-filtered quasi-isomorphic summands, the dual BGG versions of the Hodge spectral sequences

$$E_1^{a,b} := H^{a+b}(M_{H,R}^\text{tor}, Gr_F^d(BGG^*((V_{[\mu]}^\psi, R)^\text{can})))$$

(5.27)

$$\cong \bigoplus_{w \in W_{M_1}} H^{a+b-l(w)}(M_{H,R}^\text{tor}, Gr_F^d((W_{w, [\mu]}^\psi, R)^\text{can}))$$

$$\Rightarrow H^{a+b}(M_{H,R}^\text{tor}, BGG^*((V_{[\mu]}^\psi, R)^\text{can}))$$

and

$$E_1^{a,b} := H^{a+b}(M_{H,R}^\text{tor}, Gr_F^d(BGG^*((V_{[\mu]}^\psi, R)^\text{sub})))$$

(5.28)

$$\cong \bigoplus_{w \in W_{M_1}} H^{a+b-l(w)}(M_{H,R}^\text{tor}, Gr_F^d((W_{w, [\mu]}^\psi, R)^\text{sub}))$$

$$\Rightarrow H^{a+b}(M_{H,R}^\text{tor}, BGG^*((V_{[\mu]}^\psi, R)^\text{sub}))$$

do not degenerate at their $E_1$ terms. These can be proved by first reducing to the case $R = \mathbb{C}$, and by realizing both sides of these spectral sequences as summands of the corresponding Hodge spectral sequences of the toroidal compactifications of Kuga families (or certain mixed Shimura varieties) with trivial coefficients. (See [10] Ch. IV, Sec. 1–2 and [24] for the algebraic construction of toroidal compactifications of PEL-type Kuga families, and see [32] for the analytic construction of toroidal compactifications of mixed Shimura varieties.) Alternatively, one may resort to Saito’s theory of mixed Hodge modules (see [34]). (See [10] Ch. VI, p. 234 and [17] Cor. 4.2.3 for the methods of proving the degeneration.)

Remark 5.29. Suppose $R = \mathbb{C}$. As explained in the proof of Proposition 5.17, the right-hand sides of (5.25) and (5.27) (resp. (5.26) and (5.28)) are canonically isomorphic to $H^{a+b}(M_{H,\text{an}}, \tilde{V}_{[\mu]}^\psi, \text{Betti})$ (resp. $H^{a+b}_c(M_{H,\text{an}}, \tilde{V}_{[\mu]}^\psi, \text{Betti})$). As a result, the left-hand side of (5.22) (resp. the analogue for $(\tilde{V}_{[\mu]}^\psi, \text{Betti})$ gives the Hodge graded pieces of the cohomology of $H^{a+b}(M_{H,\text{an}}, \tilde{V}_{[\mu]}^\psi, \text{Betti})$ (resp. $H^{a+b}_c(M_{H,\text{an}}, \tilde{V}_{[\mu]}^\psi, \text{Betti})$).
5.6. Descending the BGG and dual BGG complexes. Up to modifying the choices of $\mathcal{O}_{F_0,(p)}$ and $R_1$ in Section 2.1, consider any $\mathcal{O}_{F_0,(p)}$-algebra $R'$ and any $R_1$-algebra $R$ satisfying the following:

**Assumption 5.30.** There exist a faithfully flat homomorphism $R' \to R$, and an $\mathcal{O} \otimes R'$-module $L_0'$, locally free over $R'$, such that $L_0' \otimes R \cong L_{0,R_1}$.

**Remark 5.31.** The upshot is that $R'$ does not have to satisfy Condition 2.5 as $R_1$ does. (It does not even have to be an $\mathcal{O}_{F_0,(p)}$-algebra for some $F_0$.)

**Remark 5.32.** If $\mathcal{O} \otimes F_0$ is a product of matrix algebras over fields (e.g., if $\mathcal{O} \otimes \mathbb{Q}$ is a field), then, up to modifying the choices of $\mathcal{O}_{F_0,(p)}$ and $R_1$ in Section 2.1, $R'$ can be taken to be any $\mathcal{O}_{F_0,(p)}$-algebra.

Let us denote by $\iota_1, \iota_2 : R \to R^{(2)} := R \otimes R$ the two natural homomorphisms. By the theory of descent, the category of $R'$-modules is equivalent to the category of $R$-modules with descent data. Namely, the datum of an $R'$-module $M'$ is equivalent to the datum of a pair $(M, \delta_M)$, where $M$ is an $R$-module, and where $\delta_M : M \otimes R^{(2)} \to M \otimes R^{(2)}$ is an isomorphism of $R \otimes R'$-modules such that the three pullbacks of $\delta_M$ to $R^{(3)} := R \otimes R \otimes R$ satisfy the usual cocycle condition.

By Assumption 5.30, and by imitating Definition 2.4 with $\mathcal{O}_{F_0,(p)}$ and $L_0$ replaced with $R'$ and $L_0'$, we obtain group functors $G', P'$, and $M'$, together with the canonical morphisms among them, such that their base changes from $R'$ to $R$ are compatibly isomorphic to $G_1 \otimes R_1, P_1 \otimes R_1, R_1 \otimes R_1$, and $M_1 \otimes R_1$, and $M' \otimes R \cong M_1 \otimes R$, and suppress them in what follows. We shall consider objects and morphisms canonical if they canonically depend on these fixed choices.

Then we can define the categories $\text{Rep}_{R'}(G'), \text{Rep}_{R'}(P')$, and $\text{Rep}_{R'}(M')$ as in Definition 2.21. Moreover, we can define the analogues over $R'$ of all the objects and maps in Sections 2.2–2.4 with the superscript “1” replaced by a prime in the notation system. For example, for $W' \in \text{Rep}_{R'}(G')$, we can define the de Rham complex

$$(\mathcal{E}_{G',R'}^*(W') \otimes \Omega^*_{M,H,R'} / S_{R'}, \nabla)$$

and the log de Rham complexes

$$(\mathcal{E}_{G',R'}^\text{can}(W') \otimes \Omega^*_{M,H,R'} / S_{R'}, \nabla)$$

and

$$(\mathcal{E}_{G',R'}^\text{sub}(W') \otimes \Omega^*_{M,H,R'} / S_{R'}, \nabla)$$

as in Definition 2.38. We can also define Hodge filtrations as in Section 3.3. Apart from the remark in the second paragraph of Section 4.1, the construction of differential operators in Section 3 works verbatim when $U(g'')$-$P_1$-modules are replaced with $U(g')$-$P'$-modules. The pullbacks of these objects from $R'$ to $R$ all carry descent data in the same way as $R'$-modules (because they are all given by complexes of sheaves of $R'$-modules).
Suppose \( V' \in \text{Rep}_{R'}(G') \), and suppose \( V := V' \otimes_{R'} R \) decomposes as
\[
V \cong \bigoplus_{i \in I_V} V_{\mu_i}, R,
\]
where \( I_V \) is an index set, and where \( \mu_i \in X_i^{r_1, <p} \) for each \( i \in I_V \) and each \( \mu_i \in [\mu_i] \).

By Definition \ref{def:4.26}, we have the canonical \( F \)-filtered complex of \( \text{U}(g')-P' \)-modules \( \text{Std}_\bullet(V') \) (resp. of \( \text{U}(g_R)-P_1 \)-modules \( \text{Std}_\bullet(V) \)), and \( \text{Std}_\bullet(V) \) carries the descent isomorphism \( \delta_{\text{Std}_\bullet(V)} : \text{Std}_\bullet(V) \otimes_{R, t_1} R^{(2)} \sim \text{Std}_\bullet(V) \otimes_{R, t_2} R^{(2)} \) canonically identifying \( \text{Std}_\bullet(V) \) as the pullback of \( \text{Std}_\bullet(V') \) from \( R' \) to \( R \).

On the other hand, we define (according to \ref{5.33}) the \( F \)-filtered complex of \( \text{U}(g_R)-P_1 \)-modules
\[
\text{BGG}_\bullet(V) := \bigoplus_{i \in I_V} \text{BGG}_\bullet(V_{\mu_i}, R).
\]

By applying Theorem \ref{5.2} to \( \mu_i \) for each \( i \in I_V \), we see that \( \text{BGG}_\bullet(V) \) is canonically \( F \)-filtered quasi-isomorphically embedded as a summand of \( \text{Std}_\bullet(V) \) in the category of \( F \)-filtered complexes of \( \text{U}(g_R)-P_1 \)-modules. Moreover, the induced complex \( \text{Gr}_R(\text{BGG}_\bullet(V)) \) is a canonical (quasi-isomorphic) summand of degree zero of \( \text{Gr}_R(\text{Std}_\bullet(V)) \) with trivial differentials. (The embedding of \( \text{BGG}_\bullet(V) \) as an \( F \)-filtered summand of \( \text{Std}_\bullet(V) \) is canonically determined by the actions of \( \text{U}(g_R) \) and \( P_1 \)). Analogues of these statements remain true when we replace \( R \) with any \( R \)-algebra.

**Theorem 5.35.** With the setting as above, there exists an \( F \)-filtered complex of \( \text{U}(g')-P' \)-modules
\[
0 \to \text{BGG}_a(V') \otimes_{R'} R \cong \text{BGG}_a(V') \otimes \bigoplus_{i \in I_V} \left( \bigoplus_{w \in W, [\mu_i]} \text{Verm}(W_{w, [\mu_i]}, R) \right)
\]
(as \( \text{U}(g_R)-P_1 \)-submodules) for each \( 0 \leq a \leq n \). Moreover, the induced complex
\[
\text{Gr}_R(\text{BGG}_\bullet(V'))
\]
is a canonical (quasi-isomorphic) summand of degree zero of \( \text{Gr}_R(\text{Std}_\bullet(V')) \) (see Definition \ref{5.1} with trivial differentials.

**Proof.** Since \( \text{U}(g_R) \) and \( P_1 \) have models over \( R' \), the descent isomorphism
\[
\delta_{\text{Std}_\bullet(V)} : \text{Std}_\bullet(V) \otimes_{R, t_1} R^{(2)} \sim \text{Std}_\bullet(V) \otimes_{R, t_2} R^{(2)}
\]
is compatible with the pullbacks of the actions of \( \text{U}(g_R) \) and \( P_1 \) under \( t_1 \) and \( t_2 \) (on the two sides). Since these actions determine the canonical summand \( \text{BGG}_\bullet(V) \otimes_{R, t_1} R^{(2)} \) of \( \text{Std}_\bullet(V) \otimes_{R, t_1} R^{(2)} \) and the corresponding canonical summand \( \text{BGG}_\bullet(V) \otimes_{R, t_2} R^{(2)} \) of \( \text{Std}_\bullet(V) \otimes_{R, t_2} R^{(2)} \), the isomorphism \( \delta_{\text{Std}_\bullet(V)} \) induces a descent isomorphism
\[
\delta_{\text{BGG}_\bullet(V)} : \text{BGG}_\bullet(V) \otimes_{R, t_1} R^{(2)} \sim \text{BGG}_\bullet(V) \otimes_{R, t_2} R^{(2)},
\]
whose three pullbacks to \( R^{(3)} \) satisfy the usual cocycle condition. Thus, the \( F \)-filtered summand \( BGG_a(V') \) of \( \text{Std}_\bullet(V) \) descends to an \( F \)-filtered summand of \( \text{Std}_\bullet(V') \), which is the desired complex \( BGG_\bullet(V') \).

Let \( V' := E_{G', R}(V') \) and \( V'^\vee := E_{G', R}(V'^\vee) \). (The notation makes sense because they are canonically dual to each other.) Let \( (V')^{\text{can}} := E_{G', R}^\text{can}(V') \), and \( (V')^{\text{sub}} := E_{G', R}^\text{sub}(V') \). (Then \( (V')^{\text{can}} \) and \( (V')^{\text{can}} \) are canonically dual to each other.) By Theorem 5.35, since the construction of differential operators in Section 4 works verbatim with \( U(g_R)-P_1 \)-modules replaced with \( U(g'_R)-P'_1 \)-modules, the same proofs of Theorem 5.9 and Corollary 5.12 give the following:

**Theorem 5.38** (cf. Theorem 5.9 and Corollary 5.12). With the setting as above, there is a canonical \( F \)-filtered complex

\[
BGG^\bullet((V'^\vee)^\text{can}),
\]

with trivial differentials on its \( F \)-graded pieces, such that

\[
\text{Gr}_F(BGG^a((V'^\vee)^\text{can})) \otimes R' \cong \text{Gr}_F(BGG^a((V'^\vee)^\text{can}))
\]

as \( \mathcal{O}_{M^K_{\mathbb{A}, \mathbb{R}}} \)-modules, together with a canonical quasi-isomorphic embedding

\[
\text{Gr}_F(BGG^\bullet((V'^\vee)^\text{can})) \hookrightarrow \text{Gr}_F((V'^\vee)^\text{can}) \otimes_{\mathcal{O}_{M^K_{\mathbb{A}, \mathbb{R}}}} \mathcal{O}_{M^K_{\mathbb{A}, \mathbb{R}}}/S_{R'}
\]

in the category of complexes of \( \mathcal{O}_{M^K_{\mathbb{A}, \mathbb{R}}} \)-modules, realizing the left-hand side as a summand of the right-hand side.

The embedding (5.40) is induced by taking \( F \)-graded pieces of a canonical \( F \)-filtered morphism

\[
BGG^\bullet((V'^\vee)^\text{can}) \to (V'^\vee)^\text{can} \otimes_{\mathcal{O}_{M^K_{\mathbb{A}, \mathbb{R}}}} \mathcal{O}_{M^K_{\mathbb{A}, \mathbb{R}}}/S_{R'}
\]

in the categories of complexes of sheaves of \( R' \)-modules, with morphisms in each degree given by differential operators (rather than morphisms of \( \mathcal{O}_{M^K_{\mathbb{A}, \mathbb{R}}} \)-modules).

By setting

\[
BGG^\bullet((V'^\vee)^\text{sub}) := BGG^\bullet((V'^\vee)^\text{can}) \otimes \mathcal{I}_D,
\]

we obtain an \( F \)-filtered complex

\[
BGG^\bullet((V'^\vee)^\text{sub}),
\]

with trivial differentials on \( F \)-graded pieces, and with other properties similar to the above (with canonical extensions replaced with subcanonical extensions).

Since Proposition 5.17 was proved by working over \( \mathbb{C} \), we also have:

**Proposition 5.42** (cf. Proposition 5.17). If the residue characteristics of \( R' \) are all zero, then the canonical morphisms (5.41) and its subcanonical analogue are (\( F \)-filtered) quasi-isomorphic embeddings.
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