LOGARITHMIC RIEHMANN–HILBERT CORRESPONDENCES
FOR RIGID VARIETIES

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ABSTRACT. On any smooth algebraic variety over a $p$-adic local field, we construct a tensor functor from the category of de Rham $p$-adic étale local systems to the category of filtered algebraic vector bundles with an integrable connection satisfying the Griffiths transversality, which we view as a $p$-adic analogue of Deligne’s classical Riemann–Hilbert correspondence. The key intermediate step is to construct extensions of the desired connections to suitable compactifications of the algebraic variety with logarithmic poles along the boundary; hence the title of the paper. As an application, we show that this $p$-adic Riemann–Hilbert functor is compatible with the classical one over all Shimura varieties, for local systems attached to representations of the associated reductive algebraic groups.

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1. Introduction

Let $X$ be a connected smooth complex algebraic variety, $X^{\text{an}}$ the associated analytic space and $X^{\text{top}}$ the underlying topological space. The classical Riemann–Hilbert correspondence establishes (tensor) equivalences among the following:

- the tensor category of finite dimensional complex representations of the fundamental group of $X^{\text{top}}$ (with any fixed choice of a based point), which by a well-known topological construction, is equivalent to the tensor category of local systems (i.e., locally constant sheaves) of finite dimensional $\mathbb{C}$-vector spaces on $X^{\text{top}}$;
- the tensor category of vector bundles of finite rank with an integrable connection on $X^{\text{an}}$; and
- the tensor category of vector bundles of finite rank with an integrable connection on $X$, with regular singularities at infinity (which we shall simply call regular integrable connections, in what follows).

The equivalence of the first and second categories is a simple consequence of the Frobenius theorem: Let $L$ be a local system on $X^{\text{top}}$. Then the associated vector bundle with an integrable connection is $(L \otimes_{\mathbb{C}} O_{X^{\text{an}}}, 1 \otimes d)$; and conversely, for a vector bundle with an integrable connection on $X^{\text{an}}$, its sheaf of horizontal sections is a local system on $X^{\text{top}}$. The equivalence of the second and third categories, however, is a deep theorem due to Deligne [Del70].

An analogous Riemann–Hilbert correspondence for varieties over a $p$-adic field is long desired but remains rather mysterious until recently. The situation is far more complicated. Let $X$ be a smooth algebraic variety over a finite extension of $\mathbb{Q}_p$. In this setting, the second and the third categories remain meaningful, and it is natural to replace the first category with the category of $p$-adic étale local systems on $X$ (or even on $X^{\text{an}}$). However, after this replacement, one cannot expect an equivalence between the first and the second categories, as can already be seen when $X$ is a point. Moreover, in general, the natural analytification functor from the third to the second categories is not an equivalence either. Nevertheless, one of the main goals of this paper is to prove the following result (as one step towards the $p$-adic Riemann–Hilbert correspondence):
Theorem 1.1. Let $X$ be a smooth algebraic variety over a finite extension $k$ of $\mathbb{Q}_p$. Then there is a tensor functor $D^\text{alg}_{\text{dR}}$ from the category of de Rham $p$-adic étale local systems $L$ on $X$ to the category of algebraic vector bundles on $X$ with regular integrable connections and decreasing filtrations satisfying the Griffiths transversality. In addition, there is a canonical comparison isomorphism

$$H^i_{\text{ét}}(X_k, L) \otimes_{\mathbb{Q}_p} B_{\text{dR}} \cong H^i_{\text{dr}}(X, D^\text{alg}_{\text{dR}}(L)) \otimes_k B_{\text{dR}}$$

compatible with the canonical filtrations and the actions of $\text{Gal}(k/k)$ on both sides.

Here $B_{\text{dR}}$ is Fontaine’s $p$-adic period ring, and $H_{\text{dr}}$ is the algebraic de Rham cohomology. Note that the comparison isomorphism in Theorem 1.1 implies that $H^i_{\text{ét}}(X_k, L)$ is a de Rham representation of $\text{Gal}(k/k)$, although we will not repeat such a remark later. The notion of de Rham $p$-adic étale local systems was first introduced by Scholze in [Sch13, Definition 8.3] (generalizing earlier work of Brinon [Bri08]) using some relative de Rham period sheaf. However, it turns out that this notion satisfies a rather surprising rigidity property: by [LZ17, Theorem 3.9], a $p$-adic étale local system $L$ on $X^{\text{an}}$ is de Rham if and only if, on each connected component of $X$, there exists some classical point $x$ such that, for some (and hence every) geometric point $\mathfrak{x}$ over $x$, the corresponding $p$-adic representation $L_{\mathfrak{x}}$ of the absolute Galois group of the residue field of $x$ is de Rham in the classical sense. In this situation, it follows that the same is also true at every classical point $x$ of $X$.

Note that the functor denoted by $D_{\text{dr}}$ in [LZ17, Theorem 3.9] is the composition of the functor $D^\text{alg}_{\text{dR}}$ in Theorem 1.1 with the analytification functor.

Theorem 1.1 includes, in particular, a new de Rham comparison isomorphism for smooth open algebraic varieties over $k$ with nontrivial coefficients. (In fact, the de Rham comparison isomorphism holds in a more general rigid analytic setting—see Corollary 5.23.) The de Rham comparison for smooth varieties over $p$-adic fields has a long history and has been proved (and shown to satisfy several natural compatibilities) in various generalities by different methods. When the variety is proper, the trivial coefficient case is contained in the works of Faltings [Fal89, Fal02], Tsuji [Tsu99], Nizioł [Niz08, Niz09], and Beilinson [Bei12] (with inputs from Bloch, Fontaine, Hyodo, Kato, Kurihara, and Messing, among others), and the nontrivial coefficient case is treated by Faltings (in the works cited above) under certain assumptions on integral models of the variety and by Scholze [Sch13, Sch10] in full generality (with inputs from Faltings, Brinon [Bri08], and Andreatta–Iovita [AI13]). For open varieties, the trivial coefficient case is contained in the same works of Faltings and Beilinson, and is also treated by Yamashita [Yam10, Yam11], Colmez–Nizioł [CN17], and most recently Li–Pan [LP18]. However, to the best of our knowledge, only Faltings has considered (in the above-mentioned works) certain nontrivial coefficients over open varieties with integral models admitting nice compactifications. Our approach to the de Rham comparison follows Scholze’s and works for general de Rham local systems over arbitrary smooth open varieties.

We shall call the functor $D^\text{alg}_{\text{dR}}$ in Theorem 1.1 the (algebraic) $p$-adic Riemann–Hilbert functor. It is natural to ask whether this functor is compatible with Deligne’s classical Riemann–Hilbert correspondence (with images in the tensor category of regular integrable connections) in a suitable sense. We shall formulate our expectation in the Conjecture 1.2 below. Let us start with some preparations.

Let $X$ be a smooth algebraic variety over a number field $E$. We fix an isomorphism $\iota: \overline{\mathbb{Q}_p} \sim \mathbb{C}$ and a field homomorphism $\sigma: E \to \mathbb{C}$, and write $\sigma X = X \otimes_{E, \sigma} \mathbb{C}$.
There is a tensor functor from the category of $p$-adic étale local systems on $X$ to the category of regular integrable connections on $\sigma X$ as follows. Note that $L|_{\sigma X}$ is an étale local system on $\sigma X$, corresponding to a $p$-adic representation of the étale fundamental group of each connected component of $\sigma X$, which is the profinite completion of the fundamental group of the corresponding connected component of $(\sigma X)^{\text{top}}$. It follows that $L|_{\sigma X} \otimes_{Q_p} \mathbb{C}$ can be regarded as a classical local system on $(\sigma X)^{\text{top}}$, denoted by $L_{\sigma}$. Then the classical Riemann–Hilbert correspondence produces a regular integrable connection on $\sigma X$.

On the other hand, the composition $E \xrightarrow{\sigma} \mathbb{C} \xrightarrow{\iota_p} \overline{\mathbb{Q}}_p$ determines a $p$-adic place $v$. Let $E_0$ be the completion of $E$ with respect to $v$, and assume that $L|_{X_{E_0}}$ is de Rham. Then we obtain $D^\text{alg}_{\text{dR}}(L|_{X_{E_0}}) \otimes_{E_0} \mathbb{C}$, which is another regular integrable connection, with an additional decreasing filtration $\text{Fil}^*$ satisfying the Griffiths transversality. We would like to compare the above two constructions. In order to do so, we need to impose a further restriction on $L$.

We say that $L$ is geometric if, for every closed point $x$ of $X$ and any geometric point $\overline{x}$ over $x$, the corresponding $p$-adic representation $L_{\overline{x}}$ of the absolute Galois group of the residue field of $x$ is geometric in the sense of Fontaine–Mazur (see [FM97, Part I, §1]). Note that geometric $p$-adic étale local systems on $X$ form a full tensor subcategory of the category of all étale local systems. If $L$ is geometric, then $L|_{X_{E_0}}$ is de Rham (by [LZ17, Theorem 3.9], as explained above). In addition, by [LZ17, Theorem 1.1], $L$ is geometric if and only if, on each connected component, there is a closed point such that the stalk of $L$ at any geometric point above this point is a geometric $p$-adic representation in the sense of Fontaine–Mazur.

**Conjecture 1.2.** The above two tensor functors from the category of geometric $p$-adic étale local systems on $X$ to the category of regular integrable connections on $\sigma X$ are canonically isomorphic. In addition, $(D^\text{alg}_{\text{dR}}(L|_{X_{E_0}}) \otimes_{E_0} \mathbb{C}, \text{Fil}^*)$ is a complex variation of Hodge structures.

Unsurprisingly, the motivation behind this conjecture is the theory of motives. Note that this conjecture is closely related to a relative version of the Fontaine–Mazur conjecture proposed in the introduction of [LZ17], but it might be more approachable because it is stated purely in terms of sheaves (which can be interpreted as realizations of some families of motives). Even so, it seems to be out of reach at the moment. Nevertheless, in the case of Shimura varieties, we can partially verify this conjecture. Let $(G, X)$ be a Shimura datum, $K \subset G(\mathbb{A}_f)$ a neat open compact subgroup, and $\text{Sh}_K = \text{Sh}_K(G, X)$ the corresponding Shimura variety, defined over the reflex field $E = E(G, X)$. Let $G^c$ be the quotient of $G$ by the maximal subtorus of the center of $G$ that is $\mathbb{Q}$-anisotropic but $\mathbb{R}$-split. Recall that there is a tensor functor from the category $\text{Rep}_{Q_p}(G^c)$ of algebraic representations of $G^c$ over $Q_p$ to the category of $p$-adic étale local systems on $\text{Sh}_K$ (see, for example, [LS17, Section 3] or [LZ17, Section 4.2]), whose essential image consists of only geometric $p$-adic étale local systems (see [LZ17, Theorem 1.2]).

**Theorem 1.3.** The conjecture holds for the $(p$-adic) étale local systems on $\text{Sh}_K$ coming from $\text{Rep}_{Q_p}(G^c)$ as above.

Note that, in Theorem 1.3, we do not need to assume that the Shimura variety is of abelian type, so these étale local systems are not (yet) known to be related to motives in general. A crucial ingredient in our proof is Margulis’s superrigidity theorem.
The argument is, therefore, rather indirect and unrelated to \( p \)-adic analytic geometry. But this is certainly not the first time that Margulis’s results on arithmetic groups played an important role in the theory of general Shimura varieties—they already did in the proof of the existence of canonical models.

As already mentioned, Theorem 1.1 also implies that the étale cohomology of these local systems are de Rham. Then Theorem 1.3, together with some other results of this paper, can be also used to determine the Hodge–Tate weights of such cohomology (see Section 7.8).

Now we shall explain our strategy to prove Theorem 1.1. As mentioned above, in \([LZ17, \text{Theorem 3.9}]\), a tensor functor \( D_{\text{an}} \text{dR} \) was constructed from the category of de Rham \( p \)-adic étale local systems on \( X_{\text{an}} \) to the category of filtered vector bundles on \( X_{\text{an}} \) with an integrable connection and a decreasing filtration satisfying the Griffiths transversality. In order to prove Theorem 1.1, a natural idea is to fix a smooth compactification \( \overline{X} \) of \( X \) such that the boundary \( D = \overline{X} - X \) (with its reduced subscheme structure) is a normal crossings divisor, and extend the filtered vector bundles with integrable connections in loc. cit. to filtered vector bundles on \( \overline{X}_{\text{an}} \) with integrable log connections (i.e., connections with log poles along the boundary divisor \( D \)). However, rather unlike the complex analytic situation in \( \text{[Del70]} \), not every integrable connection on \( X_{\text{an}} \) is extendable (see \( \text{[AB01, Chapter 4, Remark 6.8.3]} \) for some counter-example).

Instead, we proceed in another way to directly construct a functor from the category of de Rham \( p \)-adic étale local systems on \( X_{\text{an}} \) to the category of filtered vector bundles with integrable log connections on \( \overline{X}_{\text{an}} \). We shall start in the realm of logarithmic analytic geometry, and construct a log Riemann–Hilbert correspondence, which is a key step in this paper. Note that, in the classical situation over \( \mathbb{C} \), Illusie–Kato–Nakayama \([IKN05, IKN07]\) developed a theory of quasi-unipotent log Riemann–Hilbert correspondence, which as a byproduct established a correspondence between some subcategory of local systems and some subcategory of regular integrable connections, without using Deligne’s theory of canonical extensions.

We will work more generally over a smooth rigid analytic variety \( Y \) over \( k \) (viewed as an adic space over \( \text{Spa}(k, \mathcal{O}_k) \)), together with a normal crossings divisor \( D \subset Y \), and view \( Y \) as a log adic space by equipping it with the natural log structure defined by \( D \). (For applications to our previous setup, we take \( Y = \overline{X}_{\text{an}} \), and take \( D \) to be the analytification of the above boundary divisor \( \overline{X} - X \) with its reduced subscheme structure.) Important ingredients for our construction include the following:

- The Kummer étale and pro-Kummer étale topologies (see Sections 2.3 and 2.4), part of which were developed in \([Dia18]\).
- The log de Rham period sheaf \( \mathcal{O}_{\text{B\_dR, log}} \) (see Section 3), generalizing the de Rham period sheaf \( \mathcal{O}_{\text{B\_dR}} \) in \([Bri08, \text{Section 5}]\), \([Sch13, \text{Section 6}]\), and \([Sch16]\).
- A formalism of decompleting pairs and decompletion systems (see Section 4), generalizing the one introduced in \([KL16, \text{Section 5}]\).
- The calculation of eigenvalues of residues of log connections along irreducible components of boundary divisors (see Section 5.3).

We now explain our construction in more details. Every Kummer étale \( \mathbb{Z}_p \)-local system \( \mathbb{L} \) on \( Y \) induced a \( \mathbb{Z}_p \)-local system \( \mathbb{L} \) on \( Y_{\text{prokét}} \), roughly speaking via pullback (see Definition 2.84 for the precise construction). Let \( \mu : Y_{\text{prokét}} \to Y_{\text{an}} \) denote the natural projection from the pro-Kummer étale site of \( Y \) to the analytic site of \( Y \).
(Note that as a subscript “an” means the analytic site, while as a superscript it means the analytification of an a priori algebraic object.) Let $\Omega^1_{Y}(\log (D))$ denote the sheaf of differentials with log poles along $D$, as usual. The following theorem is an abbreviated version of Theorem 3.9.

**Theorem 1.4.** Let $Y$ and $\mu$ be as above. Consider the functor $D_{\text{dR,log}}$, which sends a Kummer étale $\mathbb{Z}_p$-local system $L$ on $Y$ to
\[
D_{\text{dR,log}}(L) := \mu_* (\widehat{L} \otimes_{\mathbb{Z}_p} \mathcal{O}_{D_{\text{dR,log}}}).
\]
Then $D_{\text{dR,log}}(L)$ is a vector bundle on $Y_{\text{an}}$ naturally equipped with an integrable log connection
\[
\nabla_L : D_{\text{dR,log}}(L) \to D_{\text{dR,log}}(L) \otimes_{\mathcal{O}_Y} \Omega^1_Y (\log (D))
\]
and a decreasing filtration (by coherent subsheaves) satisfying the Griffiths transversality, which extends the vector bundle $D_{\text{dR}}(L)$ with its integrable connection (also denoted by $\nabla_L$) considered in [LZ17, Theorem 3.9]. In addition, the eigenvalues of the residues of $D_{\text{dR,log}}(L)$ along the irreducible components of $D$ are rational numbers in $[0, 1]$. In particular, $(D_{\text{dR,log}}(L), \nabla_L)$ is the canonical extension of the $(D_{\text{dR}}(L), \nabla_L)$; i.e., the unique (if existent) extension of $(D_{\text{dR}}(L), \nabla_L)$ with such eigenvalues of residues (see the discussion in [AB01, Chapter 1, Section 4]).

When $L|_{Y - D}$ is a de Rham étale $\mathbb{Z}_p$-local system, $\text{gr} D_{\text{dR,log}}(L)$ is a vector bundle on $Y_{\text{an}}$ of rank $\text{rk}_{\mathbb{Z}_p}(L)$. Moreover, there are canonical isomorphisms
\[
\text{H}^i_{\text{ét}}(Y, L) \otimes_{\mathbb{Z}_p} B_{\text{dR}} \cong \text{H}^i_{\text{log dR}}(Y_{\text{an}}, D_{\text{dR,log}}(L)) \otimes_k B_{\text{dR}}
\]
and
\[
\text{H}^i_{\text{ét}}(Y, L) \otimes_{\mathbb{Z}_p} \mathbb{F} \cong \otimes_{a+b=i} \left( \text{H}^{a,b}_{\text{log Hodge}}(Y_{\text{an}}, D_{\text{dR,log}}(L)) \otimes_k \mathbb{F}(-a) \right),
\]
compatible with the canonical filtrations and the actions of $\text{Gal} (\overline{k}/k)$ on both sides.

Note that, unlike the functor $D^{\text{alg}}$ in Theorem 1.1 or the functor $D_{\text{dR}}$ in [LZ17, Theorem 3.9], the functor $D_{\text{dR,log}}$ fails to be a tensor functor in general, even when restricted to the subcategory of Kummer étale $\mathbb{Z}_p$-local systems that are de Rham over $Y - D$, as the eigenvalues of the residues of $D_{\text{dR,log}}(L_1) \otimes_{\mathcal{O}_Y} \text{dR}_{\text{dR,log}}(L_2)$ might be outside the range $[0, 1]$, and therefore $D_{\text{dR,log}}(L_1) \otimes_{\mathcal{O}_Y} D_{\text{dR,log}}(L_2)$ might not be isomorphic to $D_{\text{dR,log}}(L_1 \otimes_{\mathbb{Z}_p} L_2)$. This failure is caused by the failure of the surjectivity of the canonical morphism
\[
\mu^* (D_{\text{dR,log}}(L)) \otimes_{\mathcal{O}_{Y_{\text{proét}}}} \mathcal{O}_{D_{\text{dR,log}}} \to \widehat{L} \otimes_{\mathbb{Z}_p} \mathcal{O}_{D_{\text{dR,log}}}
\]
(obtained by adjunction) in general, even when $\text{rk}_{\mathcal{O}_Y} (D_{\text{dR,log}}(L)) = \text{rk}_{\mathbb{Z}_p}(L)$. This phenomenon is not present in the usual comparison theorems in $p$-adic Hodge theory, but is consistent with the classical complex Riemann–Hilbert correspondence. Nevertheless, we will also show in Theorem 5.24 that $D_{\text{dR,log}}$ restricts to a natural tensor functor from the subcategory of de Rham local systems with unipotent monodromy along the boundary to the subcategory of integrable log connections with nilpotent residues along the boundary.

Given Theorem 1.4, the functor $D^{\text{alg}}$ in Theorem 1.1 can be constructed as follows. Every étale $\mathbb{Z}_p$-local system $L$ on $X$ defines by analytification an étale $\mathbb{Z}_p$-local system $L_{\text{an}}$ on $X_{\text{an}}$, which can be extended to a Kummer étale $\mathbb{Z}_p$-local system $L_{\text{an}}$ on $Y := X_{\text{an}}$ (by a version of the rigid analytic Abhyankar’s lemma, see Proposition 2.56). Then the $D_{\text{dR,log}}(L_{\text{an}})$ as in Theorem 1.4 is a filtered vector bundle with
an integrable log connection on $\overline{X}^\an$, which (by rigid GAGA; see [Kop74], and also the proof of [Sch13 Theorem 9.1]) is the analytification of a filtered vector bundle $D^\alg_{\dR,\log}(\overline{X}^\an)$ with an integrable log connection on $\overline{X}$. Finally, we set

$$D^\alg_{\dR}(L) := (D^\alg_{\dR,\log}(\overline{X}^\an))|_X,$$

and show that this assignment defines the desired tensor functor $D^\alg_{\dR}$ in Theorem 1.1 which is independent of the choice of the compactification $\overline{X}$. The comparison of cohomology in Theorem 1.1 follows from the comparison of cohomology in Theorem 1.4, together with a version of the purity for étale local systems (see Theorem 2.70) and the comparison between the algebraic and rigid analytic étale cohomology of $X$ (based on results in [Ber93] and [Hub96]).

We will deduce the above theorem from a geometric log Riemann–Hilbert correspondence. Let $Y$ be as above, and let $K$ be a perfectoid field over $k$ containing all roots of unit. Let $B^+_{\dR} = B^+_{\dR}(K, O_K)$ and $B_{\dR} = B_{\dR}(K, O_K)$ (as in [LZ17 Section 3.1]), and consider the ringed spaces $\mathcal{Y}^+ = (Y^\an, O_Y \otimes B^+_{\dR})$ and $\mathcal{Y} = (Y^\an, O_Y \otimes B_{\dR})$, where $O_Y \otimes B^+_{\dR}$ and $O_Y \otimes B_{\dR}$ are sheaves on $Y^\an$ which we interpret as the rings of functions on the not-yet-defined base changes “$Y \otimes_k B^+_{\dR}$” and “$Y \otimes_k B_{\dR}$”, respectively. Let $\mu' : Y_{\text{prokét}}/Y^\an \to Y^\an$ denote the natural morphism of sites. The following theorem is an abbreviated version of Theorem 5.16.

**Theorem 1.6.** The functor

$$\mathcal{R}H^\log_{\log}(L) := R\mu'_*(\overline{L} \otimes_{\hat{\mathbb{Z}}_p} O\mathcal{B}_{\dR,\log})$$

is an exact functor from the category of Kummer étale $\mathbb{Z}_p$-local systems on $Y$ to the category of $\text{Gal}(K/k)$-equivariant vector bundles on $\mathcal{Y}$, equipped with an integrable log connection

$$\nabla : \mathcal{R}H^\log_{\log}(L) \to \mathcal{R}H^\log_{\log}(L) \otimes_{O_\mathcal{Y}} \Omega^1_{\mathcal{Y}}(\log D),$$

and a decreasing filtration (by locally free $O_Y \otimes B^+_{\dR}$-submodules) satisfying the Griffiths transversality. In addition, there is a canonical comparison isomorphism

$$H^i_{\text{ét}}(Y^\an, L) \otimes_{\mathbb{Z}_p} B_{\dR} \cong H^i_{\log \dR}(\mathcal{Y}, \mathcal{R}H^\log_{\log}(L)),$$

compatible with the canonical filtrations and the actions of $\text{Gal}(\overline{K}/k)$ on both sides.

Let

$$\mathcal{R}H^+_{\log}(L) := R\mu'_*(\overline{L} \otimes_{\hat{\mathbb{Z}}_p} \text{Fil}^0 O\mathcal{B}_{\dR,\log}).$$

This is an $O_Y \otimes B^+_{\dR}$-lattice in $\mathcal{R}H^+_{\log}(L)$, equipped with an operator

$$\nabla^+ := t\nabla : \mathcal{R}H^+_{\log}(L) \to \mathcal{R}H^+_{\log}(L) \otimes_{O_\mathcal{Y}} \Omega^1_{\mathcal{Y}}(\log D)(1),$$

where $t \in B_{\dR}$ is an element on which $\text{Gal}(K/k)$ acts via the cyclotomic character. As explained in [LZ17 Remark 3.2], this is a $t$-connection. By reduction modulo $t$, we obtain the log $p$-adic Simpson functor $\mathcal{H}_{\log}$, as first constructed by Faltings [Fal05] and Abbes–Gros–Tsuji [AGT10] in much greater generality. See Theorem 5.17 for the detailed statement.

Compared with the situation considered in [LZ17], the proof of Theorem 1.6 requires one crucial new ingredient; namely, we need to replace a decompletion statement for the toric tower by extracting $p$-power roots, which was used in [LZ17] and proved in [KL16], with a decompletion statement for the toric tower by extracting all roots. The theory in [KL16] is not enough for our purpose. We have
therefore developed a general decompletion formalism in Section 4, which might be of some independent interest. The comparison of cohomology is based on the same methods as in [Sch13], given the generalization of the primitive comparison theorem in [Dia18, Section 4] and the definition of log de Rham period sheaves.

To deduce Theorem 1.4 from Theorem 1.6 note that we have $\mathcal{D}_{\text{dR}, \log}(L) \cong (\mathcal{R}\mathcal{H}_{\log}(L))^\text{Gal}(K/k)$. By using the above mentioned decompletion statement and an argument similar to the one in [LZ17], it is not difficult to show that $\mathcal{D}_{\text{dR}, \log}(L)$ is a coherent sheaf on $Y_{\text{an}}$. However, unlike the situation in [LZ17], the existence of a log connection does not guarantee the local freeness of $\mathcal{D}_{\text{dR}, \log}(L)$. A priori, only its reflexiveness is clear. Nevertheless, there is a collection of important invariants attached to a log connection: i.e., the residues along the irreducible components of $D$. Again, by using some decompletion statement, we can show that the eigenvalues of the residues are all rational numbers in $[0, 1)$. By [AB01, Chapter 1, Proposition 4.5 and Lemma 4.6.1], the reflexiveness of $\mathcal{D}_{\text{dR}, \log}(L)$ and our calculation of the eigenvalues of residues imply that $\mathcal{D}_{\text{dR}, \log}(L)$ is indeed locally free. Finally, our result on residues plays once again a vital role in deducing the comparison of cohomology in Theorem 1.4 from the comparison of cohomology in Theorem 1.6.

Outline of this paper. Let us briefly describe the organization of this paper, and highlight the main topics in each section.

In Section 2, we recollect and develop the theory of log adic spaces. We begin by summarizing some basic definitions and constructions in Section 2.1. While the materials are similar to their counterparts in the theory of log schemes, and have been worked out in detail in [Dia18], we feel it is necessary to explain the foundational materials well enough before we embark on more serious undertakings. In Section 2.2, we develop a theory of log differentials that is sufficient for our purpose. In particular, the details of the local constructions will be useful later when we study period sheaves in local coordinates in Section 3.2. In Section 2.3, we study the Kummer étale topology. In addition to materials we summarize from [Dia18], important technical results include the rigid analytic Abhyankar’s lemma (see Proposition 2.56 and Lemma 2.59) and the purity for étale local systems (see Theorem 2.70). In Section 2.4, we study the pro-Kummer étale topology, and prove several local vanishing results.

In Section 3, we study the logarithmic de Rham period sheaves, generalizing the usual ones studied in [Bri08, Section 5], [Sch13, Section 6], and [Sch16], with a subtle difference—see Remark 3.11. In Section 3.1, we present the general definitions of these logarithmic de Rham period sheaves. In Section 3.2, we describe their structures in detail, when there are good local coordinates. In Section 3.3, we record some consequences, including the Poincaré lemma.

In Section 4, we generalize the formalism of decompleting developed in [KL16, Section 5], in order to treat the general Kummer towers. In Section 4.1, we introduce the notions of decompleting systems and decompleting pairs in general, and show that every stably decompleting pair is a decompletion system. In Section 4.2, we present several important examples of decompletion systems, which will play crucial roles later in Section 5 (in the proof of coherence and in the calculation of residues). This section can be read largely independent of the rest of the paper.

In Section 5, we establish the geometric and arithmetic versions of log $p$-adic Riemann–Hilbert correspondences, as well as the log $p$-adic Simpson correspondence, as explained above. In Section 5.1 we state the main results, which are more
detailed versions of Theorems 1.4 and 1.6 above. The proofs are given in the 
remainder of this section. In Section 5.2, we first show that we obtain coherent 
sheaves in the various constructions. The underlying ideas are similar to those in 
[LZ17], but we need the more general decompleting results in Section 4. In Section 5.3, 
we calculate the eigenvalues of residues of the connections along the boundary. 
This is the technical heart of this paper, and also relies crucially on Section 4. In Section 5.4, 
we prove that our correspondences are compatible with pullbacks and 
certain very special case of pushforwards, by using our results on residues and 
the previously known compatibilities in [Sch13] and [LZ17]. In Section 5.5, we 
prove the comparison assertions in our main theorems, by combining the primitive 
comparison theorem for Kummer étale $F_p$-local systems in [Dia18, Section 4] with 
other results we have developed earlier in this paper.

In Section 6, we finally construct the $p$-adic Riemann–Hilbert functor by com-
bining several results in earlier sections, and prove Theorem 1.1. In addition to 
the de Rham comparison in Theorem 1.1, we also establish the corresponding log 
Hodge–Tate comparison and the degeneration of (log) Hodge–de Rham spectral 
sequences, and record these latter results in Theorem 6.6.

In Section 7, we compare two constructions of filtered regular connections on 
Shimura varieties, the first one being based on the classical complex analytic con-
struction using double quotients, while the second one being based on the $p$-adic 
Riemann–Hilbert functor developed in this paper, and deduce Theorem 1.3 from 
this comparison. In Section 7.1, we begin with the overall setup. In Sections 7.2 
and 7.3, we explain the complex and $p$-adic analytic constructions of local sys-
tems and filtered vector bundles with log connections. In Section 7.4, we state our 
main comparison theorems concerning these two constructions, and record some 
consequences. In Section 7.5, we reduce the main comparison theorems to a sim-
ple technical statement concerning representations of fundamental groups, which 
will be verified in the following two subsections, using Deligne’s and Blasius’s re-

Notation and conventions. Throughout the paper, unless otherwise specified, 
$k$ denotes a nonarchimedean local field (i.e., a field complete with respect to the 
topology induced by a nonarchimedean multiplicative norm $|·|: k \to \mathbb{R}_{\geq 0}$) with 
residue field $\kappa$ of characteristic $p > 0$, and $\mathcal{O}_k$ denotes the ring of integers in $k$. We
fix an open valuation ring \( k^+ \subset \mathcal{O}_k \). Sometimes, we choose a \textit{pseudo-uniformizer} (i.e., a topological nilpotent unit) \( \varpi \) of \( k \) that is contained in \( k^+ \).

By a \textit{locally noetherian adic space} over \( k \), we mean an adic space \( X \) over \( \text{Spa}(k, k^+) \) that admits an open covering by affinoids \( U_i = \text{Spa}(A_i, A_i^+) \) where each \( A_i \) is strongly noetherian. A \textit{noetherian adic space} over \( k \) is a qcqs locally noetherian adic space over \( k \). If \( X \) is locally noetherian, we denote by \( X_{\text{an}} \) its analytic site, and by \( \lambda : X_{\text{et}} \rightarrow X_{\text{an}} \) the natural projection of sites. We will always regard rigid analytic varieties as adic spaces topologically of finite type over \( \text{Spa}(k, \mathcal{O}_k) \) (as in [Hub96]).

We say that an adic space \( X \) is \textit{étale sheafy} if \( X \) admits a well-defined étale site \( X_{\text{et}} \) and if the structure presheaf \( \mathcal{O}_{X_{\text{et}}} : U \mapsto \mathcal{O}_U(U) \) is a sheaf. Étale sheafy adic spaces include all locally noetherian adic spaces and perfectoid spaces (see [Hub96 Section 1.7] and [Sch12 Section 7], respectively).

Group cohomology will always mean continuous group cohomology.

2. Log adic spaces

In this section, we recollect and develop the theory of log adic spaces. We refer to [Dia18] for some basic definitions and results for log adic spaces.

2.1. Basics in logarithmic geometry. Let us first recall some basic definitions and constructions of log adic spaces. Since they are mostly similar to the corresponding ones for log schemes (as in [Kat89] and [Ill02]), and since many details have been written down in [Dia18], we will sometimes omit the justifications.

Monoids in this paper are always assumed to be commutative. We shall always write monoid operations multiplicatively, except when working with \( \mathbb{Z} \) finite type over \( \text{Spa}(k, \mathcal{O}_k) \) (i.e., a topological nilpotent unit) \( \varpi \). For a monoid \( P \), let \( P^\ast \) denote its subgroup of invertible elements, and let \( P = P/P^\ast \). Let \( P^{\text{gp}} \) denote its group completion.

**Definition 2.1.**

1. A monoid \( P \) is called \textit{finitely generated} if there exists a surjection \( \mathbb{Z}^n_+ \rightarrow P \) for some \( n \).
2. A monoid \( P \) is called \textit{integral} if the natural map \( P \rightarrow P^{\text{gp}} \) is injective.
3. A monoid is called \textit{fine} if it is integral and finitely generated.
4. A monoid \( P \) is called \textit{saturated} if it is integral and, for every \( a \in P^{\text{gp}} \), such that \( a^n \in P \) for some integer \( n \geq 1 \), we have \( a \in P \). A monoid that is both fine and saturated is called an \textit{fs monoid}.
5. A monoid \( P \) is called \textit{sharp} if \( P^\ast = \{1\} \).
6. An fs sharp monoid is called a \textit{toric monoid}.

For any monoid \( P \), let \( P^{\text{int}} \) denote the image of the canonical homomorphism \( P \rightarrow P^{\text{gp}} \). The functor \( P \mapsto P^{\text{int}} \) is the left adjoint of the inclusion from the category of integral monoids into the category of all monoids. Similarly, the inclusion from the category of saturated monoids into the category of integral monoids admits a left adjoint \( P \mapsto P^{\text{sat}} \), where \( P^{\text{sat}} = \{ a \in P^{\text{gp}} \mid a^n \in P, \text{ for some } n \} \). For a general monoid \( P \) that is not necessarily integral, we write \( P^{\text{sat}} \) for \( (P^{\text{int}})^{\text{sat}} \). If \( R \) is a commutative ring with unit and \( P \) is a monoid, we denote by \( R[P] \) the monoid algebra over \( R \) associated with \( P \). The image of \( a \in P \) in \( R[P] \) will sometimes be denoted by \( e^a \). For two homomorphisms of monoids \( f_1 : P \rightarrow Q_1 \) and \( f_2 : P \rightarrow Q_2 \), the \textit{amalgamated sum} \( Q_1 \oplus_P Q_2 \) is defined to be the coequalizer of \( P \rightrightarrows Q_1 \oplus Q_2 \), with the two homomorphisms given by \((f_1, 0)\) and \((0, f_2)\), respectively.
Definition 2.2. Let $X$ be an étale sheafy adic space over $k$.

1. A pre-log structure on $X$ is a pair $(\mathcal{M}_X,\alpha)$ where $\mathcal{M}_X$ is a sheaf of monoids on $X_{\text{ét}}$ and $\alpha : \mathcal{M}_X \to \mathcal{O}_{X_{\text{ét}}}$ is a morphism of sheaves of monoids.

2. A pre-log structure $(\mathcal{M}_X,\alpha)$ on $X$ is called a log structure if $\alpha^{-1}(\mathcal{O}_{X_{\text{ét}}}^\times) \to \mathcal{O}_{X_{\text{ét}}}^\times$ is an isomorphism. In this case, the triple $(X,\mathcal{M}_X,\alpha)$ is called a log adic space. When the context is clear, we simply write $(X,\mathcal{M}_X)$ or $X$.

3. For a pre-log structure $\mathcal{M}_X$ on $X$, we define the associated log structure $\mathcal{O}_X$ on $X$ to be the pushout of $\mathcal{O}_{X_{\text{ét}}}^\times \leftarrow \alpha^{-1}(\mathcal{O}_{X_{\text{ét}}}^\times) \to \mathcal{M}_X$ in the category of sheaves of monoids on $X_{\text{ét}}$.

4. A morphism $f : (Y,\mathcal{M}_Y,\alpha_Y) \to (X,\mathcal{M}_X,\alpha_X)$ of log adic spaces is a morphism $f : Y \to X$ of adic spaces together with a morphism of sheaves of monoids $f^\sharp : f^{-1}(\mathcal{M}_X) \to \mathcal{M}_Y$ compatible with $\alpha_X$ and $\alpha_Y$ in the obvious way. In this case, let $f^\flat(\mathcal{M}_X)$ be the log structure on $Y$ associated with the pre-log structure $f^{-1}(\mathcal{M}_X) \to f^{-1}(\mathcal{O}_{X_{\text{ét}}}) \to \mathcal{O}_{Y_{\text{ét}}}$. The morphism $f$ is called strict if the induced morphism $f^\flat(\mathcal{M}_X) \to \mathcal{M}_Y$ is an isomorphism.

Definition 2.3. A log adic space is called noetherian (resp. locally noetherian) if its underlying adic space is noetherian (resp. locally noetherian).

Here are some basic examples of log adic spaces we will encounter in this paper.

Example 2.4. Every étale sheafy adic space $X$ has a natural log structure with $\mathcal{M}_X = \mathcal{O}_{X_{\text{ét}}}^\times$ and $\alpha$ the identity map. We call it the trivial log structure on $X$.

Example 2.5. A log point is a log adic space whose underlying adic space is $\text{Spa}(K,K^+)$, where $K$ is a nonarchimedean field over $k$. We remark that the underlying topological space may not be a single point.

Example 2.6. In Example 2.5, if $K$ is separably closed, then the étale topos of $\text{Spa}(K,K^+)$ is equivalent to the category of sets. In this case, a log structure on $\text{Spa}(K,K^+)$ is given by a map of monoids $\alpha : M \to K$ which induces the isomorphism $\alpha^{-1}(K^\times) \cong K^\times$. For simplicity, by abuse of notation, we shall sometimes introduce a log point by writing $s = (\text{Spa}(K,K^+),M)$. We shall also use $s$ to denote the underlying adic space $\text{Spa}(K,K^+)$, when there is no danger of confusion.

Example 2.7. Suppose that the nonarchimedean base field $k$ is perfectoid (only in this example). Let $(X,\mathcal{M}_X,\alpha_X)$ be a log perfectoid space over $(k,k^+)$; i.e., a log adic space whose underlying adic space $X$ is a perfectoid space. Let $\mathcal{M}_X = \varprojlim X$ where the transition maps are given by raising each element to its $p$-th power. Let $X^p$ be the tilt of $X$. Then there is a natural map of sheaves of monoids $\alpha_X : \mathcal{M}_X \to \mathcal{O}_{X^p}$, making $(X^p,\mathcal{M}_X,\alpha_X)$ a log perfectoid space over $(k^p,k^{+p})$, called the tilt of $(X,\mathcal{M}_X,\alpha_X)$.

Example 2.8. Let $P$ be a monoid. We topologize the monoid algebra $k[P]$ such that $\mathcal{O}_k$ equipped with the $\omega$-adic topology is a ring of definition (where $\omega$ is some pseudo-uniformizer of $k$). By [Dia18, Lemma 1.40], $(k[P],k^+[P])$ is an affinoid $k$-algebra. We know that $\text{Spa}(k[P],k^+[P])$ is étale sheafy in the following cases:

1. If $P$ is finitely generated, then $(k[P],k^+[P])$ is locally noetherian and hence $\text{Spa}(k[P],k^+[P])$ is étale sheafy (see [Dia18, Lemma 1.42]). Consequently, $\text{Spa}(k(P),k^+(P))$ is also étale sheafy, where $k(P)$ is the completion of $k[P]$.

2. A monoid $P$ is called uniquely $p$-divisible if the $p$-th power map $[p] : P \to P$ is bijective. If $k$ is a perfectoid field and $P$ is uniquely $p$-divisible,
then, by [Dia18, Lemma 1.44], $k(P)$ is a perfectoid $k$-algebra. Therefore, $\text{Spa}(k(P), k^+[P])$ is an affinoid perfectoid space, hence étale sheafy.

If $X = \text{Spa}(k(P), k^+[P])$ is étale sheafy, it has a natural log structure $P^{\log}$ associated with the pre-log structure $P \to \mathcal{O}_X$ defined by $a \mapsto e^a \in k(P)$.

Example 2.9. In Example 2.8, if $P$ is a toric monoid as in Definition 2.10, then we say that $X = \text{Spa}(k(P), k^+[P])$ is an affinoid toric log adic space. This is closely related to the theory of toroidal embeddings and toric varieties (see, for example, [KKMS73, Section 7.3.2, Proposition 8], [Pn93, IV-2, 7.8.3.1], and [Hoc72, Theorem 1]) (cf. [Kat94, Theorem 4.1]).

Example 2.10. An important special case of Example 2.9 is the special case where $P \cong \mathbb{Z}_{\geq 0}^n$ for some integer $n \geq 0$. In this case, we obtain

$$X = \text{Spa}(k(P), k^+[P]) \cong \mathbb{D}^n := \text{Spa}(k(T_1, \ldots, T_n), \mathcal{O}_k(T_1, \ldots, T_n)),$$

the $n$-dimensional unit disc, with the log structure on $\mathbb{D}^n$ associated with the pre-log structure given by $\mathbb{Z}_{\geq 0}^n \to k(T_1, \ldots, T_n) : (a_1, \ldots, a_n) \mapsto T_1^{a_1} \cdots T_n^{a_n}$.

Example 2.11. Let $X$ be a normal rigid analytic variety, and let $\iota : D \hookrightarrow X$ be a closed immersion of rigid analytic varieties over $k$. By viewing $X$ as a locally noetherian adic space via [Sch12, Theorem 2.21], we can equip $X$ with the log structure defined by setting $\mathcal{M}_X = \{ f \in \mathcal{O}_{X_{\alpha}} \mid f \text{ is invertible on } X - D \}$, with $\alpha : \mathcal{M}_X \to \mathcal{O}_{X_{\alpha}}$ the natural inclusion. This makes $X$ a locally noetherian fs log adic space. The maximal open subspace of $X$ over which $\mathcal{M}_X$ is trivial is exactly $X - D$. Note that, in Example 2.10, the log structure on $X \cong \mathbb{D}^n$ can be defined alternatively as above by the closed immersion $\iota : D := \{ T_1 \cdots T_n = 0 \} \hookrightarrow \mathbb{D}^n$.

In this paper, we will be mostly interested in log adic spaces arising in the following way:

Example 2.12. Let $X$ be a smooth rigid analytic variety over $k$. A (reduced) normal crossings divisor $D$ of $X$ is defined by a closed immersion $\iota : D \hookrightarrow X$ of rigid analytic varieties over $k$ such that, (analytic) locally, $X$ and $D$ are of the form $X \cong S \times \mathbb{D}^r$ and $D = S \times \{ T_1 \cdots T_r = 0 \}$, where $S$ is a smooth connected affine space over $k$, and such that $\iota$ is the pullback of $\{ T_1 \cdots T_r = 0 \} \hookrightarrow \mathbb{D}^r$. (This definition is justified by [Kie67a, Theorem 1.18]. See also [Han15, Lemma 2.12].) In this case, we equip $X$ with the log structure defined as in Example 2.11, which is compatible with the one of $\mathbb{D}^r$ in Example 2.11 via pullback. Locally, when we have $X \cong S \times \mathbb{D}^r$ as above, with local coordinates $T_1, \ldots, T_r$ on $\mathbb{D}^r$, the log structure on $X$ is associated with the pre-log structure $\mathbb{Z}_{\geq 0}^r \to \mathcal{M}_X$ defined by $(a_1, \ldots, a_r) \mapsto \prod_{i=1}^r T_i^{a_i}$. Moreover, in this case, we also have locally an étale morphism (i.e., a composition of rational localizations and finite étale morphisms of adic spaces) $X \to \mathbb{D}^n$ such that $\iota : D \hookrightarrow X$ is the pullback of $\{ T_1 \cdots T_r = 0 \} \hookrightarrow \mathbb{D}^n$, where $T_1, \ldots, T_n$ are the coordinates of $\mathbb{D}^n$ as in Example 2.10 for some $0 \leq r \leq n$. 


In practice, we are interested in log adic spaces that are locally represented by monoids of special type (e.g., finitely generated, or uniquely $p$-divisible). To make this more precise, let us introduce the notion of charts.

**Definition 2.13.** Let $(X, \mathcal{M}_X, \alpha)$ be a log adic space. Let $P$ be a monoid and let $P_X$ denote the associated constant sheaf of monoids on $X_\et$. Note that giving a morphism $P_X \to \mathcal{M}_X$ is equivalent to giving a homomorphism $P \to \mathcal{M}_X(X)$. A *(global)* chart of $X$ modeled on $P$ is a morphism of sheaves of monoids $\theta : P_X \to \mathcal{M}_X$ such that $\alpha(\theta(P_X)) \subset \mathcal{O}_{X,\et}^+$ and such that the log structure associated with the pre-log structure $\alpha \circ \theta : P_X \to \mathcal{O}_{X,\et}$ coincides with $\mathcal{M}_X$. In other words, a chart $\theta : P_X \to \mathcal{M}_X$ on $X$ is equivalent to a strict morphism of log adic spaces

$$f : (X, \mathcal{M}_X) \to (\text{Spa}(k[P], k^+[P]), \mathcal{P}^\log),$$

where $(\text{Spa}(k[P], k^+[P]), \mathcal{P}^\log)$ is as in Example 2.8.

**Remark 2.14.** Definition 2.13 is equivalent to [Dia18, Definition 1.45] when the pre-adic space $\text{Spa}(k[P], k^+[P])$ is étale sheafy (see Example 2.8).

**Definition 2.15.** A coherent (resp. fine, resp. fs) log adic space is a log adic space $X$ that étale locally admits a chart modeled on a finitely generated (resp. fine, resp. fs) monoid.

**Lemma 2.16.** An fs log adic space always admits étale locally a chart modeled on a sharp fs monoid.

**Proof.** The proof is similar to the log scheme case, except for an additional adjustment to ensure that the image of the composition $P \to \mathcal{M}_X(X) \to \mathcal{O}_X(X)$ lands in $\mathcal{O}_X^+(X)$. See [Dia18, Proposition 1.48] for details. □

**Example 2.17.** An fs log point is a log point (as in Example 2.5) that is an fs log adic space. In the context of Example 2.6, a log point $s = (\text{Spa}(K, K^+), M)$ with $K$ separably closed is an fs log point exactly when $M/K^\times$ is fs.

**Definition 2.18.** Let $f : (Y, \mathcal{M}_Y, \alpha_Y) \to (X, \mathcal{M}_X, \alpha_X)$ be a morphism of log adic spaces. A chart of $f$ consists of charts $\theta_X : P \to \mathcal{M}_X(X)$ and $\theta_Y : Q \to \mathcal{M}_Y(Y)$ and a map $u : P \to Q$ of monoids such that the diagram

$$\begin{array}{ccc}
P & \xrightarrow{u} & Q \\
\downarrow{\theta_X} & & \downarrow{\theta_Y} \\
\text{f}^{-1}(\mathcal{M}_X) & \xrightarrow{f^\ast} & \mathcal{M}_Y \\
\end{array}$$

commutes. When the context is clear, we simply call $u : P \to Q$ the chart of $f$.

**Example 2.19.** Let $P := \mathbb{Z}_{\geq 0}^n$ and let $Q$ be a toric submonoid of $\frac{1}{m}\mathbb{Z}_{\geq 0}^n$ containing $P$, for some integer $m \geq 1$. Then the canonical map $u : P \to Q$ of fs monoids induces a morphism $f : Y := \text{Spa}(k(Q), k^+[Q]) \to X := \text{Spa}(k(P), k^+[P]) \cong \mathbb{D}^n$ of normal adic spaces, whose source and target are equipped with canonical log structures as in Examples 2.8 and 2.10 making $f : Y \to X$ a morphism of fs log adic spaces. Moreover, the canonical log structures on $X$ and $Y$ coincide with the log structures on $X$ and $Y$ defined by $D = \{T_1 \cdots T_n = 0\} \hookrightarrow X$ and its pullback to $Y$, respectively, as in Example 2.11. A chart of $f : Y \to X$ is given by the canonical charts $P \to \mathcal{M}_X(X)$ and $Q \to \mathcal{M}_Y(Y)$ and the above map $u : P \to Q$. 

By the following lemma, morphisms between coherent log adic spaces always admit charts étale locally.

**Lemma 2.20.** Let \( f : Y \to X \) be a morphism of coherent log adic space, and let \( P \to \mathcal{M}_X(X) \) be a chart modeled on a finitely generated monoid \( P \). Then, étale locally on \( Y \), there exist a chart \( Q \to \mathcal{M}_Y(Y) \) modeled on a finitely generated monoid \( Q \) and a map of monoids \( P \to Q \), which together provide a chart of \( f \). Moreover, if \( f \) is a morphism of fine (resp. fs) log adic space and if \( P \) is fine (resp. fs), we can choose \( Q \) to be fine (resp. fs) as well.

*Proof.* The construction of \( Q \to \mathcal{M}_Y(Y) \) is similar to the log scheme case, except for an additional adjustment to ensure that the image of the composition \( Q \to \mathcal{M}_Y(Y) \to \mathcal{O}_Y(Y) \) lands in \( \mathcal{O}_Y^+(Y) \). To show that in the fs case, we can choose \( Q \) to be fs, we use the fact that \( \mathcal{O}_Y^+ \) is integrally closed in \( \mathcal{O}_Y \). We refer to [Dia18, Propositions 1.50 and 1.53] for details. \( \square \)

**Lemma 2.21.** The embedding from the category of noetherian (resp. locally noetherian) fine log adic spaces over \( k \) to the category of noetherian (resp. locally noetherian) coherent log adic spaces over \( k \) admits a right adjoint \( X \mapsto X^{\int} \). The embedding from the category of noetherian (resp. locally noetherian) fs log adic spaces over \( k \) to the category of noetherian (resp. locally noetherian) fine log adic spaces over \( k \) admits a right adjoint \( X \mapsto X^{\sat} \).

*Proof.* As in the log scheme case, locally, the functor \( X \mapsto X^{\int} \) (resp. \( X \mapsto X^{\sat} \)) can be described as follows. Suppose \( X \) admits a global chart modeled on a finitely generated (resp. fine) monoid \( P \). For \( ? = \int \) (resp. \( \sat \)), we set \( X^? := X \times_{\text{Spa}(k[P_1], k^{\int}[P_1])} \text{Spa}(k[P^\int], k^{\int}[P^\int]) \). Then we can glue the local constructions together by their universal property. We refer to [Dia18, Lemma 1.54] for details. \( \square \)

**Remark 2.22.** In Lemma 2.21, the underlying adic spaces might change under the assignments \( X \mapsto X^{\int} \) and \( X \mapsto X^{\sat} \). Given any locally noetherian coherent log adic space \( X \), we write \( X^{\sat} := (X^{\int})^{\sat} \). Note that the natural morphism \( X^{\sat} \to X \) is finite.

By the following proposition, fiber products exist in the category of coherent (resp. fine, resp. fs) log adic spaces:

**Proposition 2.23.**

1. Finite fiber products exist in the category of locally noetherian (resp. locally noetherian coherent) log adic spaces. The forgetful functor from the category of locally noetherian (resp. locally noetherian coherent) log adic spaces to the category of locally noetherian adic spaces respects finite fiber products.

2. Finite fiber products exist in the category of locally noetherian fine (resp. fs) log adic spaces.

*Proof.* As in the log scheme case, this follows from Lemmas 2.20 and 2.21. We refer to [Dia18, Proposition 1.55] for details. \( \square \)

**Remark 2.24.** We emphasize that the forgetful functor from the category of locally noetherian fine (resp. fs) log adic spaces to the category of locally noetherian adic spaces does not respect fiber products. In what follows, fiber products of locally noetherian fs log adic spaces always mean fiber products in the category of locally noetherian fs log adic spaces.
We will need Nakayama’s *Four Point Lemma* in the current setting. A map of integral monoids \( u : P \rightarrow Q \) is called *exact* if \( P = (u^{SP})^{-1}(Q) \) in \( P^{SP} \). A morphism \( f : Y \rightarrow X \) of locally noetherian fs log adic spaces is called *exact* if the induced morphism \( f^*(M_X) \rightarrow M_Y \) is exact at each stalk.

**Lemma 2.25.** Let \( f : Y \rightarrow X \) and \( g : Z \rightarrow X \) be two morphisms of locally noetherian fs log adic spaces. Assume that one of them is exact. Then, given any two points \( y \in Y \) and \( z \in Z \) that are mapped to the same point \( x \in X \), there exists some point \( w \in W := Y \times_X Z \) that is mapped to \( y \in Y \) and to \( z \in Z \).

**Proof.** The proof is similar to its analogue in the setting of log schemes. See [Dia18, Proposition 1.67 and Corollary 1.68] for details. \( \square \)

Let us recall the notions of log étale and log smooth morphisms. For our purpose, we only need these notions for morphisms of locally noetherian fs log adic spaces.

**Definition 2.26.** A morphism \( f : Y \rightarrow X \) of locally noetherian fs log adic spaces is called *log smooth* (resp. *log étale*) if, étale locally on \( X \) and on \( Y \), the morphism \( f \) admits a chart \( u : P \rightarrow Q \) (with fs monoids \( P \) and \( Q \)) such that:

1. the kernel and the torsion part of the cokernel of \( u^{SP} : P^{SP} \rightarrow Q^{SP} \) (resp. the kernel and the cokernel of \( u^{SP} \)) are finite groups of order prime to the characteristic of \( k \); and

2. the induced morphism \( Y \rightarrow X \times_{\text{Spa}(k[P],k\cdot[P])} \text{Spa}(k[Q],k\cdot[Q]) \) is étale on the underlying adic spaces.

If \( f \) is strict and log smooth (resp. log étale), then the underlying map of adic spaces is smooth (resp. étale) (see [Dia18, Corollary 1.65]), and we say that \( f \) is strictly smooth (resp. strictly étale). When the context is clear, we simply say that \( f \) is smooth (resp. étale).

By [Dia18, Proposition 1.63], compositions and base changes (under morphisms between locally noetherian fs log adic spaces) of log smooth (resp. log étale) morphisms are log smooth (resp. log étale).

When \( X \) is log smooth over \( \text{Spa}(k,\mathcal{O}_k) \), where the latter is equipped with the trivial log structure as in Example 2.4, we just (abusively) say that \( X \) is log smooth over \( k \). For example, in Example 2.12, \( X \) is log smooth over \( k \). Conversely, by Proposition 2.27 below, every log smooth adic space over \( k \) with smooth underlying adic space arises in this way.

**Proposition 2.27.** Let \( X \) be a log smooth fs log adic space over \( k \). Étale locally on \( X \), there exists a toric monoid \( P \), together with a strictly étale morphism \( X \rightarrow \text{Spa}(k[P],\mathcal{O}_k[P]) \), which is a composition of rational localizations and finite étale morphisms. If the underlying adic space of \( X \) is smooth, then we may assume that \( P \cong \mathbb{Z}_{>0}^n \) for some \( n \geq 0 \), so that \( \text{Spa}(k[P],\mathcal{O}_k[P]) \cong \mathbb{D}^n \) as in Example 2.10.

**Proof.** See [Dia18, Proposition 1.67 and Corollary 1.68]. \( \square \)

**Definition 2.28.** We call the strictly étale morphism \( X \rightarrow \text{Spa}(k[P],\mathcal{O}_k[P]) \) as in Proposition 2.27 a *toric chart* of \( X \). When the underlying space of \( X \) is smooth, and when \( P \cong \mathbb{Z}_{>0}^n \), for some \( n \geq 0 \), we call the induced strictly étale morphism \( X \rightarrow \mathbb{D}^n \) a *smooth toric chart* of \( X \).

**Remark 2.29.** In Example 2.12, the (analytic) locally defined morphisms \( X \rightarrow \mathbb{D}^n \) are strictly étale and are smooth toric charts as in Definition 2.28.
2.2. Log differentials. The goal of this subsection is to develop a theory of log differentials. We first introduce log differentials for log Tate $k$-algebras.

**Definition 2.30.** (1) A pre-log Tate $k$-algebra is a triple $(A, M, \alpha)$ consisting of a Tate $k$-algebra $A$ (as in \[Sch12\] Definition 2.6], a monoid $M,$ and a homomorphism $\alpha : M \to A$ of multiplicative monoids. We sometimes denote a pre-log Tate $k$-algebra just by $(A, M),$ when the homomorphism $\alpha$ is clear from the context. We also write $\overline{M} := M / \alpha^{-1}(A^\times).$

(2) A log Tate $k$-algebra is a pre-log Tate $k$-algebra $(A, M, \alpha)$ where $A$ is complete and where the map $\alpha^{-1}(A^\times) \to A^\times$ is an isomorphism.

(3) For a pre-log Tate $k$-algebra $(A, M, \alpha),$ we define the associated log Tate $k$-algebra to be $(\hat{A}, aM, \hat{\alpha}).$ Here $\hat{A}$ denotes the completion of $A.$ Let us still denote by $\alpha$ the composition of $\alpha : M \to A$ with the natural map $A \to \hat{A}.$ Then $aM$ is the amalgamated sum $\hat{A}^\times \boxplus_{\alpha^{-1}(A^\times)} M,$ which is equipped with the canonical homomorphism $\hat{\alpha} : aM \to \hat{A}.$

(4) A homomorphism $f : (A, M, \alpha) \to (B, N, \beta)$ of pre-log Tate $k$-algebras is a continuous homomorphism $f : A \to B$ of Tate $k$-algebras, together with a homomorphism of monoids $f^\sharp : M \to N,$ such that $\beta \circ f^\sharp = f \circ \alpha.$ We say $f$ is strict if in addition $(B, N, \beta)$ is isomorphic (induced by the map $f^\sharp$) to the log Tate $k$-algebra associated to the pre-log Tate $k$-algebra $(B, M, \beta \circ f^\sharp).$ In this case sometimes we write $f^*(M)$ for $N.$ In general, any homomorphism $f : (A, M) \to (B, N)$ of log Tate $k$-algebras factors as $(A, M) \to (B, f^*(M)) \to (B, N).$

**Definition 2.31.** Let $(A, M, \alpha)$ be a log Tate $k$-algebra. We call the log structure $(M, \alpha)$ finitely generated (resp. essentially finitely generated) if $M / \alpha^{-1}(A^\times)$ (resp. $M^{gp} / a^{-1}(A^\times)$) is finitely generated.

**Remark 2.32.** By definition, a finitely generated log structure is essentially finitely generated. On the other hand, if there is a homomorphism of monoids $\theta : P \to M$ such that the log structure associated with the pre-log structure $\alpha \circ \theta : P \to A$ coincides with $\alpha : M \to A,$ and if $P$ is finitely generated, then the log structure $(M, \alpha)$ is also finitely generated.

**Definition 2.33.** Let $f : (A, M, \alpha) \to (B, N, \beta)$ be a homomorphism of pre-log Tate $k$-algebras, with $A$ and $B$ both complete. For a complete topological $B$-module $L,$ by a derivation from $(B, N, \beta)$ to $L$ over $(A, M, \alpha)$ (or an $(A, M, \alpha)$-derivation of $(B, N, \beta)$ to $L),$ we mean a continuous $A$-linear derivation $d : B \to L,$ together with a map of monoids $\delta : N \to L,$ such that $\delta(f^\sharp(m)) = 0$ and $d(\beta(n)) = \beta(n)\delta(n)$ for all $m \in M$ and $n \in N.$ We denote by $\text{Der}_A^\log(B, L)$ the set of all $(A, M, \alpha)$-derivations from $(B, N, \beta)$ to $L.$ It has a natural $B$-module structure induced by that of $L.$ If $M = A^\times$ and $N = B^\times,$ we simply denote $\text{Der}_A^\log(B, L)$ by $\text{Der}_A(B, L),$ which is the usual $B$-module of continuous $A$-derivations from $B$ to $L.$

**Remark 2.34.** In Definition 2.33 $(d, \delta)$ naturally extends to a log derivation on $(B, aN, \beta).$ Therefore, $\text{Der}_A^\log(B, L)$ remains unchanged if we replace $(B, N, \beta)$ with the associated log Tate $k$-algebra. Moreover, $\delta$ naturally extends to a group homomorphism $\delta^{gp} : (aN)^{gp} \to L.$

Now, assume that $(B, N, \beta)$ is a log Tate $k$-algebra. Let $(B \hat{\otimes}_A B)[N]$ denote the monoid algebra over $B \hat{\otimes}_A B$ associated with the monoid $N,$ and let $e^n$ denote...
its element corresponding to \( n \), for each \( n \in N \). Let \( I \) be its ideal generated by \( \{ e^{f(n)} - 1 \}_{n \in M} \) and \( \{ (\beta(n) \otimes 1) - (1 \otimes \beta(n)) e^n \}_{n \in N} \). Note that, if \( n \in \beta^{-1}(B^\times) \), then \( e^n = \beta(n) \otimes \beta(n)^{-1} \). Let \( J \) be the kernel of the homomorphism
\[
\Delta_{\log} : ((B \hat{\otimes}_A B)[N]) / I \to B
\]
sending \( b_1 \otimes b_2 \) to \( b_1 b_2 \) and all \( e^n \) to 1. Set
\[
\Omega_{B/A}^{\log} := J / J^2.
\]
Define \( d_{B/A} : B \to \Omega_{B/A}^{\log} \) and \( \delta_{B/A} : N \to \Omega_{B/A}^{\log} \) by setting
\[
d_{B/A}(b) = (b \otimes 1) - (1 \otimes b)
\]
and
\[
\delta_{B/A}(n) = e^n - 1.
\]
A short computation shows that \( d_{B/A} \) is an \( A \)-linear derivation, and that \( \delta_{B/A} \) is a homomorphism of monoids satisfying the required properties in Definition 2.33.

As observed in Remark 2.34, \( \delta_{B/A} \) naturally extends to a group homomorphism \( \delta_{B/A}^{\exp} : N^{\exp} \to \Omega_{B/A}^{\log} \) such that \( \delta_{B/A}^{\exp}(f^\times)^\exp(M^{\exp}) = 0 \). Then it follows that \( \Omega_{B/A}^{\log} \) is generated as a \( B \)-module by \( \ker(B \hat{\otimes}_A B \to B) \) and \( \{ \delta_{B/A}^{\exp}(n) \} \), where \( n \) runs over a set of representatives of generators of \( N^{\exp}/((f^\times(M))^{\exp} \beta^{-1}(B^\times)) \).

Suppose that \( f \) is topologically of finite type, and that \( N^{\exp}/((f^\times(M))^{\exp} \beta^{-1}(B^\times)) \) is finitely generated. Then \( \Omega_{B/A}^{\log} \) is a finite \( B \)-module. We shall equip \( \Omega_{B/A}^{\log} \) with its natural \( B \)-module topology, with respect to which it is complete and \( d_{B/A} \) is continuous.

**Proposition 2.37.** Under the above assumption, \( (\Omega_{B/A}^{\log}, d_{B/A}, \delta_{B/A}) \) is a universal object among all \((A, M, \alpha)\)-derivations of \((B, N, \beta)\).

**Proof.** Let \( (d, \delta) \) be a derivation from \((B, N, \beta)\) to some complete topological \( B \)-module \( L \) over \((A, M, \alpha)\). We turn the \( B \)-module \( B \hat{\otimes} L \) into a complete topological \( B \)-algebra, which we denote by \( B \ast L \), by setting \( (b, x)(b', x') = (bb', bx' + b'x) \). Note that the \( A \)-linear derivation \( d \) gives rise to a continuous homomorphism of topological \( B \)-algebras \( B \hat{\otimes}_A B \to B \ast L \) sending \( x \otimes y \) to \( (xy, xdy) \). We may further extend it to a homomorphism \((B \hat{\otimes}_A B)[N] \to B \ast L \) by sending \( e^n \) to \((1, \delta(n))\), for each \( n \in N \). By the conditions in Definition 2.33, this homomorphism factors through \((B \hat{\otimes}_A B)[N])/I \), inducing a homomorphism \((B \hat{\otimes}_A B)[N])/I \to B \ast L \) which we denote by \( \varphi \). Clearly, the composition of \( \varphi \) with the natural projection \( B \ast L \to B \) just recovers the homomorphism \((2.35)\). Therefore, \( \varphi \) induces a continuous morphism of \( B \)-modules \( \varphi : \Omega_{B/A}^{\log} = J / J^2 \to L \). Now, a careful chasing of definitions verifies that \( \varphi \circ d_{B/A} = d \) and \( \varphi \circ \delta_{B/A} = \delta \), as desired.

Given any complete topological \( B \)-module \( L \), there is a natural forgetful functor \( \text{Der}_A^{\log}(B, L) \to \text{Der}_A(B, L) \) defined by \((d, \delta) \mapsto d\). The following lemma is obvious:

**Lemma 2.38.** If \( f : (A, M, \alpha) \to (B, N, \beta) \) is a strict homomorphism of log Tate \( k \)-algebras, then the canonical morphism \( \text{Der}_A^{\log}(B, L) \to \text{Der}_A(B, L) \) is an isomorphism for every complete topological \( B \)-module \( L \). Hence, the canonical morphism \( \Omega_{B/A} \to \Omega_{B/A}^{\log} \) is an isomorphism.
Consider the following commutative diagram

\[
\begin{array}{c}
(A, M, \alpha) \xrightarrow{f} (D, O, \mu) \\
(B, N, \beta) \xrightarrow{g} (D', O', \mu')
\end{array}
\]

of solid arrows. Here \((D, O, \mu)\) is a log Tate \(k\)-algebra, \(H\) is a closed ideal of \(D\) such that \(H^2 = 0\) (so that we may regard \(1 + H\) as a submonoid of \(O\) via \(\mu : O \to D\)), \(D' = D/H, O' = O/(1 + H)\), and \(\mu' : O' \to D'\) is the induced log structure.

**Definition 2.40.** Let \((A, M, \alpha) \to (B, N, \beta)\) be a homomorphism of log Tate \(k\)-algebras. We say that \((B, N, \beta)\) is formally log smooth (resp. formally log unramified, resp. formally log étale) over \((A, M, \alpha)\) if, for any diagram as in (2.39), there exists a lifting (resp. at most one lifting, resp. a unique lifting) \(\tilde{g} : (B, N, \beta) \to (D, O, \mu)\) of \(g\) as the dotted arrow in (2.39), making the whole diagram commute.

If \(M = A^\times\) and \(N = B^\times\), then we simply say that (the underlying algebra homomorphism) \(A \to B\) is formally smooth (resp. formally unramified, resp. formally étale). (See also [Hub96, Definition 1.6.5].)

**Lemma 2.41.** If \(f : (A, M, \alpha) \to (B, N, \beta)\) is a strict homomorphism of log Tate \(k\)-algebras, then \(f\) is formally log smooth (resp. formally log unramified, resp. formally log étale) if and only if \(A \to B\) is formally smooth (resp. formally unramified, resp. formally étale).

**Remark 2.42.** For a morphism \(f : A \to B\) of Tate \(k\)-algebras topologically of finite type, by [Hub96, Proposition 1.7.11], we have that \(f\) is formally smooth (resp. formally unramified, resp. formally étale) if and only if the induced map \(\text{Spa}(B, B^\circ) \to \text{Spa}(A, A^\circ)\) is smooth (resp. unramified, resp. étale) in the sense of classical rigid analytic geometry.

In the remainder of this subsection, we shall assume that all Tate \(k\)-algebras are topologically of finite type, and all log structures are essentially finitely generated. Consequently, all log differential modules are finitely generated. The following theorem establishes the first fundamental exact sequence for log differentials:

**Theorem 2.43.**

1. A composition \((A, M, \alpha) \xrightarrow{f} (B, N, \beta)\) of log Tate \(k\)-algebras leads to a strict exact sequence

\[
C \otimes_B \Omega^\log_{B/A} \to \Omega^\log_{C/A} \to \Omega^\log_{C/B} \to 0
\]

of topological \(C\)-modules, where the first map sends \(c \otimes d_{B/A}(b)\) and \(c \otimes \delta_{B/A}(n)\) to \(cd_{C/A}(g(b))\) and \(c \delta_{C/A}(g^2(n))\), respectively, and the second map sends \(d_{C/A}(c)\) and \(\delta_{C/A}(l)\) to \(d_{C/B}(c)\) and \(\delta_{C/B}(l)\), respectively.

2. Moreover, if \((C, L, \gamma)\) is formally log smooth over \((B, N, \beta)\), then \(\Omega^\log_{B/A} \otimes_B C \to \Omega^\log_{C/A}\) is injective, and the short exact sequence

\[
0 \to C \otimes_B \Omega^\log_{B/A} \to \Omega^\log_{C/A} \to \Omega^\log_{C/B} \to 0
\]

is split in the category of topological \(C\)-modules.

3. If \((C, L, \gamma)\) is formally log unramified over \((B, N, \beta)\), then \(\Omega^\log_{C/B} = 0\).
(4) If \((C, L, \gamma)\) is formally log étale over \((B, N, \beta)\), then \(\Omega^\log_{C/A} \cong C \otimes_B \Omega^\log_{B/A} \).

Proof. Note that \(\Omega^\log_{B/A} \otimes_B C, \Omega^\log_{C/A},\) and \(\Omega^\log_{C/B}\) are all finite \(C\)-modules. Thus, to prove the exactness in \((2)\), it suffices to show that, for any finite \(C\)-module \(O\), the induced sequence

\[
\text{Hom}_C(C \otimes_B \Omega^\log_{B/A}, O) \leftarrow \text{Hom}_C(\Omega^\log_{C/A}, O) \leftarrow \text{Hom}_C(\Omega^\log_{C/B}, O) \leftarrow 0
\]

is exact. By Proposition 2.37 (and the fact that a morphism between finite modules over a complete \(k\)-algebra is automatically continuous), this sequence is nothing but

\[
\text{Der}^\log_A(B, O) \leftarrow \text{Der}^\log_A(C, O) \leftarrow \text{Der}^\log_B(C, O) \leftarrow 0,
\]

whose exactness is obvious. The strictness follows from the open mapping theorem for complete \(k\)-algebras (see [KL16, Theorem 1.2.7]).

For the rest of the proof, let \(O\) be a finite \(C\)-module, and let \((d, \delta) : (B, N, \beta) \to O\) be an \((A, M, \alpha)\)-derivation. Let \(C \ast O\) be the \(C\)-algebra defined as in the proof of Proposition 2.37. We further equip it with a log structure \((\gamma \ast \text{Id}) : L \oplus O \to C \ast O\) by setting

\[
(\gamma \ast \text{Id})(a, b) = (\delta(a), \delta(a)b),
\]

and denote the resulted log Tate \(k\)-algebra by \((C, L, \gamma) \ast L\).

We claim that there is a natural bijection between the set of \((A, M, \alpha)\)-derivations \((\tilde{d}, \tilde{\delta}) : (C, L, \gamma) \to O\) extending \((d, \delta)\) and the set of homomorphisms \(h : (C, L, \gamma) \to (C, L, \gamma) \ast L\) of log Tate \(k\)-algebras making the diagram

\[
\begin{array}{ccc}
(B, N, \beta) & \longrightarrow & (C, L, \gamma) \ast L \\
\downarrow g & & \downarrow h \\
(C, L, \gamma) & \longrightarrow & (C, L, \gamma) \ast L
\end{array}
\]

commutative. Here the upper horizontal map is a homomorphism of log Tate \(k\)-algebras sending \((b, n)\) to \(((g(b), d(b)), \gamma^*(n), \delta(n)))\), and the right vertical one is the natural projection map. To justify the claim, for each map \(h' : (C, L) \to (C \ast O, L \oplus O)\) lifting the projection \((C \ast O, L \oplus O) \to (C, L)\), let us write \(h' = (\text{Id}, d, \text{Id}, \delta)\). Then a short computation shows that \(\tilde{h}'\) is a homomorphism of log Tate \(k\)-algebras if and only if \(\tilde{d}\) is a derivation and \(\tilde{\delta}\) is a homomorphism of monoids such that \(\tilde{d}(\gamma(l)) = \gamma(l)\tilde{\delta}(l)\) for all \(l \in L\), and the claim follows.

Consequently, if \((C, L, \gamma)\) is unramified over \((B, N, \beta)\), then the natural map \(\text{Der}^\log_A(C, O) \to \text{Der}^\log_A(B, O)\) is injective for each finite \(C\)-module \(O\). In other words, \(\text{Hom}_C(\Omega^\log_{C/A}, O) \to \text{Hom}_C(C \otimes_B \Omega^\log_{B/A}, O)\) is injective. This implies that \(C \otimes_B \Omega^\log_{B/A} \to \Omega^\log_{C/A}\) is surjective, yielding (3). Similarly, if \((C, L, \gamma)\) is smooth over \((B, N, \beta)\), then \(\text{Hom}_C(\Omega^\log_{C/A}, O) \to \text{Hom}_C(C \otimes_B \Omega^\log_{B/A}, O)\) is surjective. Hence, we can justify (2) by taking \(O = C \otimes_B \Omega^\log_{B/A}\). Finally, (4) follows from (2) and (3). □

Let \(u : P \to Q\) be a homomorphism of fine monoids. Then we have the pre-log Tate \(k\)-algebra \(P \to k[P] : a \mapsto e^a\) (resp. \(Q \to k[Q] : a \mapsto e^a\), with the topology given in Example 2.8. By abuse of notation, we shall still denote by \(k(P)\) (resp. \(k(Q)\)) the resulted log Tate \(k\)-algebra.
Proposition 2.44. If the kernel and the torsion part of the cokernel (resp. the kernel and the cokernel) of \( v^{\text{gp}} : P^{\text{gp}} \to Q^{\text{gp}} \) are finite groups of orders prime to the characteristic of \( k \), then \( k\langle Q \rangle \) is formally log smooth (resp. formally log étale) over \( k\langle P \rangle \). In this case, \( \Omega_{k\langle Q \rangle/k\langle P \rangle}^{\log} \) is a free \( k\langle Q \rangle \)-module of rank equal to that of \( Q^{\text{gp}}/u^{\text{gp}}(P^{\text{gp}}) \).

Proof. Consider the commutative diagram

\[
\begin{array}{ccc}
   k\langle P \rangle & \longrightarrow & (D, O, \mu) \\
   \downarrow & & \downarrow \\
   k\langle Q \rangle & \longrightarrow & (D', O', \mu')
\end{array}
\]

as in (2.39). This gives rise to a commutative diagram of monoids

\[
\begin{array}{ccc}
P & \longrightarrow & O \\
\downarrow & & \downarrow \\
Q & \longrightarrow & O',
\end{array}
\]

which in turn induces a commutative diagram of abelian groups

\[
\begin{array}{ccc}
P^{\text{gp}} & \longrightarrow & O^{\text{gp}} \\
\downarrow & \downarrow & \downarrow \\
Q^{\text{gp}} & \longrightarrow & (O')^{\text{gp}}.
\end{array}
\]

Note that there is a natural bijection between the set of the homomorphisms of log Tate \( k \)-algebras \( k\langle Q \rangle \to (D, O, \mu) \) extending (2.45) and the set of homomorphisms of monoids \( Q \to O \) extending (2.46). By using the Cartesian diagram

\[
\begin{array}{ccc}
   O & \longrightarrow & O^{\text{gp}} \\
   \downarrow & \downarrow & \downarrow \\
   O' & \longrightarrow & (O')^{\text{gp}},
\end{array}
\]

and the fact that \( P \) and \( Q \) are fine monoids, we see that there is also a bijection between the set of the desired homomorphisms \( k\langle Q \rangle \to (D, O, \mu) \) and the set of group homomorphisms \( Q^{\text{gp}} \to O^{\text{gp}} \) extending (2.47).

Since \( H^2 = 0 \), we have \( \ker(O^{\text{gp}} \to (O')^{\text{gp}}) = 1+H \cong H \). Let \( G = Q^{\text{gp}}/u^{\text{gp}}(P^{\text{gp}}) \). Since the kernel and the torsion part of the cokernel (resp. the kernel and the cokernel) of \( u^{\text{gp}} \) are finite groups of orders prime to the characteristic of \( k \), it is straightforward to see that the set of the desired homomorphisms \( Q^{\text{gp}} \to O^{\text{gp}} \) forms a principal homogeneous space for \( \text{Hom}(G, H) \cong \text{Hom}(G/G_{\text{tor}}, H) \), where \( G_{\text{tor}} \) denotes the torsion subgroup of \( G \). This proves the first half of the proposition.

On the other hand, for any finite \( k\langle Q \rangle \)-module \( L \), by the same argument as in the proof of Theorem 2.43, there is a bijection between \( \text{Der}_{k\langle P \rangle}^{\log}(k\langle Q \rangle, L) \) and the
set of \( h : k(Q) \to k(Q) \ast L \) extending the following commutative diagram

\[
\begin{array}{ccc}
k(Q) & \longrightarrow & k(Q) \ast L \\
\downarrow & & \downarrow \\
k(Q) & \longrightarrow & k(Q) \\
\end{array}
\]

Then \( \text{Der}_{k(Q)}^{\log} (k(Q), L) \cong \text{Hom}_{k(Q)}(\Omega^{\log}_{k(Q)/k(P)}, L) \) is isomorphic to \( \text{Hom}(G/G_{\text{tor}}, L) \) as principal homogeneous spaces for \( \text{Hom}(G/G_{\text{tor}}, L) \), by the previous paragraph. Hence, \( \Omega^{\log}_{k(Q)/k(P)} \) is a free \( k(Q) \)-module of rank equal to that of \( G \), as desired. \( \square \)

The next step is to define the sheaf of log differentials for \( f \) log adic spaces topologically of finite type over \( k \).

**Definition 2.48.** Let \( f : (Y, \mathcal{M}_Y, \alpha_Y) \to (X, \mathcal{M}_X, \alpha_X) \) be a morphism of \( f \) log adic spaces, and let \( F \) be a sheaf of complete topological \( \mathcal{O}_{\text{et}} \)-modules. By a derivation of \( (Y, \mathcal{M}_Y, \alpha_Y) \) over \( (X, \mathcal{M}_X, \alpha_X) \) valued in \( F \), we mean a pair \((d, \delta)\), where \( d : \mathcal{M}_Y \to F \) is a continuous \( \mathcal{O}_{X_{\text{et}}} \)-linear derivation and \( \delta : \mathcal{M}_Y \to F \) is a morphism of sheaves of monoids, such that \( \delta(f^{-1}(\mathcal{M}_X)) = 0 \) and \( d(\alpha_Y(m)) = \alpha_Y(m)\delta(m) \), for all sections \( m \) of \( \mathcal{M}_Y \).

Fix a morphism of log Tate \( k \)-algebras \((A, M, \alpha) \to (B, N, \beta)\). We equip \( X = \text{Spa}(A, A^+) \) and \( Y = \text{Spa}(B, B^+) \) with the log structures associated with the pre-log structures \( M_X \to \mathcal{O}_{X_{\text{et}}} \) and \( N_Y \to \mathcal{O}_{Y_{\text{et}}} \), respectively, and denote by \( f \) the induced morphism of \( k \)-affinoid log adic spaces \((Y, \mathcal{M}_Y, \alpha_Y) \to (X, \mathcal{M}_X, \alpha_X)\). We denote by \( \Omega^{\log}_{Y/X} \) the coherent sheaf on \( Y_{\text{et}} \) associated with \( \Omega^{\log}_{B/A} \) (see (2.36)). For any affinoid log adic space \( \text{Spa}(C, C^+) \in Y_{\text{et}} \), by Lemma 2.41, Remark 2.42, and Theorem 2.43, the log differential \( \Omega^{\log}_{C/A} \) for the morphism \((A, M, \alpha) \to (C, N, (B \to C) \circ \beta)\) is naturally isomorphic to \( C \otimes_B \Omega^{\log}_{B/A} \cong \Omega^{\log}_{Y/X}(\text{Spa}(C, C^+)) \). Moreover, for each affinoid log adic space \( \text{Spa}(C, C^+) \in Y_{\text{et}} \), the maps \( d_{C/A} : C \to \Omega^{\log}_{C/A} \) and \( \delta_{C/A} : N \to \Omega^{\log}_{C/A} \) naturally extend to a continuous \( \mathcal{O}_{X_{\text{et}}} \)-linear derivation \( d_{Y/X} : \mathcal{O}_{Y_{\text{et}}} \to \Omega^{\log}_{Y/X} \) and a map of sheaves of monoids \( \delta_{Y/X} : N_X \to \Omega^{\log}_{Y/X} \) satisfying \( \delta_{Y/X}(f^{-1}(\mathcal{M}_X)) = 0 \) and \( d_{Y/X}(\alpha_Y(n)) = \alpha_Y(n)\delta_{Y/X}(n) \), for all \( n \) in \( N_Y \). We may further extend \( \delta_{Y/X} \) to a morphism of sheaves of monoids \( \mathcal{M}_Y \to \Omega^{\log}_{Y/X} \) such that \( \delta_{Y/X}(f^{-1}(\mathcal{M}_X)) = 0 \) and \( d_{Y/X}(\alpha_Y(m)) = \alpha_Y(m)\delta_{Y/X}(m) \), for all \( m \) in \( \mathcal{M}_Y \).

**Lemma 2.49.** The coherent sheaf \( \Omega^{\log}_{Y/X} \), together with the pair \((d_{Y/X}, \delta_{Y/X})\), is a universal object among all derivations of \((Y, \mathcal{M}_Y, \alpha_Y) \) over \((X, \mathcal{M}_X, \alpha_X)\).

**Proof.** Let \((d, \delta)\) be a derivation of \((Y, \mathcal{M}_Y, \alpha_Y) \) over \((X, \mathcal{M}_X, \alpha_X) \) valued in some complete topological \( \mathcal{O}_{Y_{\text{et}}} \)-module \( F \). For each affinoid log adic space \( \text{Spa}(C, C^+) \in Y_{\text{et}} \), the derivation \((d, \delta)\) specializes to a derivation

\[
(C, \mathcal{M}_Y(\text{Spa}(C, C^+)), \alpha_Y(\text{Spa}(C, C^+))) \to F(\text{Spa}(C, C^+))
\]

over \((A, \mathcal{M}_X(X), \alpha_X(X)) \), which in turn restricts to a derivation

\[
(C, N, (B \to C) \circ \beta) \to F(\text{Spa}(C, C^+))
\]

over \((A, M, \alpha) \). By the universal property of log differentials, it factors through a continuous \( C \)-linear morphism \( \Omega^{\log}_{C/A} \to F(\text{Spa}(C, C^+)) \). Moreover, we deduce from
the universal property of log differentials that, for any \( \text{Spa}(C_1, C_1^+) \rightarrow \text{Spa}(C_2, C_2^+) \) in \( \mathcal{Y}_{\acute{e}t} \), the diagram

\[
\begin{array}{ccc}
\Omega^\log_{C_2/A} & \longrightarrow & F\left(\text{Spa}(C_2, C_2^+)\right) \\
\downarrow & & \downarrow \\
\Omega^\log_{C_1/A} & \longrightarrow & F\left(\text{Spa}(C_1, C_1^+)\right)
\end{array}
\]

is commutative. This shows that, for all \( \text{Spa}(C, C^+) \in \mathcal{Y}_{\acute{e}t} \), the morphisms \( \Omega^\log_{C/A} \rightarrow F(\text{Spa}(C, C^+)) \) naturally extend to a continuous \( \mathcal{O}_{\mathcal{X}_{\acute{e}t}} \)-linear morphism \( \Omega^\log_{Y/X} \rightarrow F \).

Clearly, its compositions with \( d_{Y/X} \) and \( \delta_{Y/X} \) are equal to \( d \) and \( \delta \), respectively. \( \square \)

As a consequence, for \( k \)-affinoid log adic spaces \( X \) and \( Y \) topologically of finite type over \( k \), and for a morphism \( Y \rightarrow X \) admitting a chart (as in Definition 2.18) given by finitely generated monoids, there is a universal triple \( (\Omega^\log_{Y/X}, d_{Y/X}, \delta_{Y/X}) \), where \( \Omega^\log_{Y/X} \) is a coherent sheaf, among all derivations of \( Y \) over \( X \). Moreover, by Remark 2.41 and Theorem 2.43 for affinoid log adic spaces \( V \in \mathcal{Y}_{\acute{e}t} \) and \( U \in \mathcal{X}_{\acute{e}t} \) fitting into the commutative diagram

\[
\begin{array}{ccc}
V & \longrightarrow & Y \\
\downarrow & & \downarrow \\
U & \longrightarrow & X,
\end{array}
\]

we see that \( (\Omega^\log_{Y/X}, d_{Y/X}, \delta_{Y/X})|_V \) is canonically isomorphic to \( (\Omega^\log_{V/U}, d_{V/U}, \delta_{V/U}) \).

For a morphism \( f : Y \rightarrow X \) of coherent log adic spaces of topologically finite type over \( k \), by Lemma 2.20 there exist a finite index set \( I \) and finite étale coverings \( Y = \bigcup_{i \in I} Y_i \) and \( X = \bigcup_{i \in I} X_i \) such that \( f \) restricts to a morphism \( Y_i \rightarrow X_i \) which admits a chart given by finitely generated monoids, for each \( i \in I \). By previous discussions, the triples \( (\Omega^\log_{Y_i/X_i}, d_{Y_i/X_i}, \delta_{Y_i/X_i}) \) are canonically isomorphic to each other over the fiber products of \( Y_i \). Consequently, by étale descent (using [KL15 Theorem 8.2.22(d)]), we obtain a triple \( (\Omega^\log_{Y/X}, d_{Y/X}, \delta_{Y/X}) \) on \( \mathcal{Y}_{\acute{e}t} \). It is straightforward to see that \( (d_{Y/X}, \delta_{Y/X}) \) is a derivation of \( Y \) over \( X \) valued in \( \Omega^\log_{Y/X} \).

By Lemma 2.49 we immediately obtain the following:

**Lemma 2.50.** The triple \( (\Omega^\log_{Y/X}, d_{Y/X}, \delta_{Y/X}) \) is a universal object among all derivations of \( Y \) over \( X \).

Consequently, the triple \( (\Omega^\log_{Y/X}, d_{Y/X}, \delta_{Y/X}) \) is well defined, i.e., independent of the choice of étale coverings. We call \( \Omega^\log_{Y/X} \) the sheaf of log differentials of \( f \) and \( (d_{Y/X}, \delta_{Y/X}) \) the associated universal log derivations. By abuse of notation, we shall denote the pushforward of \( \Omega^\log_{Y/X} \) to \( Y_{\text{an}} \) by the same symbols. (When there is any risk of confusion, we shall denote this pushforward by \( \Omega^\log_{Y/X, \text{an}} \) instead.) If \( X = \text{Spa}(k, k^+) \), for simplicity, we shall write \( \Omega^\log_Y \) instead of \( \Omega^\log_{Y/X} \).
Proposition 2.51. Let

\[
\begin{array}{ccc}
Y' & \longrightarrow & X' \\
\downarrow f & & \downarrow \ \\
Y & \longrightarrow & X
\end{array}
\]

be a Cartesian diagram in the category of coherent (resp. fine, resp. fs) log adic spaces topologically of finite type over \( k \). Then \( f^*(\Omega_{Y/X}^{\log}) \cong \Omega_{Y'/X'}^{\log} \).

Proof. By the construction of sheaves of log differentials, the proof reduces to the case where \( Y = \text{Spa}(B, B^+) \), \( X = \text{Spa}(A, A^+) \), and \( X' = \text{Spa}(A', A'^+) \) are affinoid log adic spaces, and where \( Y \to X \) and \( X' \to X \) admit charts \( P \to Q \) and \( P \to P' \), respectively, given by finitely generated (resp. fine, resp. fs) monoids. Then it suffices to show that \( \Omega_{B^+\otimes A}^{\log}(B^+\otimes_A A', L) \cong \text{Der}_A^B(B, L)\otimes B^+\otimes A' \), for every complete topological \( B\otimes A^+ \)-module \( L \). Therefore, we have

\[
\text{Hom}_{B\otimes A}(\Omega_{B\otimes A}^{\log}(B\otimes_A B', L) \otimes B^+\otimes A') \\
\cong \text{Hom}_{B}(\Omega_{B}^{\log}(B, L)\otimes A') \\
\cong \text{Hom}_{B\otimes A}(\Omega_{B\otimes A}^{\log}(B\otimes_A A', L),
\]

yielding the desired isomorphism. \( \square \)

Theorem 2.52. (1) A composition of maps \( Y \to X \to S \) of coherent log adic spaces topologically of finite type over \( k \) naturally induces an exact sequence

\[
f^*(\Omega_{X/S}^{\log}) \to \Omega_{Y/S}^{\log} \to \Omega_{Y/X}^{\log} \to 0.
\]

(2) If \( f \) is log smooth, then \( f^*(\Omega_{X/S}^{\log}) \to \Omega_{Y/S}^{\log} \) is injective, and \( \Omega_{Y/X}^{\log} \) is a vector bundle of rank equal to the relative dimension of \( f \).

(3) If \( f \) is log étale, then \( \Omega_{Y/X}^{\log} = 0 \), in which case \( f^*(\Omega_{X/S}^{\log}) \cong \Omega_{Y/S}^{\log} \).

Proof. By the construction of sheaves of log differentials, (1) follows from Theorem 2.43. The remaining assertions (2) and (3) follow from Proposition 2.44, Proposition 2.51, Lemma 2.41, and Theorem 2.43 (2) and (4). \( \square \)

When \( X \) is log smooth over \( k \), for each integer \( j \geq 0 \), we define

\[
\Omega_{X}^{\log,j} = \wedge^j \Omega_{X}^{\log}.
\]

2.3. Kummer étale topology. In this subsection, we study the Kummer étale topology of locally noetherian fs log adic spaces, which was first introduced in Dia18. We first recall some definitions. For simplicity, we shall only define Kummer morphism between fs monoids.

Definition 2.54. (1) A homomorphism \( u : P \to Q \) of fs monoids is called Kummer if it is injective and if, for any \( a \in Q \), there is a positive integer \( n \) such that \( a^n \) lies in the image of \( u \).
(2) A morphism \( f : Y \to X \) of locally noetherian fs log adic spaces over \( k \) is called standard Kummer étale if there exists a chart \( u : P \to Q \) of \( f \) such that \( u \) is Kummer, the order of the cokernel of \( u^* \) is prime to \( \text{char}(k) \), and
\[
Y \cong X \times_{\text{Spa}(k[P], k^+[P])} \text{Spa}(k[Q], k^+[Q])
\]

(3) A morphism \( f : Y \to X \) of locally noetherian fs log adic spaces over \( k \) is called Kummer étale (resp. finite Kummer étale) if, étale locally on \( X \) and \( Y \) (resp. étale locally on \( X \)), \( f \) is a composition of a strictly étale (resp. strictly finite étale) morphism with a standard Kummer étale morphism.

Example 2.55. In Example 2.19, the map \( u : P \to Q \) of fs monoids is Kummer. If the integer \( m \geq 1 \) there can be chosen to be prime to \( \text{char}(k) \), then the morphism \( f : Y \to X \) of log adic spaces in Example 2.19 is finite Kummer étale.

An important class of finite Kummer étale coverings arise in the following way.

Proposition 2.56 (rigid Abhyankar’s lemma). Assume that the characteristic of \( k \) is zero. Let \( X \) be a smooth rigid analytic variety over \( k \), and let \( D \) be a normal crossings divisor of \( X \). We equip \( X \) with the log structure induced by \( D \) as in Example 2.12 (or rather Example 2.11). Suppose \( h : V \to U := X - D \) is a finite étale surjective morphism of rigid analytic varieties over \( k \). Then it extends to a finite surjective and Kummer étale morphism of log adic spaces \( f : Y \to X \), where \( Y \) is a normal rigid analytic variety with its log structures defined by the preimage of \( D \) as in Example 2.11.

Proof. By [Han18, Theorem 1.6] (which was based on [Lüt93, Theorem 3.1 and its proof]), \( h : V \to U \) extends to a finite ramified covering \( f : Y \to X \), for some normal rigid analytic variety \( Y \) (viewed as a noetherian adic space). Just as \( X \) is equipped with the log structure defined by \( D \) as in Example 2.11, we equip \( Y \) with the log structure defined by the preimage of \( D \). The question is whether the map \( f \) is Kummer étale (with respect to the log structures on \( X \) and \( Y \)), and this question is local in nature in the analytic topology of \( X \). As in Example 2.12 we may assume that there is a smooth connected affinoid space \( S \) over \( k \) such that \( X = S \times \mathbb{D}^r \) for some \( r \in \mathbb{Z}_{\geq 0} \), with \( D = S \times \{ T_1 \cdots T_r = 0 \} \). Thus, we can conclude the proof by combining the following Lemmas 2.57 and 2.58. \( \square \)

For simplicity, let us introduce some notation for following two lemmas. We write \( P := \mathbb{Z}_{\geq 0} \) and identify \( \mathbb{D}^r \) with \( \text{Spa}(k[P], k^+[P]) \) as in Example 2.10. For each \( m \in \mathbb{Z}_{\geq 1} \), we also write \( P \hat{m} := \frac{1}{m} \mathbb{Z}_{\geq 0} \). For each power \( \rho \) of \( p \), we denote by \( \mathbb{D}_\rho \) the (one-dimensional) disc of radius \( \rho \), so that \( \mathbb{D} = \mathbb{D}_1 \) when \( \rho = 1 \). We also denote by \( \mathbb{D}_\rho^* \) the punctured disc of radius \( \rho \), and by \( \mathbb{D}^* \) the punctured unit disc. We denote normalizations of fiber products by \( \bar{\times} \) (instead of \( \times \)).

Lemma 2.57. Suppose \( X = S \times \mathbb{D}^r \) and \( D = S \times \{ T_1 \cdots T_r = 0 \} \) as in the proof of Proposition 2.56. Assume there is some \( p \leq 1 \) such that, for each connected component \( Y' \) of \( Y \times_{\mathbb{D}^r} \mathbb{D}_\rho^* \), there exist \( d_1, \ldots, d_r \in \mathbb{Z}_{\geq 1} \) such that induced covering \( Y' \to X' := X \times_{\mathbb{D}^r} \mathbb{D}_\rho^* \) is refined (i.e., admits a further covering) by some finite ramified covering \( S \times \text{Spa}(k[P'], k^+[P']) \times_{\mathbb{D}^r} \mathbb{D}_\rho^* \to X' \), where \( P' = \bigoplus_{1 \leq i \leq r} (1 / \mathbb{Z}_{\geq 0}) \). Then, up to replacing \( k \) with a finite extension, we have \( Y' \cong S \times \text{Spa}(k[Q], k^+[Q]) \times_{\mathbb{D}^r} \mathbb{D}_\rho^* \) for some toric monoid \( Q \) (see Definition 2.1(6)) such that \( P \subset Q \subset P' \). Consequently, the morphism \( Y \times_{\mathbb{D}^r} \mathbb{D}_\rho^* \to X' = X \times_{\mathbb{D}^r} \mathbb{D}_\rho^* \)
is finite Kummer étale (see Example 2.55). Moreover, if \( mQ \subset P \) for some \( m \in \mathbb{Z}_{\geq 1} \) and \( X = X_{1} := S \times \text{Spa}(k[P_{1}^{\pm}], k^{+}[P_{1}^{\pm}][1]) \), then the finite (a priori ramified) covering \( Y' \to X \) splits completely (i.e., the source is a disjoint union of sections) and is hence strictly étale.

**Proof.** Let \( Z := S \times \text{Spa}(k[P], k^{+}[P])[1]) \times \mathbb{D}^{r}_{\rho} \). Let \( V' \) (resp. \( W' \)) be the preimage of \( Z' := U \times \mathbb{D}^{r}_{\rho} \cong S \times (\mathbb{D}^{r}_{\rho})^{r} \) in \( Y' \) (resp. \( Z' \)). Up to replacing \( k \) with a finite extension containing \( d_{j} \)-th roots of unity for all \( 1 \leq j \leq r \), the finite étale covering \( W \to U' \) is Galois with Galois group \( G' := \text{Hom}(\mathbb{Z}_{p}^{r}/(\mathbb{Z}_{p}^{r})^{r}, \bar{k}) \), and \( V' \) is (by the usual arguments, as in [Grö91, V]) the quotient of \( W' \) by some (finite) subgroup \( G' \) (as in [Han16]), which is then isomorphic to \( S \times \text{Spa}(k[Q], k^{+}[Q]) \times \mathbb{D}^{r}_{\rho} \). When \( d_{j} \) is positive integers replacing \( \mathbb{D}^{r}_{\rho} \) by some \( \mathbb{D}^{r}_{\rho} \), and hence \( \text{Spa}(k[Q], k^{+}[Q]) \times \mathbb{D}^{r}_{\rho} \) are both normal and are both finite ramified coverings of \( X' \) extending the same finite étale covering \( V' \), they are canonically isomorphic by [Han18, Theorem 1.6], as desired. Finally, for the last assertion of the lemma, it suffices to note that, for any \( Q \) as above, the connected components of \( \text{Spa}(k[Q], k^{+}[Q]) \times \mathbb{D}^{r}_{\rho} \) are all of the form \( \text{Spa}(k[\hat{P}], k^{+}[\hat{P}]) \), because \( (Q \oplus P \hat{P})^{\text{sat}} \cong P \hat{P} \) when \( mQ \subset P \). \( \square \)

**Lemma 2.58.** The (covering refinement) assumption in Lemma 2.57 holds up to replacing \( S \) with a strictly finite étale covering. Moreover, we may assume that the positive integers \( d_{1}, \ldots, d_{r} \) there (for various \( Y' \)) are no greater than the degree \( d \) of \( f : Y \to X \). Moreover, we can take \( \rho = \rho^{\left( b(d,p) \right)} \), where \( b(d,p) \) is defined as in [Lit93, Theorem 2.2], which depends on \( d \) and \( p \) but not on \( r \); and we can take \( m = d! \) in the last assertion of Lemma 2.57.

**Proof.** We shall proceed by induction on \( r \). When \( r = 0 \), the assumption in Lemma 2.57 means, for each connected component \( Y' \) of \( Y \), the strictly étale covering \( Y' \to X = S \) splits completely. This can always be achieved up to replacing \( S \) with a Galois strictly finite étale covering refining \( Y' \to X \) for all \( Y' \).

In what follows, suppose that \( r \geq 1 \), and that the lemma has been proved for all strictly smaller \( r \). Let \( \rho = \rho^{\left( b(d,p) \right)} \) be as above. Fix some \( a \in k \) such that \( |a| = \rho \).

Let \( X_{1} := S \times \mathbb{D}^{r-1} \), which we identify with the subspace \( S \times \mathbb{D}^{r-1} \times \{a\} \) of \( X = S \times \mathbb{D}^{r} \). Let \( Y_{1} := Y \times_{X} X_{1} \). Note that the degree of \( Y_{1} \to X_{1} \) is also \( d \). Let \( P_{1} := \mathbb{Z}^{r-1}_{\geq 0} \), which we identify with the monoid \( \mathbb{Z}^{r-1}_{\geq 0} \oplus \{0\} \) of \( P_{1} \). By induction, up to replacing \( S \) with a strictly finite étale covering, for each connected component \( Y_{1}' \) of \( Y_{1} \times_{\mathbb{D}^{r-1}} \mathbb{D}^{r-1}_{\rho} \), there exist \( 1 \leq d_{1}, \ldots, d_{r-1} \leq d \) such that the induced finite ramified covering \( Y_{1}' \to X_{1} \times_{\mathbb{D}^{r-1}} \mathbb{D}^{r-1}_{\rho} \) is refined by \( S \times \text{Spa}(k[P_{1}], k^{+}[P_{1}])[1]) \times_{\mathbb{D}^{r-1}} \mathbb{D}^{r-1}_{\rho} \). Hence, since \( \rho = \rho^{\left( b(d,p) \right)} \), for each connected component \( Y' \) of \( Y \times_{\mathbb{D}^{r}} \mathbb{D}^{r}_{\rho} \), by applying [Lit93, Theorem 2.2], which depends on \( d \) and \( p \) but not on \( r \); and we can take \( m = d! \) in the last assertion of Lemma 2.57. \( \square \)
Lemma 2.59. Suppose $X$ is a noetherian affinoid log adic space over $k$ modeled on a sharp fs monoid $P$. For each integer $m \geq 1$, let $P^\frac{1}{m}$ be the sharp monoid such that $P \mapsto P^\frac{1}{m}$ can be identified up to isomorphism with the $m$-th power map $[m] : P \to P$, and let $X^{\frac{1}{m}} := X \times_{\text{Spa}(k[P], k^{+}[P])} \text{Spa}(k[P^{\frac{1}{m}}], k^{+}[P^{\frac{1}{m}}])$. Suppose $f : Y \to X$ is a Kummer étale morphism of affinoid log adic spaces. Then there exists some $m$ (depending on $f$) such that $Y \times_X X^{\frac{1}{m}} \to X^{\frac{1}{m}}$ is strictly étale.

Proof. See [Dia18, Lemma 2.14].

Here are some basic facts about Kummer étale morphisms.

Lemma 2.60. 
1. Kummer étale (resp. finite Kummer étale) morphisms are stable under compositions and base changes.
2. Let $h : Z \to X$ be the Kummer étale, and $f : Y \to Z$ a morphism of log adic spaces such that the composition $g = h \circ f$ is Kummer étale. Then $f$ is also Kummer étale.

Proof. The proofs are similar to the log scheme case. See [Dia18, Proposition 2.7] for the first statement, and see [Dia18, Proposition 2.12(1)] for the second one.

Definition 2.61. Let $X$ be a locally noetherian fs log adic space over $k$. The Kummer étale site $X_{\text{ké}}$ consists of all locally noetherian fs log adic spaces that are Kummer étale over $X$. The coverings are given by the topological coverings.
It follows from Lemmas 2.25 and 2.60 that the site $X_{k\text{-}\acute{e}t}$ is well defined, and is generated by surjective étale families and standard Kummer étale coverings. Since étale morphisms are Kummer étale by definition, there is a natural projection of sites $\varepsilon_\acute{e}t : X_{k\text{-}\acute{e}t} \rightarrow X_\acute{e}t$, which is an isomorphism when the log structure on $X$ is trivial. Given any morphism $f : X \rightarrow Y$ of locally noetherian fs log adic spaces over $k$, we have a morphism of sites $f_{k\text{-}\acute{e}t} : X_{k\text{-}\acute{e}t} \rightarrow Y_{k\text{-}\acute{e}t}$, because base changes of Kummer étale morphisms are still Kummer étale, by [Dia18, Proposition 2.7].

Recall the following basic facts about the Kummer étale topology:

**Proposition 2.62.** Let $X$ be a locally noetherian fs log adic space over $k$.

1. The presheaf $\mathcal{O}_{X_{k\text{-}\acute{e}t}}$ (resp. $\mathcal{O}_{X_{k\text{-}\acute{e}t}}^+$) assigning $\mathcal{O}_Y(Y)$ (resp. $\mathcal{O}_Y(Y)^+$) to $Y \in X_{k\text{-}\acute{e}t}$ is a sheaf on $X_{k\text{-}\acute{e}t}$.
2. The natural map $\mathcal{O}_{X_{\acute{e}t}} \rightarrow R\varepsilon_{\acute{e}t, k}\mathcal{O}_{X_{k\text{-}\acute{e}t}}$ is an isomorphism. The analogous statement also holds for the natural projection of sites $\varepsilon_{an} : X_{k\text{-}\acute{e}t} \rightarrow X_{an}$.
3. The presheaf $\mathcal{M}_{X_{k\text{-}\acute{e}t}}$ assigning $\mathcal{M}_Y(Y)$ to $Y \in X_{k\text{-}\acute{e}t}$ is a sheaf on $X_{k\text{-}\acute{e}t}$.

**Proof.** The proof is similar to the log scheme case. The key step is to show the acyclicity of the Čech complex (in each case) for standard Kummer étale coverings. For details, we refer to [Dia18 Theorem 2.16] for part (1), to [Dia18 Corollary 2.18] for part (2), and to [Dia18 Proposition 2.19] for part (3). □

**Remark 2.63.** By Proposition 2.62(3), the canonical map $E \rightarrow R\varepsilon_{an,*} \varepsilon_{an}^*(E)$ is an isomorphism for every vector bundle $E$ (i.e., locally free $\mathcal{O}_{X_{an}}$-modules of finite rank) on $X_{an}$. In particular, the functor from the category of vector bundles on $X_{an}$ to the category of $\mathcal{O}_{k\text{-}\acute{e}t}$-modules is fully faithful.

As in the study of the Kummer étale topology of fs log schemes, in addition to the notion of log points already introduced in Examples 2.5 and 2.17, it is convenient to also introduce the notions of log geometric points (cf. [Dia18 Definition 2.29]; and cf. [Ill02 Section 4] for the parallel story for log schemes).

**Definition 2.64.**

1. A log geometric point is a log point $s = (\text{Spa}(l, l^+), M)$, where $l$ is a separately closed nonarchimedean field (see Example 2.6), such that the multiplication by $n$ is bijective on $\overline{M} := M/l^\times$, for every positive integer $n$ invertible in $l$.
2. Let $X$ be a locally noetherian fs log adic space. A log geometric point of $X$ is a morphism of log adic spaces $\eta : s \rightarrow X$, where $s$ is a log geometric point. If $\eta : s \rightarrow X$ is a log geometric point, a Kummer étale neighborhood of $\eta$ is a map of $X$-log adic spaces $s \rightarrow U$ where $U \rightarrow X$ is Kummer étale.

A log geometric point gives a geometric point of the underlying adic space. Conversely, we can construct a log geometric point from a classical geometric point of a locally noetherian fs log adic space as follows. Let $\pi : \text{Spa}(l, l^+) \rightarrow X$ be a geometric point. We assume that $X \rightarrow \text{Spa}(k[P], k^+[P])$ is a chart, where $P$ is an fs sharp monoid. Then we endow $\text{Spa}(l, l^+)$ with a log structure associated with the pre-log structure $P \rightarrow \mathcal{O}_X(X) \rightarrow l$. By definition (see Examples 2.6 and 2.17),

$$s := (\text{Spa}(l, l^+), P^\log)$$

is an fs log point, and the map of log adic spaces $s \rightarrow X \rightarrow \text{Spa}(k[P], k^+[P])$ is a chart of $s$. For each integer $m \geq 1$ invertible in $l$, let $P^\pm$ be the sharp monoid
such that $P \rightarrow P^\pi$ can be identified up to isomorphism with the $m$-th power map $[m] : P \rightarrow P$. Let

$$s_m := \left( s \times_{\text{Spa}(k[P], k^+[P])} \text{Spa}(k[P^\pi], k^+[P^\pi]) \right)_{\text{red}},$$

with the natural log structure modeled on $P^\pi$. Note that this is a bit different from what would have been denoted $s^\pi$ as in Lemma 2.59 because we are taking the reduced subspace of the fiber product—the underlying adic space of $s_m$ is still isomorphic to $\text{Spa}(l, l^+)$. Then the desired log geometric point above $\pi$ is given by

$$s_\infty := \lim_{\longrightarrow} s_m,$$

which is the same underlying space $\text{Spa}(l, l^+)$ endowed with the log structure associated with the pre-log structure $P^\pi := \lim_{\longrightarrow} P^\pi \rightarrow l$, where the inverse and direct limits run through all integers $m \geq 1$ invertible in $l$.

**Lemma 2.65.** Let $s \rightarrow X$ be a log geometric point of a local noetherian fs log adic space. Then the functor

$$X_{k\acute{e}t} \rightarrow (\text{Sets}) : F \mapsto F_s := \lim_{\longrightarrow} F(U),$$

where the limit is over Kummer étale neighborhoods $U$ of $s$, is a fiber functor. The fibre functors defined by log geometric points form a conservative system.

**Proof.** By Lemma 2.55, the category of Kummer étale neighborhoods $U$ of $s$ is filtered, and hence the first statement follows. Since every point of $X$ admits a log geometric point lying above it (as explained before), the second statement also follows. □

**Lemma 2.66.** Let $s = (\text{Spa}(l, l^+), M)$ be an fs log point, where $l$ is a separably closed nonarchimedean field (see Examples 2.6 and 2.17), and let $\tilde{s}$ be a log geometric point lying above $s$. Then the assignment $F \mapsto F_{\tilde{s}}$ induces an equivalence from $s_{k\acute{e}t}^\sim$ to the category of sets with continuous $\text{Hom}(\hat{M}^{gp}, \hat{\mathbb{Z}}'(1)(l))$-actions. Here

$$\hat{\mathbb{Z}}'(1)(l) := \lim_{\longrightarrow} \mu_m(l),$$

where the limit is over all integers $m \geq 1$ invertible in $l$.

**Proof.** Recall that the étale topos of $s$ is equivalent to the category of sets. By definition, the monoid $\hat{M} = M/\mathbb{Q}^\times$ is fs and sharp. We can choose a splitting $M \cong l^\times \oplus \hat{M}$. Moreover, for each integer $m \geq 1$ invertible in $l$, consider the standard Kummer étale covering $s \rightarrow s$ given by

$$M \cong l^\times \oplus \hat{M} \quad \text{Id} \oplus [m] \rightarrow M \cong l^\times \oplus \hat{M}.$$
is a natural map of monoids $M \to l$ making $x$ a log point. In addition, the monoid $P := M/l^\infty = \overline{M}_{X,x}$ is fs, and hence $x$ is an fs log point. Let $\bar{x}$ be a log geometric point constructed from $x$ as above, and let us abusively denote the composition of morphisms of log adic spaces $\bar{x} \to x \to X$ by the same symbols $\bar{x}$.

**Lemma 2.67.** Let $\mathcal{F}$ be a sheaf of finite abelian groups over $X_{\text{két}}$. Then

$$\left(R^i\varepsilon_*(\mathcal{F})\right)_x \cong H^i\left(\pi^\log_1(x, \bar{x}), \mathcal{F}_x\right).$$

**Proof.** By definition, $\left(R^i\varepsilon_*(\mathcal{F})\right)_x = \lim H^i(V_{\text{két}}, \mathcal{F})$, where the direct limit is taken over the (filtered) category of étale neighborhoods $i_V : x \to V$ of $x$ in $X$.

We claim that there is a natural equivalence of topoi

$$\mathcal{X}^\wedge_{\text{ké}} \cong \lim V_{\text{két}},$$

from which we deduce that

$$\left(R^i\varepsilon_*(\mathcal{F})\right)_x \cong \lim H^i(V_{\text{két}}, \mathcal{F}) \cong H^i\left(\pi^\log_1(x, \bar{x}), \mathcal{F}_x\right),$$

where the last isomorphism follows from Lemma 2.66.

To prove the claim, up to replacing the index category with a cofinal one, we may assume that all the $V$’s are affinoid. In addition, up to étale localization, we may fix a chart $X \to \text{Spa}(k[P], k^+[P])$ by choosing a splitting of the map $M_{X,x} \to P$. Then we form the site $\lim V_{\text{két}}$ as in [AGV73, VI, 8.2.3], where $V$ runs over the étale neighborhoods of $x$. Then we have the induced morphisms $i_V^{-1} : V_{\text{két}} \to x_{\text{két}}$ and also $i^{-1} : \lim V_{\text{két}} \to x_{\text{két}}$. It is clear that this induces an equivalence of the associated topoi, because every Kummer étale covering of $x$ can be further covered by the standard Kummer étale coverings of $x$ induced by $[n] : P \to P$ for some integers $n \geq 1$, and because coverings of the latter kind are in the essential image of $i^{-1}$. Then the claim follows (again) from Lemma 2.66.

Let $n$ be a positive integer invertible in $k$. Let $X$ and $M_{X,\text{két}}$ be as in Proposition 2.62. Then there is an exact sequence $1 \to \mu_n \to M_{X,\text{két}}^{\text{gp}} \to M_{X,\text{két}}^{\text{gp}} \to 1$ of sheaves on $X_{\text{két}}$, whose pushforward along $\varepsilon : X_{\text{két}} \to X_{\text{ét}}$ induces a map

$$\overline{M}_{\text{két}}^{\text{gp}} / n\overline{M}_{\text{két}}^{\text{gp}} \to R^1\varepsilon_*(\mu_n).$$

At every geometric point $x$ of $X$, it is clear that this map is nothing but the isomorphism in Lemma 2.67. Therefore, we obtain the following lemma:

**Lemma 2.68.** The above map is an isomorphism and, for each $i$, the canonical morphism $\wedge^i R^1\varepsilon_*(\mu_n) \to R^1\varepsilon_*(\mu_n)$ is an isomorphism.

Next, we define and study Kummer étale local systems on $X$.

**Definition 2.69.**

1. The constant sheaf (of sets) on $X_{\text{két}}$ associated with a set $M$ is the sheafification of the presheaf that assigns the set $M$ to every $Y \subset X_{\text{két}}$. A locally constant sheaf on $X_{\text{két}}$ is a sheaf $\mathcal{F}$ such that there is a covering $Y \to X$ in $X_{\text{két}}$ such that $\mathcal{F}|_V$ is a constant sheaf.

2. A $\mathbb{Z}_p$-local system (or lisse $\mathbb{Z}_p$-sheaf) on $X_{\text{két}}$ is an inverse system of $\mathbb{Z}/p^n$-modules $L = (L_n)_{n \geq 1}$ on $X_{\text{két}}$ such that each $L_n$ is a locally constant sheaf which are locally (on $X_{\text{két}}$) associated with finitely generated $\mathbb{Z}/p^n$-modules, and such that the inverse system is isomorphic in the pro-category to an inverse system in which $L_{n+1}/p^n \cong L_n$. 

(3) A \(\mathbb{Q}_p\)-local system (or \(\mathbb{Q}_p\)-sheaf) on \(X_{\text{kétt}}\) is an element of the stack associated with the fibered category of isogeny \(\mathbb{Q}_p\)-sheaves.

The main result of this subsection is the following theorem:

**Theorem 2.70.** Let \(X\) and \(D\) be as in Example 2.12, and let \(U := X - D\). Assume that the characteristic of \(k\) is zero. Let \(L\) be a torsion local system on \(U_{\text{ét}}\). Then \(\mathcal{J}_{\text{kétt}}(L)\) is a torsion local system on \(X_{\text{kétt}}\), and \(R^i\mathcal{J}_{\text{kétt}}(L) = 0\) for all \(i > 0\).

**Proof.** Let \(V \to U\) be a finite étale covering that trivializes \(L\). By Proposition 2.56, this map extends to a Kummer étale map \(f : Y \to X\), where \(Y\) is a normal rigid analytic variety with its log structures defined by the preimage of \(D\) as in Example 2.11. Then \(\mathcal{J}_{\text{kétt}}(L)|_V\) is constant, and \(\mathcal{J}_{\text{kétt}}(L)\) is a torsion local system on \(X_{\text{kétt}}\).

Given the above map \(f : Y \to X\), up to étale localization on \(X\), suppose that there exists some \(X^\otimes \to X\) as in Lemma 2.59. Then the underlying adic space of \(Z := Y \times_X X^\otimes\) is smooth, its log structure is defined by some normal crossings divisor as in Example 2.12, and the induced morphism \(Z \to X\) is Kummer étale, because these are true for \(X^\otimes\). (Alternatively, we can construct \(Z \to X\), as in the proof of Proposition 2.56, by using Lemma 2.58 and the last assertion of Lemma 2.57.) Thus, in order to show that \(R^i\mathcal{J}_{\text{kétt}}(L) = 0\) for all \(i > 0\), up to replacing \(X\) with \(Z\), we may assume that \(L = \mathbb{Z}/n\) is a constant local system.

Consider the commutative diagram:

\[
\begin{array}{ccc}
U_{\text{kétt}} & \xrightarrow{\mathcal{J}_{\text{kétt}}} & X_{\text{kétt}} \\
\downarrow \cong & & \downarrow \varepsilon \\
U_{\text{ét}} & \xrightarrow{\mathcal{J}_{\text{ét}}} & X_{\text{ét}}
\end{array}
\]

Note that, to show that \(\mathbb{Z}/n \to R\mathcal{J}_{\text{kétt}}(\mathbb{Z}/n)\) is an isomorphism, it suffices to further pushforward along \(\varepsilon\) and show that

\[
R\varepsilon_*(\mathbb{Z}/n) \to R\mathcal{J}_{\text{ét}}(\mathbb{Z}/n)
\]

is an isomorphism, since the Kummer étale localization of the isomorphism (2.72) will give the isomorphism \(\mathbb{Z}/n \to R\mathcal{J}_{\text{kétt}}(\mathbb{Z}/n)\). Indeed, let \(C\) be the cone of \(\mathbb{Z}/n \to R\mathcal{J}_{\text{kétt}}(\mathbb{Z}/n)\) (in the derived category). To show that \(C\) vanishes, it suffices to show that \(H^*(V_{\text{kétt}}, C) = 0\) for every \(V \to X\) that is the composition of an étale covering and a standard Kummer étale covering of \(X\) (defined by the chart \(X \to \mathbb{D}^n\)). Note that the complement of \(U \times_X V\) in \(V\) is a normal crossings divisor, which induces the log structure on \(V\) as in Example 2.12. Now, consider the commutative diagram (2.71), with \(U \to X\) replaced by \(U \times_X V \to V\). If we know the corresponding isomorphism (2.72) for this diagram, then we will have \(R\varepsilon_*(C|_{V_{\text{ét}}}) = 0\) on \(V_{\text{ét}}\), which will imply that \(H^*(V_{\text{kétt}}, C) = 0\).

Therefore, it suffices to prove (2.72). But this follows from Lemma 2.68 and the following Lemma 2.73.

**Lemma 2.73.** There are canonical isomorphisms \(R^1\mathcal{J}_{\text{ét}}(\mathbb{Z}/n) \cong \mathcal{M}/n\) and \(R^i\mathcal{J}_{\text{ét}}(\mathbb{Z}/n) \cong \Lambda^i R^1\mathcal{J}_{\text{ét}}(\mathbb{Z}/n)\), for all \(i\).

**Proof.** This is essentially proved in [Ber93]. Suppose \(i : D_0 \to X\) is a closed immersion of codimension \(d\) of a smooth subspace \(D_0\) into a smooth ambient space
X. By [Ber95, Theorem 2.1], we have

\[
R^i \mathbb{Z}/n(\mathbb{Z}/n) \cong \begin{cases} 
\mathbb{Z}/n, & \text{if } q = 2d; \\
(\mathbb{Z}/n)(1), & \text{for } i = 1; \\
0, & \text{for all } i > 1.
\end{cases}
\]

We apply this result to our situation. Suppose \( D = \sum_r D_r \), where the \( D_r \)'s are the irreducible components of \( D \) (i.e., the image of a connected component of the normalization of \( D \), as in [Con99]). Since the assertions to prove are étale local, we may assume that \( D = \sum_r D_r \) is a simple normal crossings divisor in the sense that all the \( D_r \)'s above are smooth. Let \( j_r : U_r = X - D_r \hookrightarrow X \) denote the open embeddings. Then, by the above-mentioned result (2.74) from [Ber95, Theorem 2.1] and by induction on \( r \), the lemma follows from the following:

- \( R^i j_{r,*} \mathbb{Z}/n(\mathbb{Z}/n) \cong \begin{cases} 
\mathbb{Z}/n, & \text{for } i = 0; \\
(\mathbb{Z}/n)(1), & \text{for } i = 1; \\
0, & \text{for all } i > 1.
\end{cases} \)
- The canonical morphisms \( \oplus_r R^i j_{r,*} \mathbb{Z}/n(\mathbb{Z}/n) \rightarrow R^i j_* \mathbb{Z}/n(\mathbb{Z}/n) \rightarrow R^i j_* Z_r(\mathbb{Z}/n), \) for all \( i \geq 0 \), are isomorphisms. \( \square \)

**Corollary 2.75.** Let \( k, X \), and \( U \) be as in Theorem 2.70. Let \( \mathbb{L} \) be an étale \( \mathbb{Z}_p \)- (resp. \( \mathbb{F}_p \))-local system on \( U_{\text{ét}} \). Then \( \mathbb{L} := j_{\text{ét}_*}(\mathbb{L}) \) is a Kummer étale \( \mathbb{Z}_p \)- (resp. \( \mathbb{F}_p \))-local system extending \( \mathbb{L} \), and there is a canonical isomorphism

\[
H^i(U_{\text{ét}}, \mathbb{L}) \cong H^i(X_{\text{ét}}, \mathbb{L}),
\]

for each \( i \geq 0 \). (We will use this isomorphism only when \( k \) is separably closed.)

**Proof.** The case of \( \mathbb{F}_p \)-local systems follows from Theorem 2.70. The case of \( \mathbb{Z}_p \)-local systems follows from taking limits of \( \mathbb{Z}_p/p^m \)-local systems over integers \( m \geq 1 \). \( \square \)

We also recall the following finiteness result and the primitive comparison theorem for the cohomology of Kummer étale \( \mathbb{F}_p \)-local systems:

**Theorem 2.76.** Assume that \( k \) is algebraically closed and of characteristic zero. Assume that \( X \) is an \( fs \) log adic space whose underlying adic space is proper over \( k \). Let \( \mathbb{L} \) be an \( \mathbb{F}_p \)-local system on \( X_{\text{ét}} \). Then \( H^i(X_{\text{ét}}, \mathbb{L}) \) is finite, and there is a canonical almost isomorphism

\[
H^i(X_{\text{ét}}, \mathbb{L}) \otimes (k^+/p) \cong H^i(X_{\text{ét}}, \mathbb{L} \otimes (\mathcal{O}_X^+/p)).
\]

**Proof.** See [Dia18, Proposition 4.4]. The proof makes use of the pro-Kummer étale topology of an \( fs \) log adic space, which will be discussed in the next subsection. \( \square \)

**Remark 2.77.** Assume that \( k \) is algebraically closed and of characteristic zero. Recall that there is no general finiteness results for étale cohomology of \( \mathbb{F}_p \)-local systems on non-proper rigid analytic varieties over \( k \), as is well known (via Artin–Schreier theory) that \( H^1(\mathcal{D}, \mathbb{F}_p) \) is infinite. However, by combining Corollary 2.75 and Theorem 2.76, we see that, if \( X \) is a smooth rigid analytic variety over \( k \) that admits a proper smooth compactification \( X \hookrightarrow \overline{X} \) such that \( \overline{X} - X \) is a divisor of normal crossing, then \( H^i(X_{\text{ét}}, \mathbb{L}) \) is finite for every \( \mathbb{F}_p \)-local system \( \mathbb{L} \) on \( X_{\text{ét}} \).
2.4. Pro-Kummer étale topologies. In this subsection, let us review the pro-Kummer étale topologies introduced in [Dia18]. Throughout the subsection, we assume that char($k$) = 0. Let pro-$X_{\text{ét}}$ denote the category of pro-objects of $X_{\text{ét}}$. We define $X_{\text{prokét}}$ as the full subcategory of pro-$X_{\text{ét}}$ consisting of objects that can be represented as a limit

$$U = \lim_{i \in I} U_i \to X$$

(over some small cofiltered category $I$ depending on $U$) such that the structural morphisms $U_i \to X$ are Kummer étale and such that the transition morphisms $U_i \to U_j$ are finite Kummer étale and surjective for all sufficiently large $j$. Such a presentation $U = \lim_{i \in I} U_i$ is called a pro-Kummer étale presentation for $U$. Note that $X_{\text{ét}}$ embeds into $X_{\text{prokét}}$ naturally, and there is a well-defined functor from $X_{\text{prokét}}$ to the category of topological spaces, given by

$$U = \lim_{i \in I} U_i \to |U| = \lim_{i \in I} |U_i|.$$ 

A morphism $U \to V$ in pro-$X_{\text{ét}}$ is called Kummer étale (resp. finite Kummer étale, resp. strictly Kummer étale) if there exists a Kummer étale (resp. finite Kummer étale, resp. strictly Kummer étale) morphism $U_0 \to V_0$ in $X_{\text{ét}}$ such that $U = U_0 \times_{V_0} V$ for some morphism $V \to V_0$. There is a notion of coverings in $X_{\text{prokét}}$, making $X_{\text{prokét}}$ a site—see [Dia18, Definition 3.2] (which generalizes [Sch16] to the current setting).

**Definition 2.78.** Let $\nu : X_{\text{prokét}} \to X_{\text{ét}}$ denote the natural projection of sites. We define the following sheaves on $X_{\text{prokét}}$:

1. The integral structure sheaf of $X$ is defined by $O_{X_{\text{prokét}}}^+ = \nu^{-1}(O_{X_{\text{ét}}}^+)$, and the structure sheaf is defined by $O_{X_{\text{prokét}}} = \nu^{-1}(O_{X_{\text{ét}}})$.
2. The completed integral structure sheaf is defined by

$$\widehat{O}_{X_{\text{prokét}}}^+ = \lim_{n} (O_{X_{\text{prokét}}}^+ / p^n),$$

and the completed structure sheaf is defined by $\widehat{O}_{X_{\text{prokét}}} = \widehat{O}_{X_{\text{prokét}}}^+ [\frac{1}{p}]$. For simplicity, when no confusion will arise, we shall write the pair as $(\widehat{O}, \widehat{O}^+)$. 
3. The tilted integral structure sheaf is defined by

$$\widehat{O}_{X_{\text{prokét}}}^+ = \lim_{\Phi} (O_{X_{\text{prokét}}}^+ / p)$$

where the transition maps $\Phi$ are $x \mapsto x^p$, and $\widehat{O}_{X_{\text{prokét}}}^+ = \widehat{O}_{X_{\text{prokét}}}^+ [\frac{1}{p}]$. For simplicity, when no confusion will arise, we shall write the pair as $(\widehat{O}^p, \widehat{O}^{p+})$.
4. Let

$$\alpha : M_{X_{\text{prokét}}} := \nu^{-1}(M_{X_{\text{ét}}}) \to O_{X_{\text{prokét}}^+},$$

$$\alpha^p : M_{X_{\text{prokét}}^p} := \lim_{a \to a^p} M_{X_{\text{prokét}}} \to \widehat{O}_{X_{\text{prokét}}}^+,$$

and

$$M_{X_{\text{prokét}}}^+ := (\alpha^p)^{-1}(\widehat{O}_{X_{\text{prokét}}}^+).$$

For simplicity, when no confusion will arise, we shall write $M, M^p, M^{p+}$ instead of $M_{X_{\text{prokét}}}, M_{X_{\text{prokét}}^p}, M_{X_{\text{prokét}}}^+$.
Definition 2.79. An object \( U \in X_{\text{pro-Kët}} \) is called log affinoid perfectoid if it has a pro-Kummer étale presentation

\[
U = \varprojlim_{i \in I} U_i = \varprojlim_{i \in I} \left( \text{Spa}(R_i, R_i^+), \mathcal{M}_i, \alpha_i \right) \to X
\]

such that the following are true:

1. Each \( U_i \) admits a global sharp fs chart \( P_i \) such that each transition map \( U_j \to U_i \) is modeled on the Kummer étale chart \( P_i \to P_j \).
2. The \( p \)-adic completion \((R, R^+)\) of \( \varprojlim_{i \in I} (R_i, R_i^+) \) is a perfectoid affinoid algebra.
3. The monoid \( P = \varprojlim_{i \in I} P_i \) is \( n \)-divisible for all integers \( n \geq 1 \).

If \( U = \varprojlim_{i \in I} U_i \in X_{\text{pro-Kët}} \) is a log affinoid perfectoid object as above, we call \( \hat{U} = \text{Spa}(R, R^+) \) the associated affinoid perfectoid space of \( U \). This is independent of the choice of the presentation \( U = \varprojlim_{i \in I} U_i \). In addition, the assignment \( U \mapsto \hat{U} \) defines a functor from the full subcategory of log affinoid perfectoid objects in \( X_{\text{pro-Kët}} \) to the category of perfectoid spaces.

For a perfectoid space \( Y \), let \( Y_{\text{pro-Kët}} \) denote its pro-étale site as in [KL15, Definition 9.1.4] (which is more general than the definition in [Sch16]; that is, \( Y_{\text{pro-Kët}} \) is the full subcategory of pro-\( Y_\text{ét} \) consisting of objects each of which can be presented as a limit \( \varprojlim_{i \in I} V_i \) such that the structural morphisms \( V_i \to Y \) are étale and such that the transition morphisms \( V_i \to V_j \) are finite étale and surjective for all sufficiently large \( j \).

The following proposition plays a fundamental role.

Proposition 2.80. (1) Let \( U \in X_{\text{pro-Kët}} \) be a log affinoid perfectoid object. If \( V \to U \) is a Kummer étale (resp. finite Kummer étale) morphism, then it is étale (resp. finite étale) and \( V \) is log affinoid perfectoid. The induced morphism \( \hat{V} \to \hat{U} \) is étale (resp. finite étale), and, the construction \( V \mapsto \hat{V} \) induces an equivalence of topoi \( \hat{U}_{\text{pro-Kët}} \cong X_{\text{pro-Kët}}/U \).

(2) Suppose \( X \) is log smooth. Then the log affinoid perfectoid objects in \( X_{\text{pro-Kët}} \) form a basis for the pro-Kummer étale topology.

Proof. Part (1) is essentially [Dia18, Lemma 3.18], and part (2) is [Dia18, Proposition 3.20]. Due to their importance, let us sketch the proofs.

We first prove (1). Let \( V \to U \) be a Kummer étale (resp. finite Kummer étale) morphism in \( X_{\text{pro-Kët}} \). In particular, there exists some Kummer étale (resp. finite Kummer étale) morphism \( V_0 \to U_0 \in X_\text{kët} \) such that \( V = V_0 \times_{U_0} U \). We may assume that \( 0 \in I \), that \( 0 \) is initial in \( I \) (up to replacing \( I \) with a cofinal subcategory), and that \( V_0 \) is affinoid. Let \( U_0^{\pm} \) be as in Lemma 2.59 such that \( V_0 \times_{U_0} U_0^{\pm} \to U_0^{\pm} \) is strictly étale (resp. strictly finite étale). Since \( P = \varprojlim_{i \in I} P_i \) is \( m \)-divisible, there is some \( i \in I \) such that \( P_0 \to P_i \) factors as \( P_0 \to \frac{1}{m} P_0 \to P_i \). Then \( V_0 \times_{U_0} U_i \to U_i \) is strictly étale (resp. strictly finite étale). We may replace \( I \) with the cofinal full subcategory of objects that receive maps from \( i \). Then \( \hat{V} := (V_0 \times_{U_0} U_i) \to \hat{U} \to U \) is strictly étale (resp. strictly finite étale), and hence so is \( \hat{V} \to \hat{U} \). This shows that we have a well-defined morphism of sites \( \hat{U}_{\text{pro-Kët}} \to (X_{\text{pro-Kët}})/U \). To show that it induces an equivalence of topoi, it suffices to note that every étale morphism \( W \to \hat{U} \) that is a composition of rational localizations and finite étale morphisms arises in the above way, by [KL15, Lemma 2.6.5 and Proposition 2.6.8].
To prove (2), we need to show that, for each $U \in X_{\text{proK}}$, we can find locally a pro-Kummer étale covering $\tilde{U} \to U$ by a log affinoid perfectoid object. By [Sch13, Proposition 3.15] (see also [Dia18, Remark 3.5]), we may base change $X$ to a perfectoid field $l$ of characteristic zero. We may also assume that:

- $U = \lim_{i \in I} U_i \to U_0$ has a final object $U_0$;
- $U_0 = \text{Spa}(R_0, R_0^+)$ is affinoid and admits (as in Definition 2.28) a toric chart $U_0 \to \text{Spa}(l(P_0), l^+(P_0))$, where $P_0$ is some sharp fs monoid; and
- all transition maps $U_j \to U_i$ are finite Kummer étale surjective.

Firstly, we construct a log affinoid perfectoid covering $\tilde{U}_0 \to U_0$. For each integer $m \geq 1$, let $U_{0,m} := U_0 \times_{\text{Spa}(l(P_0), l^+(P_0))} \text{Spa}(l(P_m^0), l^+(P_m^0)) = \text{Spa}(R_{0,m}, R_{0,m}^+)$. We claim that $\tilde{U}_0 := \lim_{m \in \mathbb{Z}_{\geq 1}} U_{0,m} \in X_{\text{proK}}$ is a log affinoid perfectoid object in $X_{\text{proK}}$. Indeed, if $(\tilde{R}_0, \tilde{R}_0^+)$ is the completion of $\lim_{m \in \mathbb{Z}_{\geq 1}} (R_{0,m}, R_{0,m}^+)$, then $\tilde{R}_0 := \left( \lim_{m \in \mathbb{Z}_{\geq 1}} P_0 \otimes_l l^+(P_0^m) \right)^\wedge \cong R_0 \otimes_l l^+(P_0^m)$ is an affinoid perfectoid algebra by Example 2.8(2) and [Sch12, Theorem 7.9]. Since each $U_{0,m}$ is modeled on $P_0^m$, the transition maps $U_{0,m'} \to U_{0,m}$ have charts $P_0^{m'} \to P_0^m$ for $m|m'$, and $\lim_{m \in \mathbb{Z}_{\geq 1}} P_0^m = P_0^+$ is $n$-divisible for all integers $n \geq 1$.

For general $U = \lim_{i \in I} U_i \to U_0$, we show that $\tilde{U} := U \times_{U_0} \tilde{U}_0 \in X_{\text{proK}}$ is a log affinoid perfectoid object. By the same argument as in the proof of (1), for each integer $i \in I$, there exists $m_i \in \mathbb{Z}_{\geq 1}$ such that $U_i \times_{U_0} U_{0,m_i} \to U_{0,m_i}$ is finite étale surjective for all $m$ divisible by $m_i$. Consider the index set $\Sigma = \{(i, m) \in I \times \mathbb{Z}_{\geq 1} | m_i \text{ divides } m\}$ ordered by partial ordering: $(i', m') \geq (i, m)$ if $i' \geq i$ and $m$ divides $m'$. Then $\tilde{U} = \lim_{(i, m) \in \Sigma} (U_i \times_{U_0} U_{0,m})$ is a pro-Kummer étale presentation such that each $U_i \times_{U_0} U_{0,m}$ has global chart $P_0^{m_i}$. Therefore, $\tilde{U}$ is log affinoid perfectoid because:

- the underlying structure ring is a direct limit of perfectoid algebras associated with $U_i \times_{U_0} \tilde{U}_0$ (which are log affinoid perfectoid as they are finite étale over $U_0$); and
- the monoid $\lim_{(i, m) \in \Sigma} P_0^{m_i} = \lim_{m \in \mathbb{Z}_{\geq 1}} P_0^m = P_0^+$ is $n$-divisible for all integers $n \geq 1$. \hfill $\Box$

Here are some consequences of the above result.

**Proposition 2.81.** Suppose $X$ is log smooth. Let $U \in X_{\text{proK}}$ be a log affinoid perfectoid object, with associated perfectoid space $\tilde{U} = \text{Spa}(R, R^+)$. Let $\text{Spa}(R^+, R^{+\otimes})$ be its tilt. Then:

1. $\hat{\mathcal{O}}_{X_{\text{proK}}}^+(U) \cong R^+$ and $\hat{\mathcal{O}}_{X_{\text{proK}}}^-(U) \cong R$.
2. $H^j(U, \hat{\mathcal{O}}_{X_{\text{proK}}}^+/p^n)$ is almost zero for all $n > 0$ and $j > 0$. Consequently, $H^j(U, \hat{\mathcal{O}}_{X_{\text{proK}}}^+/p^n)$ is almost zero for all $j > 0$.
3. $\hat{\mathcal{O}}_{X_{\text{proK}}}^-(U) \cong R^{+\otimes}$ and $\hat{\mathcal{O}}_{X_{\text{proK}}}^+(U) \cong R^\otimes$.
4. $H^j(U, \hat{\mathcal{O}}_{X_{\text{proK}}}^+/p^n)$ is almost zero for all $j > 0$.
5. $\mathcal{M}_{X_{\text{proK}}}(U) \cong \lim_{i \in I} \mathcal{M}_{U_i}(U_i)$, where $U = \lim_{i \in I} U_i$ is a pro-Kummer étale presentation.
Proof. See [Dia18] Propositions 3.22, 3.23, and 3.25. Given Proposition 2.80, the proofs are similar to the proofs of the corresponding statements in the pro-étale setting as in [Sch13]. □

Proposition 2.82. Let $U = \text{Spa}(A,A^+)$ be a log affinoid perfectoid object. Then the following are true:

1. If $L$ is a $\mathbb{F}_p$-local system on $X_{\text{prok}^\text{ét}}/U$, then $\check{H}^j(X_{\text{prok}^\text{ét}}/U, L \otimes (O^+_{X_{\text{prok}^\text{ét}}}/p))$ is almost zero for all $j > 0$, and $(L \otimes (O^+_{X_{\text{prok}^\text{ét}}}/p))(U)$ is an almost finitely generated projective $A^{+\alpha}/p$-module. If $U' = \text{Spa}(A',A'^+)$ is another log affinoid perfectoid, with a map $U' \to U$ in $X_{\text{prok}^\text{ét}}$, then
   $$(L \otimes (O^+_{X_{\text{prok}^\text{ét}}}/p))(U') \cong (L \otimes (O^+_{X_{\text{prok}^\text{ét}}}/p))(U) \otimes A^{+\alpha}/p (A^{+\alpha}/p).$$

2. For every locally free $\hat{O}_{X_{\text{prok}^\text{ét}}/U}$-module of finite rank and every $j > 0$, we have $H^j(X_{\text{prok}^\text{ét}}/U, F) = 0$.

Proof. Given Proposition 2.80, the proof of part 1 (resp. part 2) is almost identical to that of [Sch13] Lemma 4.12 (resp. [LZ17] Proposition 2.3), except with the input [Sch13] Lemma 4.10 there replaced with Proposition 2.81 here. □

Proposition 2.83. There is a base $\mathcal{B}$ of $X_{\text{prok}^\text{ét}}$ such that, for every $p$-torsion locally constant sheaf $\mathbb{L}$ on $X_{k^\text{ét}}$ and every $i > 0$, we have $H^i(X_{\text{prok}^\text{ét}}/U', \mathbb{L}) = 0$ for all $V \in \mathcal{B}$.

Proof. Let $U$ be a log affinoid perfectoid object, and let $\hat{U} = \text{Spa}(A,A^+)$ denote the associated affinoid perfectoid space. By passing to a covering, we may assume that $A$ is integral. Let $(A_\infty,A^+_\infty)$ be a universal covering of $A$ (i.e., $A_\infty$ is the union of all finite étale extensions $A_j$ of $A$ in a fixed algebraic closure of the fractional field of $A$, and $A^+_\infty$ is the integral closure of $A^+$ in $A_\infty$). Let $(\hat{A}_\infty,\hat{A}^+_\infty)$ denote the $p$-adic completion of $\lim_{\leftarrow j} \text{Spa}(A_j,A^+_j)$, and let $\hat{U}_\infty := \text{Spa}(\hat{A}_\infty,\hat{A}^+_\infty)$. Then this $\hat{U}_\infty$ is affinoid perfectoid. As explained in the proof of Proposition 2.80, there is some $V \to U$ in $X_{\text{prok}^\text{ét}}$, with $V = \lim_{\leftarrow j} V_j$ log affinoid perfectoid, such that $\hat{V} \cong \hat{U}_\infty \to \hat{U}$. Note that $\mathbb{L}|_V$ is a trivial local system. (Indeed, for any finite Kummer étale covering $Y \to X$ trivializing $\mathbb{L}$, the pullback $W := Y \times X V \to V$ and the induced morphism $\hat{W} \to \hat{V}$ are strictly finite étale by Proposition 2.80, and so $\hat{W} \to \hat{V}$ has a section by assumption.) Then we have $H^i(X_{\text{prok}^\text{ét}}/V, \mathbb{L}) \cong H^i(\check{\nu}^*(\mathbb{L})) \cong H^i(\hat{\nu}^*(L) \otimes \mathbb{L}) = 0$, for all $i > 0$, where the first isomorphism follows from Proposition 2.80, the second isomorphism follows from [Sch13] Corollary 3.17(i) (note that the locally noetherian assumption there on $X$ is not needed), and the last equality follows verbatim from the last paragraph of the proof of [Sch13] Theorem 4.9. □

Definition 2.84. Let $\mathbb{Z}_p = \lim_{\longleftarrow n} (\mathbb{Z}/p^n)$ as a sheaf of rings on $X_{\text{prok}^\text{ét}}$, and let $\mathbb{Q}_p = \mathbb{Z}_p[1/p]$. A $\mathbb{Z}_p$-local system on $X_{\text{prok}^\text{ét}}$ is a sheaf of $\mathbb{Z}_p$-modules on $X_{\text{prok}^\text{ét}}$ that is locally (on $X_{\text{prok}^\text{ét}}$) isomorphic to $\mathbb{Z}_p \otimes_{\mathbb{Z}} L$ for some finitely generated $\mathbb{Z}_p$-modules $L$. The notion of $\mathbb{Q}_p$-local system on $X_{\text{prok}^\text{ét}}$ is defined similarly.

Let $\nu : X_{\text{prok}^\text{ét}} \to X_{k^\text{ét}}$ denote the natural projection of sites.
Lemma 2.85. (1) The functor sending $L = (L_n)_{n \geq 1}$ to $\hat{L} = \varprojlim_n \nu^*(L_n)$ is an equivalence of categories from the category of $\mathbb{Z}_p$-local systems on $X_{k\acute{e}t}$ to the category of $\hat{\mathbb{Z}}_p$-local systems on $X_{\text{prok}\acute{e}t}$. Moreover, $\hat{L} \otimes_{\hat{\mathbb{Z}}_p} \hat{\mathbb{Q}}_p$ is a $\hat{\mathbb{Q}}_p$-local system.

(2) For all $i > 0$, we have $R^i \varprojlim_n \nu^*(L_n) = 0$.

Proof. Part (1) is clear from the definition. Part (2) follows from [Sch13, Lemma 3.18] and Proposition 2.83. □

3. Logarithmic de Rham period sheaves

In this section, we define and study the logarithmic de Rham period sheaves, generalizing the usual ones studied in [Bri08, Section 5], [Sch13, Section 6], and [Sch16]. We shall assume that $k$ is a nonarchimedean local field of characteristic zero and residue characteristic $p > 0$.

3.1. Definitions. Let $(R, R^+)$ be a perfectoid affinoid algebra over a perfectoid field over $k$, with $(R^\flat, R^\flat^+)$ its tilt. Recall that there are the period rings

$A_{\text{inf}}(R, R^+) := W(R^\flat^+)$

and

$B_{\text{inf}}(R, R^+) := A_{\text{inf}}(R, R^+)/[\frac{1}{p}]$.

It is well known that there is a natural surjective map

$\theta : A_{\text{inf}}(R, R^+) \rightarrow R^+$,

whose kernel is a principal ideal generated by some $\xi \in A_{\text{inf}}(R, R^+)$, which is not a zero divisor (see, for example, [Sch13, Lemma 6.3]). Let

$B_{\text{dR}}^+(R, R^+) := \varprojlim_r \left( B_{\text{inf}}(R, R^+)/[\xi^r] \right)$

and

$B_{\text{dR}}(R, R^+) := B_{\text{dR}}^+(R, R^+)[\xi^{-1}]$.

We shall equip $B_{\text{dR}}(R, R^+)$ with the filtration given by

$\text{Fil}^r B_{\text{dR}}(R, R^+) := \xi^r B_{\text{dR}}^+(R, R^+)$,

for all $r \in \mathbb{Z}$. This filtration is separated, complete, and independent of the choice of $\xi$. Therefore, for all $r \in \mathbb{Z}$, we have a canonical isomorphism

$\text{gr}^r B_{\text{dR}}(R, R^+) \cong \xi^r R$.

Let us record the following lemma for later use.

Lemma 3.3. For every integer $r \geq 1$, the ring $A_{\text{inf}}(R, R^+)/[\xi^r]$ is $p$-adically separated and complete.

Proof. Note that $R^+$ is $p$-torsion-free, and hence that $(\xi) \cap (p) = (p\xi)$. Since $\xi$ is not a zero divisor, by induction, we obtain that $(\xi^r) \cap (p) = (p\xi^r)$. This implies that $(\xi^r)$ is $p$-adically closed, yielding the lemma. □

Now let $X$ be a locally noetherian fs log adic space over $k$. By following [Sch13, Section 6], the above constructions can be sheafified.

Definition 3.4. We define the following sheaves on $X_{\text{prok}\acute{e}t}$:
(1) Let \( \mathbb{A}_{\inf} = W(\hat{\mathcal{O}}_{\mathbb{A}_{\prokét}}^+) \) and \( \mathbb{B}_{\inf} = \mathbb{A}_{\inf}[\frac{1}{p}] \), where the latter is equipped with a natural map \( \theta : \mathbb{B}_{\inf} \to \hat{\mathcal{O}}_{\mathbb{A}_{\prokét}}^+ \).

(2) Let \( \mathbb{B}_{\text{dR}}^+ = \lim_{\leftarrow t} \mathbb{B}_{\inf} / (\ker \theta)^t \), and \( \mathbb{B}_{\text{dR}} = \mathbb{B}_{\text{dR}}^+ [t^{-1}] \), where \( t \) denotes any generator of \( (\ker \theta)\mathbb{B}_{\text{dR}}^+ \). (We will make a choice of \( t \) in (3.18) below.)

(3) The filtration on \( \mathbb{B}_{\text{dR}}^+ \) is given by \( \text{Fil}^j \mathbb{B}_{\text{dR}}^+ = (\ker \theta)^j \mathbb{B}_{\text{dR}}^+ \). It induces a filtration on \( \mathbb{B}_{\text{dR}} \) given by \( \text{Fil}^j \mathbb{B}_{\text{dR}} := \bigoplus_{j \geq -r} t^{-s} \text{Fil}^j \mathbb{B}_{\text{dR}}^+ \).

In the following proposition, let \( k \) be a perfectoid field (over \( \mathbb{Q}_p \)). Assume that \( \wp/p \in (k^+)^\times \). Let \( \mathbb{A}_{\inf}(k, k^+) = W(k^+) \). Let \( \mathbb{w}^p = (\wp, \wp^\frac{1}{p}, \ldots) \) be a pseudo-uniformizer of \( k^p \). In the following situation, we consider the almost setting with respect to this ring and the ideal generated by \( [(\wp^m)^{\frac{1}{pm}}] \) for all integers \( m \geq 1 \).

**Proposition 3.5.** Suppose \( U \in X_{\prokét} \) is log affinoid perfectoid, with associated perfectoid space \( \hat{U} = \text{Spa}(R, R^+) \).

1. We have a canonical isomorphism \( \mathbb{A}_{\inf}(U) \cong \mathbb{A}_{\inf}(R, R^+) \), and similar isomorphisms for \( \mathbb{B}_{\inf}, \mathbb{B}_{\text{dR}}^+ \), and \( \mathbb{B}_{\text{dR}} \).

2. \( H^j(U, \mathbb{A}_{\inf}) \) and \( H^j(U, \mathbb{B}_{\inf}) \) are almost zero for all \( j > 0 \).

3. \( H^j(U, \mathbb{B}_{\text{dR}}^+) = 0 \) and \( H^j(U, \mathbb{B}_{\text{dR}}) = 0 \) for all \( j > 0 \).

**Proof.** The proof is essentially the same as in the one of [Sch13] Theorem 6.5, with [Sch13] Lemma 5.10 there replaced with Proposition 2.81 here.

**Remark 3.6.** By the previous discussion and Proposition 3.5 (for \( \mathbb{A}_{\inf} \)), the element \( t \) in Definition 3.4 exists locally on \( X_{\prokét} \) and is not a zero divisor. Therefore, the sheaf \( \mathbb{B}_{\text{dR}} \) and its filtration are indeed well defined.

Now assume that \( k \) is a perfectoid field (over \( \mathbb{Q}_p \)) containing all roots of unity.

**Corollary 3.7.** \( \text{gr}^* \mathbb{B}_{\text{dR}} \cong \bigoplus_{r \in \mathbb{Z}} (\hat{\mathcal{O}}_{X_{\prokét}}(r)) \).

**Proof.** This follows from [3.2] and Proposition 3.5 as in [Sch13] Corollary 6.4. □

Now we assume that \( X \) is log smooth over \( k \). We shall construct \( \Omega^\prokét, \log \), a logarithmic version of the geometric de Rham period sheaves \( \Omega_{\mathbb{B}_{\text{dR}}^+, \log} \) introduced in [Bri08 Sch13 Sch16]. By Proposition 2.80, log affinoid perfectoid objects form a basis \( B \) for \( X_{\prokét} \). We will define \( \Omega_{\mathbb{B}_{\text{dR}}, \log} \) to be the sheaf on \( X_{\prokét} \) associated with a presheaf on \( B \).

We adopt the notation in Definition 2.79. Let \( U = \varprojlim_{i \in I} U_i \in X_{\prokét} \) be a log affinoid perfectoid object, with \( U_i = (\text{Spa}(R_i, R_i^+), \mathcal{M}_i, \alpha_i) \) for each \( i \in I \). By Proposition 2.81 (\( (\hat{\mathcal{O}}(U), \hat{\mathcal{O}}^+(U)) \)) is the \( p \)-adic completion \( (R, R^+) \) of the direct limit \( \varinjlim_{i \in I} (R_i, R_i^+) \), which is perfectoid, and \( (\hat{\mathcal{O}}^p(U), \hat{\mathcal{O}}^{p+}(U)) \) is the tilt \( (R^p, R^{p+}) \) of \( (R, R^+) \). Let us write \( \mathcal{M}_i := \mathcal{M}_i(U_i) \), for each \( i \in I \):

\[
\mathcal{M} := \mathcal{M}_{X_{\prokét}}(U) = \varprojlim_{i \in I} \mathcal{M}_i;
\]

and

\[
\mathcal{M}^p := \mathcal{M}_{X_{\prokét}}^p(U) = \varinjlim_{a \to ap} \mathcal{M}_p.
\]
Recall that we have $\alpha_i : M_i \to R_i$, for each $i \in I$, and $\alpha^\ast : M^+ \to R^+$. For each $i \in I$, let $(R_i \otimes_{W(\kappa)} W(R^+)) [M_i \times_M M^+]$ denote the monoid algebra over $R_i \otimes_{W(\kappa)} W(R^+)$ associated with the monoid $M_i \times_M M^+$, and let $e^a$ denote its element corresponding to $a = (a', a'') \in M_i \times_M M^+$. Let

$$S_i := (R_i \otimes_{W(\kappa)} W(R^+)) [M_i \times_M M^+]/(\alpha_i(a') - [\alpha^\ast(a'')] e^a)_{a = (a', a'') \in M_i \times_M M^+}.$$  

(3.8)

By abuse of notation, we shall sometimes also write $e^a = \frac{\alpha_i(a')}{[\alpha^\ast(a'')]}$ in $S_i$. There is a natural map

$$\theta_{\log} : S_i \to R$$

induced by the natural map $R_i \to R$ and the map $\theta : k_{\text{inf}}(R, R^+) \to R^+$ in (3.1) such that $\theta_{\log}(e^a) = 1$, which is well defined because $\theta([\alpha^\ast(a'')]) = \alpha_i(a')$ in $R$, for all $(a', a'') \in M_i \times_M M^+$. Let

$$\tilde{S}_i := \lim_{\leftarrow r} (S_i/(\ker \theta_{\log})^r).$$

Note that $\lim_{\leftarrow i \in I} \tilde{S}_i$ depends only on $U \in X_{\text{prok'}}$, but not on the presentation.

**Definition 3.10.**

1. We define the geometric de Rham period sheaf $\mathcal{O}B^+_{\text{dR, log}}$ on $X_{\text{prok'}}$ to be the sheaf associated with the presheaf sending $U$ to the above $\lim_{\leftarrow i \in I} \tilde{S}_i$. We equip $\mathcal{O}B^+_{\text{dR, log}}$ with a filtration given by

$$\text{Fil}^r \mathcal{O}B^+_{\text{dR, log}} := (\ker \theta_{\log})^r \mathcal{O}B^+_{\text{dR, log}}.$$  

2. We define the filtration on $\mathcal{O}B^+_{\text{dR, log}}[t^{-1}]$, where $t$ is the same as in Definition 3.4, 2, by setting

$$\text{Fil}^r (\mathcal{O}B^+_{\text{dR, log}}[t^{-1}]) := \sum_{s \geq r} t^{-s} \text{Fil}^{r+s} \mathcal{O}B^+_{\text{dR, log}}.$$  

3. Let $\mathcal{O}B_{\text{dR, log}}$ be the completion of $\mathcal{O}B^+_{\text{dR, log}}[t^{-1}]$ with respect to the above filtration, equipped with the induced filtration. Then we have

$$\text{Fil}^r \mathcal{O}B_{\text{dR, log}} = \lim_{\leftarrow s \geq 0} (\text{Fil}^r (\mathcal{O}B^+_{\text{dR, log}}[t^{-1}]) / \text{Fil}^{r+s} (\mathcal{O}B^+_{\text{dR, log}}[t^{-1}])).$$

and $\mathcal{O}B_{\text{dR, log}} = \cup_{r \in \mathbb{Z}} \text{Fil}^r \mathcal{O}B_{\text{dR, log}}$.

Note that $\text{Fil}^0 \mathcal{O}B_{\text{dR, log}}$ is a sheaf of rings and $\mathcal{O}B_{\text{dR, log}} = (\text{Fil}^0 \mathcal{O}B_{\text{dR, log}})[t^{-1}]$. (However, $\mathcal{O}B_{\text{dR, log}} \neq \mathcal{O}B^+_{\text{dR, log}}[t^{-1}]$ in general.)

**Remark 3.11.** Even if the log structure is trivial, our definition of $\mathcal{O}B_{\text{dR, log}}$ (which is then just $\mathcal{O}B_{\text{dR}}$) is slightly different from the ones in [Bri08 Section 5], [Sch13 Section 6], and [Sch16]. The key difference is that we have performed a further completion with respect to the filtration. This modification is necessary because the sheaves $\mathcal{O}B_{\text{dR}}$ defined in loc. cit. are not complete with respect to the filtrations—we thank Koji Shimizu for pointing out this—and therefore could not allow one to relate the Galois cohomology computations for $\mathcal{O}B_{\text{dR}}$ to those for the associated graded pieces. We will see in Corollary 3.39 below that the modified geometric de Rham period sheaf also satisfies the Poincaré lemma and, with the new definition of $\mathcal{O}B_{\text{dR}}$, all the previous arguments in loc. cit. (and also those in [LZ17]) remain essentially unchanged.
The period sheaf $O^+_{dR, log}$ is equipped with a natural log connection. Recall that we denote by $\nu : X_{prokét} \to X_{két}$ and $\varepsilon_{et} : X_{két} \to X_{ét}$ the natural projections of sites, and by $\Omega_{X}^{log,j}$ the sheaf of $j$-th exterior differentials with log poles on $X_{ét}$ as in (2.53). By abuse of notation, we also denote $\nu^* (\varepsilon_{et}^*(\Omega_X^{log,j}))$ by $\Omega_X^{log,j}$.

Note that there is a unique $B_{inf}(U)$-linear log connection
\begin{equation}
\nabla : S_i \to S_i \otimes_R \Omega_X^{log}(U_i)
\end{equation}
extending $d : R_i \to \Omega_X^{log}(U_i)$ and $\delta : M_i \to \Omega_X^{log}(U_i)$ such that
\begin{equation}
\nabla (\alpha_i (a')) = \frac{\alpha_i (a')}{[\alpha^p(a'')] [\alpha^p(a'')]} \delta (a'),
\end{equation}
for all $(a', a'') \in M_i \times_M M^{p+}$. Essentially by definition, we have
\begin{equation}
\nabla ((\ker(\theta_{log})^r) \subset (\ker(\theta_{log}))^{-1} \otimes_R \Omega_X^{log}(U_i)
\end{equation}
for all $r \geq 1$. By taking $\ker(\theta_{log})$-adic completion and direct limit, the above log connection (3.12) uniquely extends to a $B_{dR}^+$-linear log connection
\begin{equation}
\nabla : \Omega^+_{dR, log} \to \Omega^+_{dR, log} \otimes O_{X_{prokét}} \Omega_X^{log}.
\end{equation}
Since $t \in B^+_{dR}$, (3.14) further extends to a $B_{dR}$-linear log connection
\begin{equation}
\nabla : \Omega^+_{dR, log}[t^{-1}] \to \Omega^+_{dR, log}[t^{-1}] \otimes O_{X_{prokét}} \Omega_X^{log},
\end{equation}
satisfying
\begin{equation}
\nabla (\text{Fil}^r(\Omega^+_{dR, log}[t^{-1}])) \subset (\text{Fil}^r(\Omega^+_{dR, log}[t^{-1}])) \otimes O_{X_{prokét}} \Omega_X^{log}
\end{equation}
for all $r \in \mathbb{Z}$. Therefore, (3.15) also extends to a $B_{dR}$-linear log connection
\begin{equation}
\nabla : \Omega_{dR, log} \to \Omega_{dR, log} \otimes O_{X_{prokét}} \Omega_X^{log},
\end{equation}
satisfying
\begin{equation}
\nabla (\text{Fil}^r \Omega_{dR, log}) \subset (\text{Fil}^r \Omega_{dR, log}) \otimes O_{X_{prokét}} \Omega_X^{log}
\end{equation}
for all $r \in \mathbb{Z}$.

3.2. Local study of $B_{dR}$ and $\Omega^+_{dR, log}$. In this subsection, we study $B_{dR}$ and $\Omega^+_{dR, log}$ when there are good local coordinates. These results are similar to [Bri08, Section 5], [Sch13, Section 6], and [Sch16]. Let us first fix some notation. We assume that $k$ is a finite extension of $\mathbb{Q}_p$. Fix a compatible system of roots of unity $\{\zeta_m\}_{m \geq 1}$ in $\mathcal{O}_K$ such that $\zeta_m = \zeta^m_m$ whenever $m | n$. Let $k_m = k(\zeta_m)$, for all $m \geq 1$; let $k_{\infty} = \cup_m k_m$ in $\hat{K}$; and let $\hat{k}_{\infty}$ be the $p$-adic completion of $k_{\infty}$. Then $\hat{k}_{\infty}$ is a perfectoid field. Let $\hat{k}^{\circ}_{\infty}$ denote its tilt. Let
\[ A_{inf} := A_{inf}(\hat{k}_{\infty}, \mathcal{O}_{\hat{k}_{\infty}}) \]
For all $m \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\geq 1}$, define
\[ \epsilon^m_n = (\zeta_m, \zeta_m^p, \zeta_m^{p^2}, \ldots) \in \mathcal{O}_{\hat{k}_{\infty}}, \]
with
\[ [\epsilon^m_n] \in A_{inf}. \]
For each \( y \in \mathbb{Q} \), we write \( y = \frac{m}{n} \) with \( m \in \mathbb{Z} \) and \( n \in \mathbb{Z}_{>1} \), and write \( \zeta^y := \zeta^m \) and \( \epsilon^y := \epsilon^m \). It is clear that these expressions are well defined. Note that \( \epsilon^y := \frac{\epsilon - 1}{\epsilon^m - 1} \) is a pseudo-uniformizer of \( \hat{k}_{k_{\infty}} \).

Let

\[ \xi := \frac{[\epsilon] - 1}{[\epsilon^m] - 1} \in A_{\inf}. \]

It generates the kernel of \( \theta : A_{\inf} \to \mathcal{O}_{\hat{k}_{\infty}} \). Also, let

\[ B_{dR}^+ := \mathbb{B}_{dR}(\hat{k}_{\infty}, \mathcal{O}_{\hat{k}_{\infty}}) = \lim_{\frac{\xi}{r} \to 0} \theta(A_{\inf}[[\frac{1}{p}]]/(\xi^r)). \]

Let

\[ (3.17) \]

\[ t := \log(|\xi|) \in B_{dR}^+. \]

Let \( k \to B_{dR}^+ \) be the unique embedding such that the composition \( k \to B_{dR}^+ \to \hat{k}_{\infty} \) is the natural one.

We assume that \( X = \text{Spa}(R, R^+) \) is an affinoid \log-\adic space over \( k \) which admits a \textit{smooth toric chart} \( X \to \mathbb{D}^n \cong \text{Spa}(k(T_1, \ldots, T_n), \mathcal{O}_k(T_1, \ldots, T_n)) \), as in Definition \[2.28\] for some \( n \geq 0 \). Note that \( \mathbb{D}^n \) has a \textit{pro-Kummer étale covering}

\[ \lim_{\frac{\xi}{m} \to 0} \mathbb{D}^n_m \to \mathbb{D}^n \]

defined as follows. For each \( m \geq 1 \), let \( \mathbb{D}^n_m \) be the \log-\adic space

\[ \text{Spa}(k_m(T_1^\frac{1}{m}, \ldots, T_n^\frac{1}{m}), \mathcal{O}_{k_{\infty}}(T_1^\frac{1}{m}, \ldots, T_n^\frac{1}{m})) \]

endowed with the \log structure associated with the \pre-log structure

\[ \mathbb{Z}^n_{\geq 0} \to k_m(T_1^\frac{1}{m}, \ldots, T_n^\frac{1}{m}) \]

sending \( (a_1, \ldots, a_n) \) to \( T_1^{a_1} \cdots T_n^{a_n} \). For \( m' \mid m \), the transition map \( \mathbb{D}^n_m \to \mathbb{D}^n_{m'} \) is finite \textit{Kummer étale} with the associated map on monoids given by \( \mathbb{Z}^n_{\geq 0} \to \mathbb{Z}^n_{\geq 0} \), sending \( (a_1, \ldots, a_n) \) to \( (\frac{m'}{m}a_1, \ldots, \frac{m'}{m}a_n) \). Consider

\[ \mathbb{D}^n := \lim_{\frac{\xi}{m} \to 0} \mathbb{D}^n_m \in \mathbb{D}^n_{\text{proK}}. \]

It turns out that \( \mathbb{D}^n \) is \log affinoid perfectoid with associated perfectoid space

\[ \hat{\mathbb{D}}^n = \text{Spa}(\hat{k}_{\infty}(T_1^\frac{1}{\infty}, \ldots, T_n^\frac{1}{\infty}), \mathcal{O}_{\hat{k}_{\infty}}(T_1^\frac{1}{\infty}, \ldots, T_n^\frac{1}{\infty})). \]

For later use, let

\[ T_i^\frac{1}{m} := (T_i^\frac{1}{m}, T_i^\frac{1}{m}, \ldots) \in (\mathcal{O}_{\hat{k}_{\infty}}(T_1^\frac{1}{m}, \ldots, T_n^\frac{1}{m}))^b. \]

Similarly, for each \( m \geq 1 \), let

\[ T_i^\frac{1}{m'} := (T_i^\frac{1}{m'}, T_i^\frac{1}{m'}, \ldots) \in (\mathcal{O}_{\hat{k}_{\infty}}(T_1^\frac{1}{m'}, \ldots, T_n^\frac{1}{m'}))^b. \]

Let \( \tilde{X} \) be the pullback of \( \hat{\mathbb{D}}^n \) along \( X \to \mathbb{D}^n \). Then \( \tilde{X} \in X_{\text{proK}} \) is also \log affinoid perfectoid. We write

\[ \text{Spa}(R_m, R_m^+) = X_m := X \times_{\mathbb{D}^n} \mathbb{D}^n_m \]

for each \( m \geq 1 \), and let \( (\hat{R}_{\infty}, \hat{R}_{\infty}^+) \) be the \textit{p}-adic completion of \( \lim_{\frac{\xi}{m} \to 0} (R_m, R_m^+) \). Then \( \hat{X} = \text{Spa}(\hat{R}_{\infty}, \hat{R}_{\infty}^+) \) is the associated perfectoid space of \( \tilde{X} \).
Let $\Gamma = \text{Aut}(\mathbb{D}^n/\mathbb{D}_k^n) \cong (\hat{\mathbb{Z}}(1))^n$. For each $i = 1, \ldots, n$, there is a unique $\gamma_i \in \Gamma$ such that, for every $j = 1, \ldots, n$ and every $m \geq 1$,

$$
\gamma_i T^{1/2}_j = \xi^{b_i}_m T^{1/2}_j.
$$

Then $\{\gamma_1, \ldots, \gamma_n\}$ is a set of topological generators of $\Gamma$.

For every $r \geq 1$, the $p$-adic topology on $A_{\inf}/\xi^r$ is separated and complete by Lemma 3.3. We equip $B_{\inf}^+/\xi^r$ with the structure of a Tate algebra (in the sense of [Sch12, Definition 2.6]) by declaring that $A_{\inf}/\xi^r$ is a ring of definition with the $p$-adic topology. Then we can also view

$$
(B_{\inf}^+/\xi^r)(T^{1/2}_1, \ldots, T^{1/2}_n) = (B_{\inf}^+/\xi^r)\hat{\otimes}_k k[T^{1/2}_1, \ldots, T^{1/2}_n]
$$

as a Tate algebra. We define a continuous action of $\Gamma$ on this algebra by requiring that $\gamma_j(T_j^{1/2}) = (\xi^r)^{b_j} T_j^{1/2}$. After reduction modulo $\xi$, it reduces to the natural action of $\Gamma$ on $\hat{k}(T^{1/2}_1, \ldots, T^{1/2}_n)$ via the Galois group of $D^n_{m, k}\otimes \mathbb{D}_k^n$.

By [Hub96, Corollary 1.7.3(iii)], there is a finitely generated $O_k[T_1, \ldots, T_n]$-algebra $R_0^+$ such that $R_0 \subset R_0^+ [1/p]$ is étale over $k[T_1, \ldots, T_n]$ and such that $R_0^+$ is the $p$-adic completion of $R_0^+$. Then $\hat{\otimes}_k (B_{\inf}^+/\xi^r)$ is the $p$-adic completion of $R_0 \otimes_k (B_{\inf}^+/\xi^r)$. By étaleness, the action of $\Gamma$ on $(B_{\inf}^+/\xi^r)[T_1, \ldots, T_n]$ given by $\gamma_j(T_j) = (\xi^r)^{b_j} T_j$ uniquely extends to a continuous action on $R_0 \otimes_k B_{\inf}^+$, whose reduction modulo $\xi$ is the trivial action. Therefore, $\gamma_i$ naturally acts on $R \otimes_k B_{\inf}^+/\xi^r$, whose reduction modulo $\xi$ is trivial.

**Lemma 3.19.** There is a unique $\Gamma$-equivariant embedding

$$
R \otimes_k (B_{\inf}^+/\xi^r) \to B_{\inf}^+(X)/\xi^r
$$

whose reduction modulo $\xi$ coincides with the natural embedding $R \otimes \hat{k}_\infty \to \hat{R}_\infty$. In addition, it induces a $\Gamma$-equivariant isomorphism

$$
B_{\inf}^+(X)/\xi^r \cong (R \otimes_k (B_{\inf}^+/\xi^r)) \hat{\otimes} (B_{\inf}^+/\xi^r)(T_1, \ldots, T_n)(B_{\inf}^+/\xi^r)(T^{1/2}_1, \ldots, T^{1/2}_n).
$$

**Proof.** The existence of the map can be shown by an argument similar to the one in the proof of [Sch13, Proposition 6.10]. We sketch it for the reader’s convenience. Recall that the $p$-adic topology on $\mathcal{A}_{\inf}(X)/\xi^r$ is separated and complete (by Lemma 3.3) and $B_{\inf}^+(X)/\xi^r$, equipped with the topology such that $\mathcal{A}_{\inf}(X)/\xi^r$ is a ring of definition, is a Tate algebra over $k$. Therefore, to construct the desired embedding in the lemma, it suffices to construct a $\Gamma$-equivariant continuous map $R_0 \otimes_k (B_{\inf}^+/\xi^r) \to B_{\inf}^+(X)/\xi^r$ whose reduction modulo $\xi$ coincides with the natural map $R_0 \otimes \hat{k}_\infty \to \hat{R}_\infty$. Since $R_0 \otimes_k (B_{\inf}^+/\xi^r)$ is étale over $(B_{\inf}^+/\xi^r)[T_1, \ldots, T_n]$, there is a unique such map extending the map $(B_{\inf}^+/\xi^r)[T_1, \ldots, T_n] \to B_{\inf}^+(X)/\xi^r$ sending $T_i$ to $[T_i^{1/2}]$. The $\Gamma$-equivariance is clear.

Hence, for each $m \geq 1$, the above map induces a map

$$
(3.20) \quad (R \otimes_k (B_{\inf}^+/\xi^r)) \otimes (B_{\inf}^+/\xi^r) \otimes (B_{\inf}^+/\xi^r)(T_1, \ldots, T_n) \to B_{\inf}^+(X)/\xi^r
$$

sending $T^{1/2}_i$ to $[T^{1/2}_i]$, which is compatible with the filtrations induced by the powers of the ideals generated by $\xi$. We need to show that it induces an isomorphism after taking direct limit over all $m$ and the $p$-adic completion at the left-hand side. It
suffices to prove this at the level of associated graded pieces, and so it suffices to show that
\[(R\hat{\otimes}_k R_\infty) \otimes_{R_\infty} (T_1, \ldots, T_n) \hat{\otimes}_k R_\infty (T_1^{s^{1/b}}, \ldots, T_n^{s^{1/b}}) \cong \hat{R}_\infty.
\]
But this is well known. (See, for example, [Sch13, Lemmas 4.5 and 6.18]. The result there is stated for the pro-étale tower of tori $\mathbb{T}^n \to \mathbb{T}^n$, but its proof remains valid for the pro-Kummer étale tower of discs $\mathbb{D}^n \to \mathbb{D}^n$ here.) \qed

We will apply the following result to the calculation of residues of vector bundles on $X$ along boundary divisors. Consider the following situation. Let $Z_j \subset X$ be the smooth divisor defined by $T_j = 0$. Then the smooth toric chart $X \to \mathbb{D}^n$ induces a map $Z_j \to \mathbb{D}^{n-1}$, which is also a composition of rational localizations and finite étale maps. Therefore, $Z_j$ is equipped with a log structure such that $Z_j \to \mathbb{D}^{n-1}$ is a smooth toric chart. In addition, we have a log affinoid perfectoid object
\[
\hat{Z}_j \coloneqq Z_j \times_{\mathbb{D}^{n-1}} \mathbb{D}^{n-1} \in (Z_j)_{\text{pro\-ét}}.
\]
Let $\hat{Z}_j$ denote the associated perfectoid space. Note that there is a closed immersion $\iota_j : \hat{Z}_j \to \hat{X}$ of perfectoid spaces given by
\[
(\hat{R}_\infty, \hat{R}_\infty^+) \to (\hat{R}_\infty^+/\hat{R}_\infty^+), \quad (\hat{R}_\infty^+/\hat{R}_\infty^+) \mapsto (\hat{R}_\infty^+/\hat{R}_\infty^+),
\]
where the two instances of $(T_1^{s^{1/b}})_{s \in \mathbb{Q}_{>0}}$ denote the $p$-adic completions of the ideals generated by $(T_1^{s^{1/b}})_{s \in \mathbb{Q}_{>0}}$ in $\hat{R}_\infty^+$ and $\hat{R}_\infty^+$, respectively. By Lemma 3.19 we have the following:

**Corollary 3.21.** The map $\iota_j : \hat{Z}_j \to \hat{X}$ induces a canonical isomorphism
\[
\mathbb{B}^+_{\text{dR}}(\hat{X})/(\xi^r, (T_j^{s^{1/b}})_{s \in \mathbb{Q}_{>0}}) \cong \mathbb{B}^+_{\text{dR}}(\hat{Z}_j)/\xi^r,
\]
where $(T_j^{s^{1/b}})_{s \in \mathbb{Q}_{>0}}$ similarly denotes the $p$-adic completion of the ideal of $\mathbb{B}^+_{\text{dR}}(\hat{X})/\xi^r$

generated by $(T_j^{s^{1/b}})_{s \in \mathbb{Q}_{>0}}$.

Next, we give an explicit description of $\mathcal{O}_X \mathbb{B}^+_{\text{dR,log}}$ on the localized site $X_{\text{pro\-ét}/\hat{X}}$. Note that $T_j/\langle T_j \rangle = -1 \in \ker \theta_{\log}$, and therefore we have
\[
\log \left( \frac{T_j}{\langle T_j \rangle} \right) := \log \left( 1 + \left( \frac{T_j}{\langle T_j \rangle} - 1 \right) \right) = \sum_{a=1}^{\infty} \frac{(-1)^{a-1}}{a} \left( \frac{T_j^\hat{\circ}}{\langle T_j \rangle} - 1 \right)^a \in \mathcal{O} \mathbb{B}^+_{\text{dR,log}}(\hat{X}).
\]
For any ring $S$, any $S$-module $A$, and any integer $b \geq 0$, we abusively denote by $A[[X_1, \ldots, X_n]]^{(\geq b)}$ the $S$-submodule of $A[[X_1, \ldots, X_n]]$ consisting of those power series with (lowest-degree) leading terms having degrees at least $b$. Then we define a filtration $\text{Fil}^{a}$ on $\mathbb{B}^+_{\text{dR}}(\hat{X})[[X_1, \ldots, X_n]]$ by
\[
\text{Fil}^{a} (\mathbb{B}^+_{\text{dR}}(\hat{X})[[X_1, \ldots, X_n]]) := \bigcup_{a+b=r} ((\text{Fil}^{b} \mathbb{B}^+_{\text{dR}}(\hat{X})[[X_1, \ldots, X_n]]^{(\geq b)})
\]
**Proposition 3.23.** The natural map
\[
\mathbb{B}^+_{\text{dR}}(\hat{X})[[X_1, \ldots, X_n]] \to \mathcal{O}_X \mathbb{B}^+_{\text{dR,log}}(\hat{X})
\]
sending $X_j$ to $\log(T_j/\langle T_j \rangle)$ is an isomorphism of sheaves on $X_{\text{pro\-ét}/\hat{X}}$, compatible with the filtration on $\mathbb{B}^+_{\text{dR}}(\hat{X})[[X_1, \ldots, X_n]]$ defined in (3.22), and the natural filtration on $\mathcal{O}_X \mathbb{B}^+_{\text{dR,log}}$ induced by the powers of $\ker \theta_{\log}$.  

This is a logarithmic analogue of [Brîoś, Proposition 5.2.2] and [Sch13, Proposition 6.10]. The idea of the proof is similar to loc. cit., although the technical details are slightly more involved.

**Proof of Proposition 3.23** Note that there is a natural isomorphism
\[ \mathbb{B}^{+}_{\text{dR}}|_{\tilde{X}}[[x_1, \ldots, x_n]] \to \mathbb{B}^{+}_{\text{dR}}|_{\tilde{X}}[[X_1, \ldots, X_n]] \]
sending \( x_j \) to \( \exp(X_j) - 1 \), with the inverse given by sending \( X_j \) to \( \log(1 + x_j) \). Here \( \exp(X_j) \) and \( \log(1 + x_j) \) are the usual power series in \( X_j \) and \( x_j \), respectively. Therefore, it suffices to show that the map
\[ (3.25) \quad \mathbb{B}^{+}_{\text{dR}}|_{\tilde{X}}[[x_1, \ldots, x_n]] \to O\mathbb{B}^{+}_{\text{dR, log}}|_{\tilde{X}}, \]
sending \( x_j \) to \( T_j/[T_j^2] - 1 \), is an isomorphism of sheaves on \( X_{\text{pro-Kummer}}/\tilde{X} \). We need to construct the map
\[ (3.26) \quad O\mathbb{B}^{+}_{\text{dR, log}}|_{\tilde{X}} \to \mathbb{B}^{+}_{\text{dR}}|_{\tilde{X}}[[x_1, \ldots, x_n]] \]
that is the inverse of (3.25). Similar to the proof of [Sch13, Proposition 6.10], we first show that there is a map of sheaves of algebras over \( X_{\text{pro-Kummer}}/\tilde{X} \), denoted by
\[ (3.27) \quad \psi : O_X_{\text{pro-Kummer}}|_{\tilde{X}} \to \mathbb{B}^{+}_{\text{dR}}|_{\tilde{X}}[[x_1, \ldots, x_n]], \]
satisfying the following two properties:

1. \( \psi(T_j^m) = (T_j^m)(1 + x_j)^{\frac{m}{n}} \), where \( (1 + x_j)^{\frac{m}{n}} \) is the usual power series in \( x_j \); and
2. the composition \( O_X_{\text{pro-Kummer}}|_{\tilde{X}} \to \mathbb{B}^{+}_{\text{dR}}|_{\tilde{X}}[[x_1, \ldots, x_n]] \to O\mathbb{B}^{+}_{\text{dR, log}}|_{\tilde{X}} \) is the natural one.

For any log affinoid perfectoid \( U \in X_{\text{pro-Kummer}}/\tilde{X} \), as in Definition 2.79, there is a pro-Kummer étale presentation \( U = \lim_{\leftarrow} U_i \), with \( U_i = (\text{Spa}(A_i, A_i^+), \mathcal{M}_i, \alpha_i) \). We would like to construct a compatible family of maps
\[ (3.28) \quad \psi_i : A_i \to \mathbb{B}^{+}_{\text{dR}}(U)[[x_1, \ldots, x_n]], \]
indexed by \( i \in I \). For each \( i \in I \), by Lemma 2.59 there is some \( m_i \) such that \( U_i \times_{\text{Spa}(A_i, A_i^+)} \mathbb{D}^n_{m_i} \to \mathbb{D}^n_{m_i} \) is strictly étale. In addition, since there is a map \( U \to \tilde{X} \), there exists some \( U_{\nu^i} \to U_i \) that factors through \( U_i \to U_i \times_{\text{Spa}(A_i, A_i^+)} \mathbb{D}^n_{m_i} \). In particular, the map \( A_i \to A_{\nu^i} \) factors as \( A_i \to O(U_i \times_{\mathbb{D}^n_{m_i}} A_i^+) \to A_{\nu^i} \). Now, consider the natural map
\[ (3.29) \quad k_{m_i}([T_1^{m_i}], \ldots, [T_n^{m_i}]) \to \mathbb{B}^{+}_{\text{dR}}(U)[[x_1, \ldots, x_n]] \]
sending \( T_j^{m_i} \) to \( ([T_j^{m_i}])(1 + x_j)^{\frac{1}{m_i}} \). As argued in the proof of Lemma 3.19, we can uniquely lift it to a ring homomorphism
\[ (3.29) \quad \psi(U_i \times_{\mathbb{D}^n_{m_i}}) \to \mathbb{B}^{+}_{\text{dR}}(U)[[x_1, \ldots, x_n]], \]
whose composition with
\[ \mathbb{B}^{+}_{\text{dR}}(U)[[x_1, \ldots, x_n]] \to O(U) \]
is the natural one induced by \( U \to U_{\nu^i} \to U_i \times_{\mathbb{D}^n_{m_i}} \). Then the desired map (3.28) is the composition of (3.29) with \( A_i \to O(U_i \times_{\mathbb{D}^n_{m_i}} A_i^+) \). We can readily check that the \( \psi_i \)'s, for all \( i \in I \), are compatible with each other and define the desired \( \psi \) in (3.27). By composing (3.27) with \( \mathbb{B}^{+}_{\text{dR}}[[x_1, \ldots, x_n]] \to \mathbb{B}^{+}_{\text{dR}} \), we obtain
\[ (3.30) \quad \psi : O_X_{\text{pro-Kummer}}|_{\tilde{X}} \to \mathbb{B}^{+}_{\text{dR}}|_{\tilde{X}}, \]
which was essentially constructed in Lemma 3.19.

Next, we define a multiplicative map of sheaves of monoids

\[(3.31)\quad \beta : \mathcal{M}^+|\tilde{X} \to \mathbb{B}^+_\text{dR}|\tilde{X}[x_1, \ldots, x_n],\]

satisfying

\[(3.32)\quad \beta(a) [\alpha^\beta(a)] = v(\alpha(a^\beta))\]

for all \(a \in \mathcal{M}^+\), where we denote by \(a \mapsto a^\beta\) the natural projection \(\mathcal{M}^+ \to \mathcal{M}\). Concretely, the sheaf \(\mathcal{M}^+|\tilde{X}\) is generated by \(\mathbb{Q}_{\geq 0}\) (regarded as a constant sheaf) and \(\lim_{f \to f^p} \mathcal{O}_{X|X_{\text{prok}}}|\tilde{X}\). If we write (locally)

\[(3.33)\quad \alpha^\beta(a) = h \prod_{i=1}^n (T_i^{\varphi})^{s_i}\]

for some \(h \in \lim_{f \to f^p} \mathcal{O}_{X|X_{\text{prok}}}|\tilde{X} \subset \mathcal{O}_{X|X_{\text{prok}}}^\times\) and \((s_i) \in \mathbb{Q}_{\geq 0}^n\), then

\[\alpha(a^\beta) = (\alpha^\beta(a))^\beta = h^\beta \prod_{i=1}^n T_i^{s_i},\]

and we set

\[\beta(a) = \frac{v(h^\beta)}{h} \prod_i (1 + x_i)^{s_i} \in \mathbb{B}^+_\text{dR}|\tilde{X}[x_1, \ldots, x_n].\]

This expression is independent of the expression of \(\alpha^\beta(a)\) in (3.33) and satisfies (3.32). This defines the desired map \(\beta\) in (3.31). Moreover, it is obvious that the composition of \(\beta\) with

\[\mathbb{B}^+_\text{dR}(U)[x_1, \ldots, x_n] \xrightarrow{x_i \mapsto 0, \xi \mapsto 0} \mathcal{O}_{X_{\text{prok}}}|\tilde{X}\]

is nothing but the constant 1.

Now, note that \((\mathbb{B}^+_\text{dR}(U)/\xi^r)[x_1, \ldots, x_n]/(x_1, \ldots, x_n)^r\) is \(p\)-adically separated and complete. Thus, for every \(i \in I\), the maps \(v_i\) and \(\beta\) induce a well-defined map from the ring \(S_i/\xi^r\) defined in (3.8) to \((\mathbb{B}^+_\text{dR}(U)/\xi^r)[x_1, \ldots, x_n]/(x_1, \ldots, x_n)^r\), by sending \(\alpha_i(a')/\alpha_i(a'')\) to \(\beta(a'')\). Its composition with the projection

\[(\mathbb{B}^+_\text{dR}(U)/\xi^r)[x_1, \ldots, x_n]/(x_1, \ldots, x_n)^r \xrightarrow{x_i \mapsto 0, \xi \mapsto 0} R\]

is exactly the map \(\theta_{\text{log}}\) in (3.9). Therefore, the above map further induces a map

\[S_i/(\ker \theta_{\text{log}})^r \to \mathbb{B}^+_\text{dR}(U)[x_1, \ldots, x_n]/(\xi, x_1, \ldots, x_n)^r.\]

By taking \(\ker(\theta_{\text{log}})^r\)-adic completion and direct limit, we arrive at the map (3.26). It remains to check that the two compositions between (3.25) and (3.26) are both the identity map. The composition of (3.25) followed by (3.26) is the identity map, since it is the direct and inverse limit (over all \(i \in I\) and \(r \geq 1\)) of the compositions

\[\mathbb{B}^+_\text{dR}(U)[x_1, \ldots, x_n]/(\xi, x_1, \ldots, x_n)^r \to S_i/(\ker \theta_{\text{log}})^r\]

\[\to \mathbb{B}^+_\text{dR}(U)[x_1, \ldots, x_n]/(\xi, x_1, \ldots, x_n)^r,\]

all of which are the identity maps. It remains to check that (3.25) is surjective.

Let

\[\bar{R}_i := (R_i \otimes_{W(\kappa)} R_i)[M_i]/((\alpha_i(a') \otimes 1) - (1 \otimes \alpha_i(a'))e^a)_{a' \in M_i},\]
where \((R_i \hat{\otimes}_W \kappa) R_i)[M_i]\) denotes the monoid algebra over \(R_i \hat{\otimes}_W \kappa) R_i\) associated with \(M_i\), and where \(e^{a'}\) denotes the image of \(a' \in M_i\), as before. We regard it as an \(R_i\)-algebra via the map to the second factor of \(R_i \hat{\otimes}_W \kappa) R_i\). Note that, for each \(r \geq 1\), the map \((3.30)\) induces a map

\[
\tilde{R}_i \to S_i/\xi^r
\]

sending \(\frac{1}{1 \otimes [a(a')]} e^a\), for each \(a = (a', a'') \in M_i \times M M^p\). Given any \(a \in M_i\), the existence of such an \(a = (a', a'')\) follows from the \(p\)-divisibility of \(M\). Clearly, \(\frac{1}{1 \otimes [a(a')]}\) is a unit in \(S_i/\xi^r\), and \(\frac{1}{1 \otimes [a(a')]} e^a\) is independent of the choice of \(a\). In particular, \(e^a\) belongs to the image, for every \(a \in M_i \times M M^p\). Let \(J_i\) denote the kernel of the map \(\tilde{R}_i \to R_i\) induced by the multiplication map \(R_i \hat{\otimes}_W \kappa) R_i \to R_i\) and by sending \(e^{a'}\) to \(1\) for all \(a \in M_i\). By the definition in \((2.36)\) and the later results in Section \(2.2\), we know that \(J_i/\xi^2 \cong \Omega^\log_{\kappa_{\bar{W}}/\kappa}[1/p]\) is a finite \(R_i\)-module. It follows that \(\tilde{R}_i/\xi^r\) is a finite \(R_i\)-module, and the induced morphism

\[
(\tilde{R}_i/\xi^r) \hat{\otimes}_{R_i} (B^+/\kappa) (U)/\xi^r) \cong (\tilde{R}_i/\xi^r) \hat{\otimes}_{R_i} (B^+/\kappa) (U)/\xi^r) \to S_i/(J_i^r, \xi^r),
\]

where \(J_i\) is regarded as an ideal of \(S_i\) via \((3.34)\), is surjective.

We claim that the morphism

\[
R_i[x_1, \ldots, x_n]/(x_1, \ldots, x_n)^r \to \tilde{R}_i/\xi^r
\]

mapping \(h\) to \(1 \otimes h\) for all \(h \in R_i\), and mapping \(x_j\) to \(\frac{1}{1 \otimes \theta^j} - 1\) for all \(1 \leq j \leq n\), is an isomorphism. The case \(r = 1\) is clear. By Proposition \(2.37\) and Theorem \(2.43(3)\), the morphism

\[
(x_1, \ldots, x_n)/(x_1, \ldots, x_n)^2 \to J_i/\xi^2
\]

(induced by \((3.35)\)) is an isomorphism. For \(r \geq 2\), by taking the \((r-1)\)-th symmetric power on both sides, we obtain

\[
(x_1, \ldots, x_n)^{r-1}/(x_1, \ldots, x_n)^r \cong J_i^{r-1}/J_i^r.
\]

Thus, we can conclude the claim by induction.

By base change under the map \(R_i \to B^+/\kappa\) (see \((3.34)\)), and by the surjectivity of \((\tilde{R}_i/\xi^r) \hat{\otimes}_{R_i} (B^+/\kappa) (U)/\xi^r) \to S_i/(J_i^r, \xi^r)\), we obtain a composition of surjective maps

\[
B^+/\kappa(U)[x_1, \ldots, x_n]/(\xi^r, (x_1, \ldots, x_n)^r) \to S_i/(\xi^r, J_i^r) \to S_i/(\ker \theta^\log)^r.
\]

By taking the limit over all \(r \geq 1\), we see that \((3.25)\) is indeed surjective.

Finally, it is easy to check that both maps \((3.25)\) and \((3.25)\) preserve the filtrations, and the proposition follows.

\[\square\]

**Corollary 3.36.** Over \(X_{\text{prokét}}/\bar{X}\), the isomorphism \((3.24)\) induces isomorphisms

\[
\text{Fil}^r \mathcal{O}_{B^+/\kappa} \cong t^r \mathcal{O}_{B^+/\kappa}\{W_1, \ldots, W_n\}
\]

\[
:= \left\{ t^r \sum_{\Lambda \in \mathbb{Z}_{\geq 0}} b_\Lambda W^\Lambda \in \mathcal{O}_{B^+/\kappa}[W_1, \ldots, W_n] \mid b_\Lambda \to 0, \text{ \(t\)-adically, as \(\Lambda \to \infty\)} \right\},
\]

where we have the variable

\[
W_j := t^{-1} X_j,
\]
for each $1 \leq j \leq n$, and the monomial
\begin{equation}
W^\Lambda := W_1^{\Lambda_1} \cdots W_n^{\Lambda_n},
\end{equation}
for each exponent $\Lambda = (\Lambda_1, \ldots, \Lambda_n) \in \mathbb{Z}_{\geq 0}^n$. (Here we denote by $\{W_1, \ldots, W_n\}$ the ring of power series that are $t$-adically convergent, which is similar to the notation $\langle W_1, \ldots, W_n \rangle$ for the ring of power series that are $p$-adically convergent.) Therefore,
\begin{equation}
\text{gr}^r \mathcal{O}_{\mathbb{B}_{\text{dR}, \log}} \cong t^r \hat{\mathcal{O}}_{\text{pro-Ker}}[W_1, \ldots, W_n],
\end{equation}
and
\begin{equation}
\text{gr}^* \mathcal{O}_{\mathbb{B}_{\text{dR}, \log}} \cong \hat{\mathcal{O}}_{\text{pro-Ker}}[t^\pm, W_1, \ldots, W_n].
\end{equation}

3.3. Consequences. Let us record the following consequences of Proposition \ref{prop:main-consequence}.

Let us resume the assumption on $X$ in Example \ref{ex:main-example}.

We have the Poincaré lemma in Section \ref{section:poincare}.

\begin{corollary}
\begin{enumerate}
\item The complex
\begin{equation}
0 \to \mathbb{B}_{\text{dR}}^+ \to \mathcal{O}_{\mathbb{B}_{\text{dR}, \log}}^+ \otimes \mathcal{O}_{\text{pro-Ker}} \Omega_X^{\log, 1} \otimes \mathcal{O}_{\text{pro-Ker}} \Omega_X^{\log, 2} \otimes \cdots
\end{equation}
is exact.
\item The above statement holds with $\mathbb{B}_{\text{dR}}^+$ and $\mathcal{O}_{\mathbb{B}_{\text{dR}, \log}}^+$ replaced with $\mathcal{O}_{\text{dR}}^+$ and $\mathcal{O}_{\mathcal{B}_{\text{dR}, \log}}^+$, respectively.
\item For each $r \in \mathbb{Z}$, the subcomplex
\begin{equation}
0 \to \text{Fil}^r \mathcal{O}_{\text{dR}} \to \text{gr}^r \mathcal{O}_{\mathbb{B}_{\text{dR}, \log}} \otimes \mathcal{O}_{\text{pro-Ker}} \Omega_X^{\log, 1} \otimes \mathcal{O}_{\text{pro-Ker}} \Omega_X^{\log, 2} \otimes \cdots
\end{equation}
of the complex for $\mathcal{O}_{\text{dR}}$ and $\mathcal{O}_{\mathcal{B}_{\text{dR}, \log}}$ is also exact.
\item For each $r \in \mathbb{Z}$, the quotient complex
\begin{equation}
0 \to \mathcal{O}_{X_{\text{pro-Ker}}} \to \mathcal{O}_{\text{Clog}} \otimes \mathcal{O}_{\text{pro-Ker}} \Omega_X^{\log, 1} \otimes \mathcal{O}_{\text{pro-Ker}} \Omega_X^{\log, 2} \otimes \cdots
\end{equation}
of the previous complex is exact, and can be identified with the complex
\begin{equation}
0 \to \mathcal{O}_{X_{\text{pro-Ker}}} \to \mathcal{O}_{\text{Clog}} \otimes \mathcal{O}_{\text{pro-Ker}} \Omega_X^{\log, 1}(-1) \otimes \mathcal{O}_{\text{pro-Ker}} \Omega_X^{\log, 2}(-2) \otimes \cdots
\end{equation}
over $\mathcal{O}_{X_{\text{pro-Ker}}}$, where
\begin{equation}
\mathcal{O}_{\text{Clog}} := \text{gr}^0 \mathcal{O}_{\mathbb{B}_{\text{dR}, \log}}.
\end{equation}
\end{enumerate}
\end{corollary}

Proof. Étale locally on $X$, we may assume that there exists a smooth toric chart as in Section \ref{section:toric} and then pass to $X$ pro-Kummer étale locally. We shall make use of the explicit descriptions in Proposition \ref{prop:main-consequence} and Corollary \ref{cor:exactness-consequence}. On $X$, it suffices to prove the exactness using $\mathbb{B}_{\text{dR}}^+|_X[[X_1, \ldots, X_n]]$ and $\mathbb{B}_{\text{dR}}|_X[W_1, \ldots, W_n]$ in place of $\mathbb{B}_{\text{dR}, \log}$ and $\mathcal{O}_{\mathbb{B}_{\text{dR}, \log}}$, respectively. Note that
\begin{equation}
\Omega_X^{\log} = \bigoplus_{i=1}^n \left( \mathcal{O}_{X_{\text{pro-Ker}}} \frac{dT_i}{T_i} \right),
\end{equation}
and hence (because of (3.13))
\[ \nabla(X_i) = \nabla \left( \log \frac{T_i}{T_i} \right) = \frac{dT_i}{T_i} \]
and (because of (3.37))
\[ \nabla(W_i) = t^{-1} \frac{dT_i}{T_i}. \]
The exactness follows from a straightforward calculation. (Note that the $t$-adic convergence condition on power series is not affected by taking anti-derivatives.) \(\square\)

By combining Corollaries 3.7 and 3.39, we also obtain the following logarithmic Faltings’s extension:

**Corollary 3.43.** We have a short exact sequence of sheaves of $\hat{\mathcal{O}}_{X_{\text{prokét}}}$-modules
\[ 0 \to \hat{\mathcal{O}}_{X_{\text{prokét}}}(1) \to \text{gr}^1 \mathcal{O}_{\mathcal{E}X_{\text{prokét}}, \log} \to \hat{\mathcal{O}}_{X_{\text{prokét}}} \otimes_{\hat{\mathcal{O}}_{X_{\text{prokét}}}} \Omega^1_{X, \log} \to 0. \]

In the remainder of this subsection, let us fix a perfectoid field $K$ containing $\hat{k}_{\infty}$, and introduce some more sheaves.

**Definition 3.44.** (1) As in [LZ17, Section 3.1], let
\[ B_{\text{dr}}^+ = \mathcal{B}_{\text{dr}}^+(K, \mathcal{O}_K) \]
(which replaces (3.17) from now on), and let
\[ B_{\text{dr}} = \mathcal{B}_{\text{dr}}(K, \mathcal{O}_K). \]
When $K \subset \hat{k}$, they are the $\text{Gal}(\hat{k}/K)$-invariants of the classical Fontaine’s rings. Let $t = \log(|\epsilon|) \in B_{\text{dr}}^+$ be as in (3.18). The embedding $k \to K$ lifts uniquely to an embedding $k \to B_{\text{dr}}^+$.

(2) For each integer $r \geq 1$, we define $\mathcal{O}_X \hat{\otimes} B_{\text{dr}}^+ / t^r$ to be the sheaf on $X_{\text{an}}$ associated with the presheaf which assigns to every rational open subset $U = \text{Spa}(A, A^+)$ of $X$ the ring $A \hat{\otimes}_k (B_{\text{dr}}^+ / t^r)$. Then we define
\[ \mathcal{O}_X \hat{\otimes} B_{\text{dr}}^+ = \lim_{\leftarrow r} (\mathcal{O}_X \hat{\otimes} B_{\text{dr}}^+ / t^r) \]
and
\[ \mathcal{O}_X \hat{\otimes} B_{\text{dr}} = (\mathcal{O}_X \hat{\otimes} B_{\text{dr}}^+) / [t^{-1}]. \]
(3) The filtrations on $\mathcal{O}_X \hat{\otimes} B_{\text{dr}}^+$ and $\mathcal{O}_X \hat{\otimes} B_{\text{dr}}$ are defined by setting
\[ \text{Fil}^s(\mathcal{O}_X \hat{\otimes} B_{\text{dr}}^+) = t^s (\mathcal{O}_X \hat{\otimes} B_{\text{dr}}^+) \]
and
\[ \text{Fil}^s(\mathcal{O}_X \hat{\otimes} B_{\text{dr}}) = t^s (\mathcal{O}_X \hat{\otimes} B_{\text{dr}}^+) / [t^{-1}]. \]
(4) By replacing rational open subsets $U \subset X$ in (2) with general étale maps $U \to X$, we also define similarly the sheaves $\mathcal{O}_{X_{\text{an}}} \hat{\otimes} (B_{\text{dr}}^+ / t^r)$, for all integers $r \geq 1$; $\mathcal{O}_{X_{\text{an}}} \hat{\otimes} B_{\text{dr}}^+$; and $\mathcal{O}_{X_{\text{an}}} \hat{\otimes} B_{\text{dr}}$ on $X_{\text{ét}}$. They are equipped with similarly defined filtrations.
Remark 3.47. Note that these sheaves were introduced in [LZ17, Section 3.1] in a slightly different way. Namely, in loc. cit., they were considered as sheaves on $X_{K,\mathrm{an}}$ and $X_{K,\mathrm{ét}}$. See Remark 5.3 below for a comparison.

It is straightforward to verify that the arguments in the proofs of [LZ17] Lemmas 3.1 and 3.2 also apply in this situation, and give the following:

Lemma 3.48. (1) If $U = \text{Spa}(A, A^+) \subset X$ is étale, then we have

$$H^i(U_{\text{ét}}, \mathcal{O}_{X_{\text{an}}} \hat{\otimes} (B_{\text{dR}}^+/t^r)) = \begin{cases} A \hat{\otimes}_k (B_{\text{dR}}^+/t^r), & \text{if } i = 0; \\ 0, & \text{if } i > 0. \end{cases}$$

(2) There is a canonical isomorphism $\text{gr}^r(\mathcal{O}_{X_{\text{an}}} \hat{\otimes} B_{\text{dR}}^+) \cong \mathcal{O}_{X_{\text{an}}} \hat{\otimes}_k K(r)$.

(3) Recall that we denote by $\lambda : X_{\text{ét}} \to X_{\text{an}}$ the natural projection of sites. Then there are natural isomorphisms

$$\mathcal{O}_X \hat{\otimes} B_{\text{dR}}^+/t^r) \cong \lambda_*(\mathcal{O}_{X_{\text{an}}} \hat{\otimes} (B_{\text{dR}}^+/t^r)) \cong R\lambda_*(\mathcal{O}_{X_{\text{an}}} \hat{\otimes} (B_{\text{dR}}^+/t^r)),$$

which in turn induce isomorphisms

$$\mathcal{O}_X \hat{\otimes} B_{\text{dR}}^+ \cong \lambda_*(\mathcal{O}_{X_{\text{an}}} \hat{\otimes} B_{\text{dR}}^+) \cong R\lambda_*(\mathcal{O}_{X_{\text{an}}} \hat{\otimes} B_{\text{dR}}^+).$$

Now consider the natural projection of sites

$$\mu' : X_{\text{prokét}}/X_K \to X_{K,\text{an}}.$$

Corollary 3.50. For any $-\infty \leq a < b \leq \infty$, there is a natural isomorphism

$$(\mathcal{O}_X \hat{\otimes} B_{\text{dR}})_{[a,b]} \cong R\mu'_*(\mathcal{O}_{B_{\text{dR}}}^{[a,b]}).$$

Proof. By Lemma 3.48, it suffices to prove the analogous statement for the projection $\nu' : X_{\text{prokét}}/X_K \to X_{\text{ét}}$ (instead of $\mu' : X_{\text{prokét}}/X_K \to X_{\text{an}}$). By the same argument as in the proof of [LZ17] Lemma 3.7, there is a natural map $\mathcal{O}_{X_{\text{an}}} \hat{\otimes} B_{\text{dR}}^+ \to \nu'_*(\mathcal{O}_{B_{\text{dR}}}^{[a,b]} \hat{\otimes} B_{\text{dR}}^+)$. Since the question is local, we may assume that there exists a smooth toric chart $X \to \mathbb{D}^n$ as in Section 3.2. Then, by using Corollary 3.36 the proof is similar to the one of [Sch13, Proposition 6.16(i)].

4. A FORMALISM OF DECOMPLETION

In this section, we generalize the formalism of decompleting developed in [KL16, Section 5], in order to treat the general Kummer towers.

4.1. The results. Throughout this subsection, let $k$ be a $p$-adic analytic field, i.e., a field over $\mathbb{Q}_p$ equipped with a complete multiplicative norm extending the standard one on $\mathbb{Q}_p$. Note that, by [KL15, Lemma 2.2.12], a finite projective module over a Banach ring $R$ admits a canonical structure of Banach $R$-modules (up to equivalence). Thus, in what follows, we do not distinguish between finite projective modules and finite projective Banach modules when the base ring is a Banach ring.
**Definition 4.1.** Let $I$ be a small filtered index category with an initial object $0$. Let $\{A_i\}_{i \in I}$ be a direct system in the category of $k$-Banach algebras with submetric transition morphisms. Let $\Gamma$ be a profinite group continuously acting on the direct system $\{A_i\}_{i \in I}$. Let $\hat{A}_\infty$ be the completed direct limit of the $A_i$'s. We call the pair $((A_i)_{i \in I}, \Gamma)$ a decompletion system (resp. weakly decompletion system) if for any finite projective (resp. finite free) $\Gamma$-module $L_\infty$ over $\hat{A}_\infty$—i.e., a finite projective (resp. finite free) $\hat{A}_\infty$-module equipped with a continuous semilinear $\Gamma$-action—the following are true:

1. For some $i \in I$, there exists a finite projective (resp. finite free) $\Gamma$-module $L_i$ over $A_i$ together with a $\Gamma$-equivariant continuous $A_i$-linear morphism $\iota_i : L_i \to L_\infty$

such that the induced map $\iota_i \otimes 1 : L_i \otimes_{A_i} \hat{A}_\infty \to L_\infty$

is an isomorphism. We shall call such a pair $(L_i, \iota_i)$ a model of $L_\infty$ over $A_i$. Moreover, for any two models $(L_{i,1}, \iota_{i,1})$ and $(L_{i,2}, \iota_{i,2})$ over $A_i$, there exists some $i' \geq i$ such that the two models $(L_{i,1} \otimes_{A_i} A_{i'}, \iota_{i,1} \otimes 1)$ and $(L_{i,2} \otimes_{A_i} A_{i'}, \iota_{i,2} \otimes 1)$ over $A_{i'}$ are isomorphic. That is, there exists a $\Gamma$-equivariant isomorphism $\zeta : L_{i,1} \otimes_{A_i} A_{i'} \to L_{i,2} \otimes_{A_i} A_{i'}$ of Banach $A_{i'}$-modules such that $(\iota_{i,2} \otimes 1) \circ \zeta = \iota_{i,1} \otimes 1$.

2. A model $(L_i, \iota_i)$ over $A_i$ is called good if the natural map $H^*(\Gamma, L_i) \to H^*(\Gamma, L_\infty)$

is an isomorphism. Then, for any model $(L_i, \iota_i)$ over $A_i$, there exists some $i_0 \geq i$ such that for any $i' \geq i_0$, the model $(L_i \otimes_{A_i} A_{i'}, \iota_i \otimes 1)$ is good.

**Definition 4.2.** Let $\{A_i\}_{i \in I}$ and $\Gamma$ be as in Definition 4.1. Moreover, let $\{\Gamma_i\}_{i \in I^{op}}$ be an inverse system of profinite groups, with compatibly induced actions of $\Gamma_i$ on $A_i$, for all $i$, such that $\Gamma_0 = \Gamma$ and $\Gamma_i$ is an open normal subgroup of $\Gamma$, for any $i \leq i'$. The pair $((A_i)_{i \in I}, \{\Gamma_i\}_{i \in I^{op}})$ is called decompleting if it satisfies the following:

1. For each $i \in I$, the $\Gamma_0$-action on $A_i$ is isometric. Moreover, all transition morphisms $A_i \to A_{i'}$ are isometric instead of merely submetric.

2. (Splitting.) The natural projection $\hat{A}_\infty \to \hat{A}_\infty/A_i$ admits an isometric section as Banach modules over $A_i$.

3. (Uniform strict exactness.) There exists some $c > 0$ such that for each $i \in I$ and each cocycle $f \in C^*(\Gamma_i, \hat{A}_\infty/A_i)$, there exists some cochain $g$ with $\|g\| \leq c\|f\|$ such that $dg = f$. In particular, $\hat{A}_\infty/A_i$ has totally trivial $\Gamma_i$-cohomology (and hence has totally trivial $\Gamma_0$-cohomology by the Hochschild–Serre spectral sequence).

The pair $((A_i)_{i \in I}, \{\Gamma_i\}_{i \in I^{op}})$ is called stably decompleting if it satisfies the following:

(a) $A_i$'s and $\hat{A}_\infty$ are all stably uniform (in the sense of Remark 2.8.5);

and

(b) $\Gamma_0$ acts trivially on $A_0$, and the base change of $((A_i)_{i \in I}, \{\Gamma_i\}_{i \in I^{op}})$ under every rational localization of $A_0$ is decompleting.

The main result of this section is the following theorem.

**Theorem 4.3.** (1) A decompleting pair is a weakly decompletion system.
Lemma 4.5. M

Let \( s \in \text{uniform strict exactness condition (3) in Definition 4.2.} \) For each \( s \geq 1 \), there exists some \( c(s) > 0 \) such that for each \( i \in I \) and each cocycle \( f \in C^s(\Gamma_i, \tilde{A}_\infty/A_i) \), there exists some cochain \( g \in C^{s-1}(\Gamma_i, 
\tilde{A}_\infty/A_i) \) with \( \|g\| \leq c(s)\|f\| \) such that \( dg = f \).

We first establish some basic estimates for a decompleting pair. In the following, let \( \{A_i\}_{i \in I}, \{\Gamma_i\}_{i \in I=\omega} \) be a decompleting pair as in Definition 4.2.

**Lemma 4.5.** For any \( i \in I \) and \( f \in C^s(\Gamma_i, \tilde{A}_\infty/A_i) \), there exists \( g \in C^{s-1}(\Gamma_i, \tilde{A}_\infty/A_i) \) with \( \|g\| \leq c\|df\| \) and \( \|h\| \leq \max\{c\|f\|, c^2\|df\|\} \) such that \( dg = df, \ dh = f - g, \) and \( \|f - dh\| = \|g\| \leq c\|df\| \).

**Proof.** Since \( df \) is a cocycle, the existence of \( g \) follows from uniform strict exactness. Since \( df - g = 0 \), by uniform strict exactness alone, there exists \( h \in C^{s-1}(\Gamma_i, A_{\infty}/A_i) \) with \( \|h\| \leq \|f - g\| \leq c \max\{\|f\|, \|g\|\} \leq \max\{c\|f\|, c^2\|df\|\} \) such that \( dh = f - g \), and so that \( \|f - dh\| = \|g\| \leq c\|df\| \), as desired. \( \square \)

**Lemma 4.6.** Let \( A \to B \) be an isometric morphism of k-Banach algebras. Suppose the natural projection \( \pi : B \to B/\mathcal{A} \) admits an isometric section \( s : B/\mathcal{A} \to B \) as Banach modules over \( A \). Then, for all \( b_1, b_2 \in B \), we have

\[
|\pi(b_1b_2)| \leq \max\{|b_2||\pi(b_1)|, |b_1||\pi(b_2)|\}.
\]

**Proof.** Note that the isomorphism \( \text{Id} \oplus s : A \oplus B/\mathcal{A} \to B \) of Banach \( A \)-modules is isometric, where the left hand side is equipped with the supremum norm. Indeed, for \( e = a + s(b) \in B \), we have \( e \leq \max\{|a|, |s(b)|\} \) and \( |\cdot|_A \) and \( |\cdot|_{B/\mathcal{A}} \) denote the norms on \( A \) and \( B/\mathcal{A} \), respectively. Next, note that \( s(b) = 0 \) because \( \pi(e) = b \), and so \( |e - s(b)| \leq |e| \). Thus, \( |e| = \max\{|a|, |s(b)|\} \). Now write \( b_i = a_i + s(\pi(b_i)) \) for \( i = 1, 2 \), so that \( \pi(b_1b_2) = a_1\pi(b_1) + a_2\pi(b_2) + s(\pi(b_1))\pi(b_2) \). Then \( |a_1\pi(b_2)| \leq |a_1||\pi(b_2)| \leq |b_1||\pi(b_2)| \). Similarly, \( |a_2\pi(b_1)| \leq |b_2||\pi(b_1)| \). Finally,

\[
|\pi(s(\pi(b_1))s(\pi(b_2)))| \leq |s(\pi(b_1))|s(\pi(b_2))| \leq |s(\pi(b_1))||\pi(b_2)| = |\pi(b_1)||\pi(b_2)|.
\]

Since \( |\pi(b_1)||\pi(b_2)| \leq \min\{|b_2||\pi(b_1)|, |b_1||\pi(b_2)|\} \), the lemma follows. \( \square \)

**Lemma 4.7.** Fix \( c \geq 1 \) as in Definition 4.2. Let \( f \) be a cocycle in \( C^1(\Gamma_0, GL_1(\tilde{A}_\infty)) \). Suppose that, for some \( i \in I \), we have \( |f(\gamma) - 1| \leq \frac{1}{2c} \) for all \( \gamma \in \Gamma_i \), \( \|f\| \leq \frac{1}{c^2} \), where \( \bar{f} \) is the image of \( f \) in \( C^1(\Gamma_i, M_i(\tilde{A}_\infty/A_i)) \) (which is merely a cochain). (We shall also denote similar images by overlines in the proof below.) Then the restriction of \( f \) to \( \Gamma_i \) is equivalent to a cocycle in \( C^1(\Gamma_i, GL_1(A_i)) \).

**Proof.** We claim that there exists some \( \zeta \in M_i(\tilde{A}_\infty) \) with \( |\zeta| \leq c||f|| \) such that the cocycle

\[
f' : \gamma \mapsto \gamma(1 + \zeta)f(\gamma)(1 + \zeta)^{-1}
\]

satisfies \( |f'(\gamma) - 1| \leq \frac{1}{2c} \) for all \( \gamma \in \Gamma_i \). Let \( \bar{f} \leq \frac{1}{c^2} \) in \( C^1(\Gamma_i, M_i(\tilde{A}_\infty/A_i)) \).\( \bar{f} \) is a cocycle in \( C^1(\Gamma_0, GL_1(\tilde{A}_\infty)) \) with \( |\zeta_n| \leq c||\bar{f}|| \leq \frac{1}{c^2} \) such that

\[
\gamma\left(\prod_{i=1}^n(1 + \zeta_i)\right)f(\gamma)\left(\prod_{i=1}^n(1 + \zeta_i)\right)^{-1} \leq \frac{||\bar{f}||}{2^n}.
\]
Put $\varsigma_\infty = \prod_{i=1}^{\infty} (1 + \varsigma_i) \in \text{GL}_d(\hat{A}_\infty)$. It follows that the cocycle
\[
\tilde{f} : \gamma \mapsto \gamma(\varsigma_\infty) f(\gamma) \varsigma_\infty^{-1}
\]
take values in $\text{M}_d(A_i) \cap \text{GL}_d(\hat{A}_\infty)$ and $|\tilde{f}(\gamma) - 1| \leq \frac{1}{2c} \leq \frac{1}{2}$ for $\gamma \in \Gamma_i$. This yields that $\tilde{f}$ is a cocycle in $C^1(\Gamma_i, \text{GL}_d(A_i))$, and the lemma follows.

It remains to prove the claim. Note that $f(\gamma_1 \gamma_2) = \gamma_1(f(\gamma_2))f(\gamma_1)$ because $f$ is cocycle in $C^1(\Gamma_0, \text{GL}_d(\hat{A}_\infty))$. By Lemma 4.6 we have
\[
|d\overline{f}(\gamma_1, \gamma_2)| = |\gamma_1 f(\gamma_2) + f(\gamma_1) - f(\gamma_1 \gamma_2)| = |(\gamma_1 f(\gamma_2) - 1)(f(\gamma_1) - 1)| \leq \frac{\|\overline{f}\|}{2c}.
\]
By Lemma 4.5 there exist $\overline{h} \in \text{M}_d(\hat{A}_\infty/A_i)$ and $\overline{\varsigma} \in C^1(\Gamma_i, \text{M}_d(\hat{A}_\infty/A_i))$ with
\[
(4.8) \quad |\overline{h}| \leq \max\{c\|\overline{f}\|, c^2\|d\overline{f}\|\} = c\|\overline{f}\| \leq \frac{1}{2c} \leq \frac{1}{2}
\]
and
\[
(4.9) \quad \|\overline{\varsigma}\| \leq \frac{\|\overline{f}\|}{2}
\]
such that $d\overline{h} = \overline{\varsigma} - \overline{f}$. That is,
\[
(4.10) \quad \overline{\varsigma}(\gamma) - \overline{f}(\gamma) = \gamma(\overline{h}) - \overline{h}
\]
for all $\gamma \in \Gamma_i$. By the splitting condition (see Definition 4.2(2)), and by (4.8), we can lift $\overline{h}$ to some $h \in \text{M}_d(\hat{A}_\infty)$ with $|h| = |\overline{h}| \leq c\|\overline{f}\|$.

For $\gamma \in \Gamma_i$, a short computation shows that
\[
|\gamma(1 + h) f(\gamma)(1 + h)^{-1} - 1| \leq \frac{1}{2c}
\]
and
\[
(4.11) \quad |\gamma(1 + h) f(\gamma)(1 + h)^{-1} - \gamma(1 + h) f(\gamma)(1 - h)| \leq |h|^2 \leq \frac{1}{2c} (c\|\overline{f}\|) = \frac{\|\overline{f}\|}{2}.
\]
On the other hand, by using (4.10) and Lemma 4.6 we get
\[
|\gamma(1 + h) f(\gamma)(1 - h) - \overline{\varsigma}(\gamma)| = |\overline{f}(\gamma) - \overline{\varsigma}(\gamma) + \gamma(h)(f(\gamma) - 1) - (f(\gamma) - 1)h + \gamma(\overline{h}) - \overline{h} - \gamma(h) f(\gamma) h| \\
= |\gamma(h)(f(\gamma) - 1) - (f(\gamma) - 1)h - \gamma(h) f(\gamma) h| \leq \frac{\|\overline{f}\|}{2}.
\]
By combining (4.10), (4.11), and (4.12), we obtain the desired estimate
\[
|\gamma(1 + h) f(\gamma)(1 + h)^{-1}| \leq \frac{\|\overline{f}\|}{2},
\]
which proves the claim.

\[\square\]

**Lemma 4.13.** Let $L$ be a finite free $\Gamma_0$-module over $A_i$ for some $i \in I$. Then there exists some $i_0 \geq i$ such that, $L \otimes_{A_i} (\hat{A}_\infty/A_{i'})$ has totally trivial $\Gamma_{i'}$-cohomology for every $i' \geq i_0$, and therefore has totally trivial $\Gamma_0$-cohomology by the Hochschild–Serre spectral sequence.
Proof. Fix $c \geq 1$ as in Definition 4.2. Fix a basis $e_1, \ldots, e_l$ of $L$ over $A_i$, and equip $L$ with the supremum norm. We can find some $i_0 \geq i$ such that $| (\gamma - 1)(e_j)| \leq \frac{1}{2^c}$ for all $\gamma \in \Gamma_{i_0}$ and $1 \leq j \leq l$. In the following, we will show that $L \otimes_{A_i} (\hat{A}_\infty/A_{i'})$ has totally trivial $\Gamma_{i'}$-cohomology for every $i' \geq i_0$.

Let $f$ be a cocycle in $C^s(\Gamma_{i'}, L \otimes_{A_i} (\hat{A}_\infty/A_{i'}))$. Write $f = \sum_{j=1}^l (e_j \otimes f_j)$ with $f_j \in C^s(\Gamma_{i'}, \hat{A}_\infty/A_{i'})$ for all $1 \leq j \leq l$. It follows that the norm of

$$\sum_{j=1}^l (e_j \otimes df_j) = \sum_{j=1}^l (e_j \otimes df_j) - df = \sum_{j=1}^l (e_j \otimes df_j) - d\left(\sum_{j=1}^l (e_j \otimes f_j)\right)$$

is bounded by $\frac{\|f\|}{2^c}$. Thus, for each $j$, we have $\|df_j\| \leq \frac{\|f\|}{2^c}$. By Lemma 4.5, there exist $g_j \in C^s(\Gamma_{i'}, \hat{A}_\infty/A_{i'})$ and $h_j \in C^{s-1}(\Gamma_{i'}, \hat{A}_\infty/A_{i'})$ with $\|g_j\| \leq \frac{\|f\|}{2^c}$ and $\|h_j\| \leq c\|f\|$ such that $dh_j = f_j - g_j$. Put $h = \sum_{j=1}^l (e_j \otimes h_j)$. Then $\|h\| \leq c\|f\|$, and the norm of

$$f - dh = \sum_{j=1}^l (e_j \otimes g_j) + \left(\sum_{j=1}^l (e_j \otimes dh_j) - d\left(\sum_{j=1}^l (e_j \otimes h_j)\right)\right)$$

is bounded by $\frac{\|f\|}{2^c}$. By iterating this process, we can construct a sequence of cochains $H_1, H_2, \ldots \in C^{s-1}(\Gamma_{i'}, L \otimes_{A_i} (\hat{A}_\infty/A_{i'}))$ satisfying $\|H_n\| \leq \frac{c\|f\|}{2^c n}$ and

$$\|f - dH_1 - \cdots - dH_n\| \leq \frac{\|f\|}{2^c n},$$

for all $n$. It follows that $f = dH$ for $H = \sum_{i=1}^{\infty} H_i \in C^{s-1}(\Gamma_{i'}, L \otimes_{A_i} (\hat{A}_\infty/A_{i'}))$, yielding $H^s(\Gamma_{i'}, L \otimes_{A_i} (\hat{A}_\infty/A_{i'})) = 0$, as desired. \hfill $\square$

Proof of Theorem 4.3. Let $L_\infty$ be a finite free $\Gamma_0$-module over $\hat{A}_\infty$. If $(L_i, \iota_i)$ is a model of $L_\infty$ over $A_i$, then by Lemma 4.13, there exists some $i_0 \geq i$ such that $L_i \otimes_{A_i} (\hat{A}_\infty/A_{i'})$ has totally trivial $\Gamma_{i'}$-cohomology for every $i' \geq i_0$. This verifies Definition 4.1.2. For two models $(L_{i,1}, \iota_{i,1})$ and $(L_{i,2}, \iota_{i,2})$ of $L_\infty$ over $A_i$, by Lemma 4.13, we get that for some $i' \geq i$,

$$\text{Hom}_{A_i}(L_{i,1}, L_{i,2}) \otimes_{A_i} A_{i'} = \text{Hom}_{A_{i'}}(L_{i,1} \otimes_{A_i} A_{i'}, L_{i,2} \otimes_{A_i} A_{i'})$$

$$\rightarrow \text{Hom}_{\hat{A}_\infty}(L_\infty, L_\infty) = \text{Hom}_{A_{i'}}(L_{i,1}, L_{i,2}) \otimes_{A_i} \hat{A}_\infty$$

becomes an isomorphism after taking $\Gamma_0$-invariants. This implies that

$$(\iota_{i,1} \otimes 1)(L_{i,1} \otimes_{A_i} A_{i'}) = (\iota_{i,2} \otimes 1)(L_{i,2} \otimes_{A_i} A_{i'})$$

in $L_\infty$.

It remains to prove the existence of a model. By choosing an $\hat{A}_\infty$-basis of $L_\infty$, the $\Gamma_0$-module structure of $L_\infty$ amounts to a cocycle in $C^1(\Gamma_0, GL_1(\hat{A}_\infty))$, where $l$ is the rank of $L_\infty$. By Lemma 4.7, for some $i \in I$, as a $\Gamma_i$-module, $L_\infty$ admits a free model $(L_i, \iota_i)$ over $A_i$. By the previous paragraph, for some sufficiently large $i' \geq i$, all of the models $\gamma L_i$, for $\gamma \in \Gamma_0/\Gamma_i$, become identical after base change from $A_i$ to $A_{i'}$. Thus, the base change of $(L_i, \iota_i)$ to $A_{i'}$ is a model of $L_\infty$, as desired. \hfill $\square$

In the remainder of this subsection, we assume that the pair $\{(A_i)_{i \in I}, (\Gamma_i)_{i \in I'}\}$ is stably decompleting.
Lemma 4.14. Let $L_i$ be a finite projective $\Gamma_0$-module over $A_i$ for some $i \in I$. Then there exists some $i_0 \geq i$ such that, $L \otimes_{A_i} (\hat{A}_\infty / A'_i)$ has totally trivial $\Gamma_i$-cohomology for every $i' \geq i_0$, and therefore has totally trivial $\Gamma_0$-cohomology by the Hochschild–Serre spectral sequence.

Proof. We may choose a finite covering $\mathfrak{B}$ of $\text{Spa}(A_i, A_i^n)$ by rational subsets such that the restriction of $L_i$ to each rational subset in $\mathfrak{B}$ is free. Since the pair $(\{A_i\}_{i \in I}, \{\Gamma_i\}_{i \in I'})$ is stably decompetting, it follows from Lemma 4.13 that, for some sufficiently large $i_0 \geq i$, the restrictions of $L \otimes_{A_i} (\hat{A}_\infty / A'_i)$ to all the rational subsets as well as their intersections have totally trivial $\Gamma_i$-cohomology, for every $i' \geq i_0$. By the Tate’s sheaf property for the structure sheaf of a stably uniform adic Banach algebra (see, again, [KL15, Theorems 2.7.7 and 2.8.10]), $\hat{A}_\infty / A'_i$ satisfies Tate’s acyclic condition with respect to $\mathfrak{B}$. Since $L$ is finite projective over $A_i$, the same property holds for $L \otimes_{A_i} (\hat{A}_\infty / A'_i)$. We therefore conclude that $L \otimes_{A_i} (\hat{A}_\infty / A'_i)$ has totally trivial $\Gamma_i$-cohomology.

Proof of Theorem 4.3(2). Let $L_\infty$ be a finite projective $\Gamma_0$-module over $\hat{A}_\infty$. Given Lemma 4.14, we may proceed as in the proof of Theorem 4.3(1) to verify Definition 4.1(2) and the second half of Definition 4.1(1). It remains to show the existence of a model over some $A_i$. By [KL16, Lemma 5.6.8], the $\hat{A}_\infty$-module $L_\infty$ is given by the base change to $\hat{A}_\infty$ of a finite projective $A_i$-module $L_i$ (without considering the $\Gamma_0$-action). Choose a finite covering $\mathfrak{B}$ of $\text{Spa}(A_i, A_i^n)$ by rational subsets such that the restriction of $L_i$ to each rational subset in $\mathfrak{B}$ is free. Consequently, the restriction of $L_\infty$ to each rational subset in $\mathfrak{B}$ is free. By Theorem 4.3(1), by enlarging $i$, the restriction of $L_\infty$ to every rational subset in $\mathfrak{B}$ admits a model. Moreover, by enlarging $i$, we may further ensure that these models coincide on overlaps of rational subsets in $\mathfrak{B}$. Thus, they glue to a model of $L_\infty$ over $A_i$ by the Kiehl gluing property for stably uniform adic Banach algebras (see, again, [KL15, Theorems 2.7.7 and 2.8.10]).

4.2. Examples. In Sections 4.2.1–4.2.3 below, we present three examples of decomposition systems, which will be used in Section 5 below.

4.2.1. Geometric towers. Let $K$ be a perfectoid field containing $\hat{k}_\infty$. Let

$$Y = \text{Spa}(A, A^n) \to \mathbb{D}_K^n \cong \text{Spa}(K(T_1, \ldots, T_n), \mathcal{O}_K(T_1, \ldots, T_n))$$

be a smooth toric chart over $K$ (cf. Definition 2.28). For $m \geq 1$, let

$$Y_m = \text{Spa}(A_m, A_m^n) = Y \times_{\mathbb{D}_K} \text{Spa}(K(T_1^m, \ldots, T_n^m), \mathcal{O}_K(T_1^m, \ldots, T_n^m)).$$

Let $(\hat{A}_\infty, \hat{A}_\infty^\times)$ be the completed direct limit of $(A_m, A_m^\times)$. Put $Y_\infty = \lim_{\leftarrow} Y_m$, whose associated affinoid perfectoid space is $\hat{Y}_\infty = \text{Spa}(\hat{A}_\infty, \hat{A}_\infty^\times)$. Put

$$\Gamma_0 := \hat{\mathbb{Z}} \gamma_1 + \cdots + \hat{\mathbb{Z}} \gamma_n \cong \hat{\mathbb{Z}}^n,$$

which acts the algebras $A_m$’s and $\hat{A}_\infty$ by

$$\gamma_i T_j^m = \zeta_{m,i}^j T_j^m$$

(and $\gamma_i$ acts trivially on $A$ and $K$). It follows that $Y_\infty$ is a Galois pro-Kummer étale covering of $Y_m$ with Galois group the subgroup $\Gamma_m = (1 + m \hat{\mathbb{Z}})^n$ of $\Gamma_0 \cong \hat{\mathbb{Z}}^n$. 

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Lemma 4.15. Fix \( m \geq 3 \) and \((i_1, \ldots, i_n) \in \mathbb{Q}^n \setminus \frac{1}{m} \mathbb{Z}^n \). Then, for any cocycle
\[
f \in C^\bullet(\Gamma_m, KT_1^{i_1} \cdots T_n^{i_n}),
\]
there exists a cochain \( g \) with \( \|g\| \leq |\zeta_m - 1|^{-1}\|f\| \) such that \( dg = f \).

Proof. Up to replacing \( f \) with a scalar multiple, we may suppose that \( \|f\| = 1 \), so that \( f \in C^\bullet(\Gamma_m, \mathcal{O}_K T_1^{i_1} \cdots T_n^{i_n}) \). Suppose that \( i_j \notin \frac{1}{m} \mathbb{Z} \). Then a short calculation shows that \( \zeta_m - 1 \) annihilates
\[
H^\bullet((1 + m\mathbb{Z})\gamma_j, (\mathcal{O}_K/p\mathcal{O}_K)T_1^{i_1} \cdots T_n^{i_n} \cdot)
\]
Hence, \( \zeta_m - 1 \) also annihilates
\[
H^\bullet(\Gamma_m, (\mathcal{O}_K/p\mathcal{O}_K)T_1^{i_1} \cdots T_n^{i_n} \cdot)
\]
by the Hochschild–Serre spectral sequence. By using the long exact sequence of \( H^\bullet(\Gamma_m, \cdot) \) associated with the short exact sequence
\[
0 \to p\mathcal{O}_K T_1^{i_1} \cdots T_n^{i_n} \to \mathcal{O}_K T_1^{i_1} \cdots T_n^{i_n} \to (\mathcal{O}_K/p\mathcal{O}_K)T_1^{i_1} \cdots T_n^{i_n} \to 0,
\]
we see that there exists a cochain \( g \in C^\bullet(\Gamma_m, \mathcal{O}_K T_1^{i_1} \cdots T_n^{i_n}) \) such that
\[
dg - (\zeta_m - 1)f \in C^\bullet(\Gamma_m, p\mathcal{O}_K T_1^{i_1} \cdots T_n^{i_n}).
\]
Thus,
\[
f - (g/(\zeta_m - 1)) \in C^\bullet(\Gamma_m, (p/(\zeta_m - 1))\mathcal{O}_K T_1^{i_1} \cdots T_n^{i_n}).
\]
Note that \( \nu_p(p/(\zeta_m - 1)) > 0 \) because \( m \geq 3 \). By iterating this process, we obtain a cochain
\[
g \in C^\bullet(\Gamma_m, (\zeta_m - 1)^{-1}\mathcal{O}_K T_1^{i_1} \cdots T_n^{i_n})
\]
such that \( dg = f \), yielding the lemma. \( \square \)

Proposition 4.16. The pair \( (\{A_m\}_{m \geq 3}, \{\Gamma_m\}_{m \geq 3}) \) is stably decompleting.

Proof. Obviously the \( Y_m \)'s and \( \hat{\mathbb{Y}}_\infty \) are uniform. Since a rational subset of \( Y \) is strictly étale over \( \mathbb{D}^n \), it reduces to showing that the pair \( (\{A_m\}_{m \geq 3}, \{\Gamma_m\}_{m \geq 3}) \) is decompleting. To begin with, the condition [1] of Definition 4.2 is clearly satisfied. The splitting condition [2] of Definition 4.2 follows from the splitting
\[
K\langle T_1^{\frac{1}{m}}, \ldots, T_n^{\frac{1}{m}} \rangle = K\langle T_1^{\frac{1}{m}}, \ldots, T_n^{\frac{1}{m}} \rangle \oplus \left(K\langle T_1^{\frac{1}{m}}, \ldots, T_n^{\frac{1}{m}} \rangle/K\langle T_1^{\frac{1}{m}}, \ldots, T_n^{\frac{1}{m}} \rangle \right).
\]
It remains to check the uniform strict exactness condition [3] of Definition 4.2. For any \( C > 1 \), by [BGR84], Section 2.7, Proposition 3], there exists a Schauder basis \( \{e_j\}_{j \in J} \) of \( A \) over \( K \) such that
\[
\max_{j \in J}\{|a_j e_j|\} \leq C\left|\sum_{j \in J} a_j e_j\right|,
\]
for every convergent sum \( \sum_{j \in J} a_j e_j \). Let \( S_m = (\mathbb{Q} \cap [0, \frac{1}{m}])^n \setminus (0, \ldots, 0) \). Then \( \{T_1^{i_1} \cdots T_n^{i_n}\}_{(i_1, \ldots, i_n) \in S_m} \) forms an orthonormal basis of the Banach module
\[
K\langle T_1^{\frac{1}{m}}, \ldots, T_n^{\frac{1}{m}} \rangle/K\langle T_1^{\frac{1}{m}}, \ldots, T_n^{\frac{1}{m}} \rangle
\]
over \( K\langle T_1^{\frac{1}{m}}, \ldots, T_n^{\frac{1}{m}} \rangle \). It follows that \( \{T_1^{i_1} \cdots T_n^{i_n} e_j\}_{(i_1, \ldots, i_n, e_j) \in S_m \times J} \) forms a \( K \)-Schauder basis of \( \hat{A}_\infty/A_m \) such that
\[
\max\{|a_{i_1 \cdots i_n} T_1^{i_1} \cdots T_n^{i_n} e_j|\} \leq C\left|\sum_{i_1 \cdots i_n} a_{i_1 \cdots i_n} T_1^{i_1} \cdots T_n^{i_n} e_j\right|,
\]
for every convergent sum $\sum a_{i_1,\ldots,i_n} T_1^{i_1} \cdots T_n^{i_n} e_j$. Using this basis and Lemma 4.15 we see that the uniform strict exactness holds for
\[
c = C \max_{m \geq 3} \{ |\zeta_m - 1|^{-1} \} = \begin{cases} p^{\frac{1}{p-1}} C, & \text{if } p > 2; \\ \sqrt{2} C, & \text{if } p = 2, \end{cases}
\]
as desired. \hfill \Box

**Theorem 4.17.** The pair $(\{A_m\}_{m \geq 3}, \Gamma_0)$ is a decomposition system.

**Proof.** Combine Proposition 4.16 and Theorem 4.3. □

4.2.2. **Arithmetic towers.** Let $A$ be an affinoid algebra over $k$. For $m \geq 1$, put $A_m = A_{k_m}$. Then $\hat{\Delta}_\infty = A_{\hat{k}_\infty}$. Put $\Gamma_m = \text{Gal}(k_\infty/k_m)$.

**Proposition 4.18.** There exists some sufficiently large $m_0$ such that the pair $(\{A_m\}_{m \geq m_0}, \{\Gamma_m\}_{m \geq m_0})$ is stably decompleting.

**Proof.** Using a $k$-Schauder basis as in the proof of Proposition 4.16, it suffices to show that the pair $(\{k_m\}_{m \geq m_0}, \{\Gamma_m\}_{m \geq m_0})$ is decompleting for some sufficiently large $m_0$. The only nontrivial part is the uniform strict exactness, which is part of the classical Tate–Sen formalism (see [BC08, Proposition 4.1.1]). □

**Theorem 4.19.** For some sufficiently large $m_0$, the pair $(\{A_m\}_{m \geq m_0}, \Gamma_0)$ is a decomposition system.

**Proof.** Combine Proposition 4.18 and Theorem 4.3. □

4.2.3. **Deformation of geometric towers.** In this example, we shall follow the setup in Section 3.2. Recall that we have a $\Gamma$-equivariant isomorphism \(3.20\), which induces, for every $k \geq 1$ and $m \geq 1$, a $\Gamma$-equivariant embedding
\[
\mathbb{B}_{r,m} := \left( R \hat{\otimes}_k (B_{dR}^+ / \xi^r) \right) \otimes_{(B_{dR}^+ / \xi^r)(T_1,\ldots,T_n)} (B_{dR}^+ / \xi^r)(T_1^\frac{1}{p}, \ldots, T_n^\frac{1}{p}) \rightarrow \mathbb{B}_{dR}^+ (\hat{X}) / \xi^r.
\]

**Lemma 4.20.** The natural projection $\theta : B_{dR}^+ / \xi^r \to \hat{k}_\infty$ admits a section $s$ in the category of $k$-Banach spaces satisfying $|s| \leq 2|\varpi|^{-1}$, where $\varpi$ is a uniformizer of $k$.

**Proof.** By using [BGRS, Section 2.7, Proposition 3], we can find a Schauder basis $\{e_j\}_{j \in J}$ of $\hat{k}_\infty$ over $k$ such that
\[
\max_{j \in J} \{|b_j e_j|\} \leq 2 \left| \sum_{j \in J} b_j e_j \right|,
\]
for every convergent sum $\sum_{j \in J} b_j e_j$. Moreover, we may rescale $e_j$ such that $|\pi| < |e_j| \leq 1$ for all $j \in J$. Clearly, we can lift each $e_j$ to some element $\tilde{e}_j$ in $A_{im}$. Then we define $s$ by mapping each convergent sum $\sum_{j \in J} b_j e_j$ to $\sum_{j \in J} b_j \tilde{e}_j$. □

**Proposition 4.21.** For each $m \geq 1$, put $\Gamma_m = (1 + m\hat{\mathcal{Z}})^n$. For every $r \geq 1$, the pair $(\{\mathbb{B}_{r,m}\}_{m \geq 3}, \{\Gamma_m\}_{m \geq 3})$ is decompleting.

**Proof.** The only nontrivial part is the uniform strict exactness. We prove it as follows. Let
\[
\bar{\mathbb{B}}_{r,\infty} = \left( R \hat{\otimes}_k (B_{dR}^+ / \xi^r) \right) \hat{\otimes}_{(B_{dR}^+ / \xi^r)(T_1,\ldots,T_n)} (B_{dR}^+ / \xi^r)(T_1^\frac{1}{p}, \ldots, T_n^\frac{1}{p})
\]
be the completed direct limit of \( \{B_{r,m}\}_{m \geq 3} \) (which is canonically isomorphic to \( B_{\text{dir}}^+(X)/\xi' \), by Lemma 3.19). Note that \((\{B_{1,m}\}_{m \geq 3}, \{\Gamma_m\}_{m \geq 3})\) is just the geometric tower considered in Section 4.2.1 with \( Y = X_{k_{\infty}} \), which is stably deforming by Proposition 4.16. Let \( c \geq 1 \) be a constant for the uniform strict exactness there. We shall show by induction that \((\{B_{r,m}\}_{m \geq 3}, \{\Gamma_m\}_{m \geq 3})\) satisfies the uniform strict exactness for the constant \((2|\xi|^{-2})^{r-1}c'\). The case \( r = 1 \) has already been verified.

For any \( r > 1 \), let \( f \) be a cocycle in \( C^*(\Gamma_m, \tilde{B}_{r,\infty}/B_{r,m}) \). Then its image \( \mathcal{f} \) in \( C^*(\Gamma_m, \tilde{B}_{1,\infty}/B_{1,m}) \) satisfies \( \|\mathcal{f}\| \leq |\xi|^{-1}\|f\| \). Let \( \mathcal{g} \) be a cocycle satisfying \( d\mathcal{g} = \mathcal{f} \) with \( \|\mathcal{g}\| \leq c\|\mathcal{f}\| \leq c|\xi|^{-1}\|f\| \). By Lemma 4.20, we can lift \( \mathcal{g} \) to a cocycle \( \tilde{g} \in C^*(\Gamma_m, \tilde{B}_{r,\infty}/B_{r,m}) \) with \( \|\tilde{g}\| \leq 2|\xi|^{-1}\|\mathcal{g}\| \leq 2|\xi|^{-2}c|\mathcal{g}| \). Now it is straightforward to see that there is a cochain \( f_1 \in C^*(\Gamma_m, \tilde{B}_{r-1,\infty}/B_{r-1,m}) \) such that \( f - d\tilde{g} = \xi f_1 \) via the isometry \( B_{\text{dir}}^+/\xi^{r-1} \cong \xi B_{\text{dir}}^+/\xi^r \) induced by multiplication by \( \xi \), and \( \|f_1\| = \|f - d\tilde{g}\| \leq 2|\xi|^{-2}\|\mathcal{g}\| \). By the inductive hypothesis, we can find a cochain \( g_1 \in C^*(\Gamma_m, \tilde{B}_{r-1,\infty}/B_{r-1,m}) \) satisfying \( dg_1 = f_1 \) with \( \|g_1\| \leq (2|\xi|^{-2})^{r-2}c^{-1}\|f_1\| \leq (2|\xi|^{-2})^{r-1}c\|f\| \). Now put \( g = \tilde{g} + \xi g_1 \); here again \( \xi g_1 \) is a cochain in \( C^*(\Gamma_m, \tilde{B}_{r,\infty}/B_{r,m}) \) via the isometry \( B_{\text{dir}}^+/\xi^{r-1} \cong \xi B_{\text{dir}}^+/\xi^r \). Then it is clear that \( dg = f \) and \( \|g\| \leq \max \{\|\tilde{g}\|, \|g_1\|\} \leq (2|\xi|^{-2})^{r-1}c\|f\| \). \( \square \)

**Theorem 4.22.** The pair \((\{B_{r,m}\}_{m \geq 3}, \Gamma)\) forms a decomposition system.

**Proof.** Let \( L_\infty \) be a finite projective \( \Gamma \)-module over \( \tilde{B}_{r,\infty} \). Note that if \((L_m, \iota_m)\) is a model of \( L_\infty \) over \( B_{r,m} \), then \((\xi^{s-1}L_m/\xi^sL_m, \tau_m)\) is a model of \( L_\infty/\xi L_\infty \) over \( B_{1,m} \), for every \( 1 \leq s \leq r \). Since \((\{B_{1,m}\}_{m \geq 3}, \Gamma)\) is a decomposition system, we can find an \( m_0 \geq 3 \) such that, for every \( m' \geq m_0 \) and every \( 1 \leq s \leq r \), the base change of \((\xi^{s-1}L_m/\xi^sL_m, \tau_m)\) to \( B_{r,m'} \) is a good model, yielding that the base change of \((L_m, \iota_m)\) to \( B_{r,m'} \) is a good model. Thus, we have proved that Definition 4.12 holds. Moreover, by the same argument as in the proof of Theorem 4.3(1), this property further ensures that any two models over \( B_{r,m} \) becomes identical in \( L_\infty \) after base change to \( B_{r,m'} \) for some sufficiently large \( m' \geq m \).

It remains to show the existence of a model of \( L_\infty \). Firstly, by the same argument as in the proof of Theorem 4.3(2), for some \( m \), we can find a finite covering of \((X_m)_{k_{\infty}} \) by rational subsets such that the restriction of \( L_\infty/\xi L_\infty \) to every rational subset in the covering is free. Note that, for an affinoid space \( Y \) over \( k \), the analytic topology of \( Y_{k_{\infty}} \) is generated by the base change of rational subsets of \( Y_{k'} \) with \( |k'| : k| < \infty \). Hence, by replacing \( X \) with \( X_m \) and replacing \( k \) with a finite extension, we may assume that there exists a finite covering \( X = \bigcup_{\iota \in \mathcal{I}} \text{Spa}(R_{i,} R_{\iota}^+) \) by rational subsets such that the restriction of \( L_\infty/\xi L_\infty \) to each \( \text{Spa}(R_{i,} R_{\iota}^+)_{k_{\infty}} \) is free. It follows that the base change of \( L_\infty \) to \( R_{\iota} \otimes_k (B_{\text{dir}}^+/\xi^r) \), denoted by \( L_{\infty,i} \), is free, because \( \xi \) is a nilpotent element. By Proposition 4.21 and Theorem 4.3(1), for some sufficiently large \( m \), each \( L_{\infty,i} \) admits a free model \((L_{m,i,\iota,m,i})\) over \((R_{i,} (B_{\text{dir}}^+/\xi^r))^\infty \otimes_{(B_{\text{dir}}^+/\xi^r)/(T_{1,1}, \ldots, T_{r,1})} (B_{\text{dir}}^+/\xi^r)/(T_{1,1}^{1/\xi}, \ldots, T_{r,1}^{1/\xi})) \). Moreover, by further enlarging \( m \), we may ensure that these models coincide on the overlaps of rational subsets in the covering. Thus, we conclude that these models glue to a desired model of \( L_\infty \), by applying [LZ17 Proposition 3.3]. \( \square \)
5. Log Riemann–Hilbert Correspondences

In this section, we establish our log $p$-adic Riemann–Hilbert correspondences, as well as the log $p$-adic Simpson correspondence.

5.1. Statements of theorems. We first state the geometric version of the log Riemann–Hilbert correspondence. Let $X$ be as in Example 2.12, where $k$ is a finite extension of $\mathbb{Q}_p$. For a $\mathbb{Q}_p$-local system $L$ on $X_{k\acute{e}t}$, let $\mathcal{L}$ be the corresponding $\mathbb{Q}_p$-local system on $X_{\text{prok\acute{e}t}}$ as in Definition 2.84. Let $K$ be a perfectoid field containing $k_{\infty}$.

As in [LZ17, Section 3.1], it is convenient to consider the ringed spaces

\begin{equation}
\mathcal{X}^+ = (X_{\text{an}}, \mathcal{O}_X \hat{\otimes} B_\text{dR}^+)
\end{equation}

and

\begin{equation}
\mathcal{X} = (X_{\text{an}}, \mathcal{O}_X \hat{\otimes} B_{\text{dR}}),
\end{equation}

where $\mathcal{O}_X \hat{\otimes} B_\text{dR}^+$ and $\mathcal{O}_X \hat{\otimes} B_{\text{dR}}$ are as in Definition 3.4.2, which should be interpreted as the (not-yet-defined) base changes of $X$ under $k \to B_\text{dR}^+$ and $k \to B_{\text{dR}}$, respectively.

Following [LZ17, Definition 3.5], we call a locally free $\mathcal{O}_X \hat{\otimes} B_\text{dR}^+$-module of finite rank a vector bundle on $\mathcal{X}^+$. By varying the open subsets of $X$, these objects form a stack on $X_{\text{an}}$. By passing to the $t$-isogeny category, we obtain the stack of vector bundles on open subsets of $\mathcal{X}$. Then the category of vector bundles on $\mathcal{X}$ is the groupoid of global sections of the stack. Clearly, there is a faithful functor from the category of vector bundles on $\mathcal{X}$ to the category of $\mathcal{O}_X \hat{\otimes} B_{\text{dR}}$-modules.

Remark 5.3. Since the base changes of objects of $X_{\text{an}}$ and $X_{\acute{e}t}$ generate $X_{K,\text{an}}$ and $X_{K,\acute{e}t}$, respectively, we see that the categories of finite locally free $\mathcal{O}_X \hat{\otimes} (B_\text{dR}^+/t^r)$-modules (resp. $\mathcal{O}_X \hat{\otimes} B_\text{dR}^+$-modules) and finite locally free $\mathcal{O}_{X_{\text{an}}} \hat{\otimes} (B_\text{dR}^+/t^r)$-modules (resp. $\mathcal{O}_{X_{\text{an}}} \hat{\otimes} B_\text{dR}^+$-modules) are naturally equivalent to the categories introduced in [LZ17, Definition 3.5]. For example, the category of locally free $\mathcal{O}_X \hat{\otimes}_k K$-modules (i.e., gr$^0(\mathcal{O}_X \hat{\otimes}_k B_\text{dR}^+$)-modules) on $X_{\text{an}}$ is equivalent to the category of vector bundles on $X_{K,\text{an}}$.

Thanks to Remark 5.3, the arguments in the proofs of [LZ17, Proposition 3.3 and Corollary 3.4] are also applicable in the current setting and give the following:

Lemma 5.4. (1) If $X = \text{Spa}(A, A^+)$ is affinoid, then the categories

- of finite projective $A \hat{\otimes}_k B_\text{dR}^+$-modules;
- of locally free $\mathcal{O}_X \hat{\otimes} B_\text{dR}^+$-modules of finite rank; and
- of locally free $\mathcal{O}_{X_{\text{an}}} \hat{\otimes} B_\text{dR}^+$-modules of finite rank;

are all equivalent to each other.

(2) Recall that we denote by $\lambda : X_{\acute{e}t} \to X_{\text{an}}$ the natural projection of sites. Then $\lambda_*$ induces an equivalence from the category of finite locally free $\mathcal{O}_{X_{\text{an}}} \hat{\otimes} (B_\text{dR}^+/t^r)$-modules (resp. $\mathcal{O}_{X_{\text{an}}} \hat{\otimes} B_\text{dR}^+$-modules) to the category of finite locally free $\mathcal{O}_X \hat{\otimes} (B_\text{dR}^+/t^r)$-modules (resp. $\mathcal{O}_X \hat{\otimes} B_\text{dR}^+$-modules).

Hence, for each vector bundle $\mathcal{E}$ on $X_{\text{an}}$, the sheaf $\mathcal{E} \hat{\otimes}_k B_\text{dR}^+$ (resp. $\mathcal{E} \hat{\otimes}_k B_{\text{dR}}$) (with its obvious meaning) is a vector bundle on $\mathcal{X}^+$ (resp. $\mathcal{X}$). More generally, if $\mathcal{E}$ is a
vector bundle on $X_{an}$, and if $\mathcal{M}$ is a vector bundle on $\mathcal{X}^+$ (resp. $\mathcal{X}$), then we may regard $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{M}$ as a vector bundle on $\mathcal{X}^+$ (resp. $\mathcal{X}$).

**Definition 5.5.** Recall that we denote by $\Omega_{X,log}^j$ the sheaf of log differentials on $X_{an}$ (see (2.53)). Then we can also define the sheaves of relative log differentials

$$\Omega_{X/B_{dR}}^{log,j} := \Omega_{X,log}^j \otimes_{\mathcal{O}_{X}} B_{dR}$$

and

$$\Omega_{X/B_{dR}}^{log,j} := \Omega_{X,log}^j \otimes_{\mathcal{O}_{X}} B_{dR}.$$  

There is a natural $B_{dR}$-linear differential map

$$d : \mathcal{O}_{\mathcal{X}} \rightarrow \Omega_{\mathcal{X}/B_{dR}}^{log,j}$$

extending the one on $\mathcal{O}_X$, which further extends to a $B_{dR}$-linear map

$$d : \mathcal{O}_{\mathcal{X}} \rightarrow \Omega_{\mathcal{X}/B_{dR}}^{log,j}.$$  

**Definition 5.6.**

1. A log connection on a vector bundle $\mathcal{E}$ on $\mathcal{X}$ is a $B_{dR}$-linear map of sheaves $\nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{O}_{\mathcal{X}}} \Omega_{\mathcal{X}/B_{dR}}^{log,j}$ satisfying the usual Leibniz rule. We say that $\nabla$ is integrable if $\nabla^2 = 0$, in which case we have the log de Rham complex

$$DR_{log}(\mathcal{E}) : 0 \rightarrow \mathcal{E} \otimes_{\mathcal{O}_{\mathcal{X}}} \Omega_{\mathcal{X}/B_{dR}}^{log,1} \rightarrow \mathcal{E} \otimes_{\mathcal{O}_{\mathcal{X}}} \Omega_{\mathcal{X}/B_{dR}}^{log,2} \rightarrow \cdots.$$  

For simplicity, we shall often write in this case that

$$DR_{log}(\mathcal{E}) = ([\mathcal{E} \otimes_{\mathcal{O}_{\mathcal{X}}} \Omega_{\mathcal{X}/B_{dR}}^{log,j}, \nabla]).$$

Then we also have the log de Rham cohomology

$$H^i_{log,R^{dR}}(\mathcal{X}, \mathcal{E}) := H^i(X, DR_{log}(\mathcal{E})).$$

2. Let $t = \log([\varepsilon]) \in B_{dR}^+$. A log $t$-connection on a vector bundle $\mathcal{E}^+$ on $\mathcal{X}^+$ is a $B_{dR}^+$-linear map of sheaves

$$\nabla^+ : \mathcal{E}^+ \rightarrow \mathcal{E}^+ \otimes_{\mathcal{O}_{\mathcal{X}^+}} \Omega_{\mathcal{X}^+/B_{dR}}^{log,j}$$

satisfying the (modified) Leibniz rule $\nabla^+(fe) = (te) \otimes df + f \nabla^+(e)$, for all $f \in \mathcal{O}_{\mathcal{X}^+}$ and $e \in \mathcal{E}^+$. We say $\nabla^+$ is integrable if $(\nabla^+)^2 = 0$, in which case we have a similar log de Rham complex (as above).

3. A log Higgs bundle on $X_K$ is a vector bundle $\mathcal{E}$ on $X_{K,an}$ equipped with an $\mathcal{O}_{X_K}$-linear map of sheaves

$$\theta : E \rightarrow E \otimes_{\mathcal{O}_{X_K}} \Omega_{X_K}^{log,1}(-1),$$

such that $\theta \wedge \theta = 0$. We have the log Higgs complex

$$Higgs_{log}(\mathcal{E}) : 0 \rightarrow E \otimes_{\mathcal{O}_{X_K}} \Omega_{X_K}^{log,1}(-1) \rightarrow E \otimes_{\mathcal{O}_{X_K}} \Omega_{X_K}^{log,2}(-2) \rightarrow \cdots.$$  

For simplicity, we shall often write in this case that

$$Higgs_{log}(\mathcal{E}) = ([E \otimes_{\mathcal{O}_{X_K}} \Omega_{X_K}^{log,j}, \theta]).$$

Then we also have the log Higgs cohomology

$$H^i_{log,Higgs}(X_{K,an}, E) := H^i(X_{K,an}, Higgs_{log}(\mathcal{E})).$$
(4) A **log connection** on a coherent sheaf $E$ on $X$ is a $k$-linear map of sheaves

$$\nabla : E \rightarrow E \otimes_{\mathcal{O}_X} \Omega^1_X$$

satisfying the usual Leibniz rule. We say that $\nabla$ is **integrable** if $\nabla^2 = 0$, in which case we have the **log de Rham complex**

$$DR_{\log}(E) : 0 \rightarrow E \overset{\nabla}{\rightarrow} E \otimes \mathcal{O}_X \Omega^1_X \overset{\nabla}{\rightarrow} E \otimes \mathcal{O}_X \Omega^2_X \rightarrow \cdots.$$ 

For simplicity, we shall often write in this case that $DR_{\log}(E) = (E \otimes \mathcal{O}_X \Omega^\bullet_X, \nabla)$.

Then we also have the log de Rham cohomology

$$H^i_{\log dR}(X, E) := H^i(X, DR_{\log}(E)).$$

Suppose $E$ is equipped with a decreasing filtration by coherent sub-sheaves $\text{Fil}^r E$ satisfying the (usual) Griffiths transversality condition

$$\nabla(\text{Fil}^r E) \subset (\text{Fil}^{r-1} E) \otimes \mathcal{O}_X \Omega^\bullet_X,$$

for all $r$. Then the complex $DR_{\log}(E)$ admits a filtration $\text{Fil}^\bullet DR_{\log}(E)$ defined by

$$\text{Fil}^r DR_{\log}(E) := ((\text{Fil}^{r-1} E) \otimes \mathcal{O}_X \Omega^\bullet_X, \nabla),$$

with the two $\bullet$ equal to each other, and with $\nabla$ respecting the filtration and inducing $\mathcal{O}_X$-linear morphisms on the graded pieces because of (5.8). Then the graded pieces form a complex $gr DR_{\log}(\mathcal{L})$ with $\mathcal{O}_X$-linear differentials, and we also have the **log Hodge cohomology**

$$H^{a,b}_{\log \text{Hodge}}(X, E) := H^{a+b}(X, gr\, DR_{\log}(E)).$$

The cohomology (5.7) and (5.10) are related by the **(log) Hodge–de Rham spectral sequence** (associated with the above filtration (5.9))

$$E_1^{a,b} = H^{a,b}_{\log \text{Hodge}}(X_{\text{an}}, E) \Rightarrow H^{a+b}_{\log \text{dR}}(X_{\text{an}}, E).$$

The following lemma is clear.

**Lemma 5.12.** The functor

$$(\mathcal{E}^+, \nabla^+) \mapsto (\mathcal{E}, \nabla, \{\text{Fil}^r \mathcal{E}\}_{r \geq 0}) := (\mathcal{E}^+ \otimes_{B^+_{\text{dR}}} B_{\text{dR}}, t^{-1} \nabla^+, \{t^r \mathcal{E}^+\}_{r \geq 0})$$

is an equivalence of categories from the category of integrable log $t$-connections on $X^+$ to the category of integrable log connections $(\mathcal{E}, \nabla)$ on $X$ that are equipped with filtrations $\{\text{Fil}^r \mathcal{E}\}_{r \geq 0}$ by locally free $\mathcal{O}_X \otimes B^+_{\text{dR}}$-modules satisfying the Griffith transversality condition:

$$\nabla(\text{Fil}^r \mathcal{E}) \subset (\text{Fil}^{r-1} \mathcal{E}) \otimes \mathcal{O}_{X^+} \Omega^\bullet_{X^+/B^+_{\text{dR}}}.$$

On the other hand, we have the following:

**Lemma 5.14.** The functor

$$(\mathcal{E}^+, \nabla^+) \mapsto (E, \theta) := (\mathcal{E}^+/t, \nabla^+),$$

where $\nabla^+$ abusively also denotes its induced map on $\mathcal{E}^+/t$, is a functor from the category of integrable log $t$-connections to the category of log Higgs bundles on $X_K$. 

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Let $\mu^*: X_{\text{prok}\et}/X_K \to X_{\text{an}}$ be the natural projection of sites, as in \cite[(3.49)]{AGT16}. For any $\mathbb{Q}_p$-local system $\mathcal{L}$ on $X_{\text{k}\et}$, consider

\begin{equation}
\mathcal{R}_{\log}(\mathcal{L}) := R\mu^*(\hat{\mathcal{L}} \otimes_{\mathcal{O}_p} \mathcal{O}_{\text{dR}, \log}).
\end{equation}

**Theorem 5.16.** (1) The assignment $\mathcal{L} \mapsto \mathcal{R}_{\log}(\mathcal{L})$ is an exact functor from the category of $\mathbb{Q}_p$-local systems on $X_{\text{k}\et}$ to the category of $\text{Gal}(K/k)$-equivariant vector bundles on $X$ equipped with an integrable log connection

$$\nabla: \mathcal{R}_{\log}(\mathcal{L}) \to \mathcal{R}_{\log}(\mathcal{L}) \otimes_{\mathcal{O}_X} \Omega^\log_{X/\text{dR}},$$

and a decreasing filtration (by locally free $\mathcal{O}_X \otimes B^+_{\text{dR}}$-submodules) satisfying the Griffith transversality as in \cite[(5.13)]{Fal05}.

(2) For each irreducible component $D_0$ (defined as in \cite{Con99}) of the normal crossings divisor $D$, let $\text{Res}_{D_0}(\nabla)$ denote the residue of the log connection $\nabla$ along $D_0$ (see Section 5.3 below for the definition of residues). Then all the eigenvalues of $\text{Res}_{D_0}(\nabla)$ are in $\mathbb{Q} \cap [0, 1)$.

(3) Assume that $X$ is proper over $k$ and $K = \overline{k}$. Let $\mathcal{L}$ be a $\mathbb{Z}_p$-local system on $X_{\text{k}\et}$. Then there is a canonical $\text{Gal}(K/k)$-equivariant isomorphism

$$H^i(X_{K,\text{et}}, \mathcal{L}) \otimes_{\mathbb{Z}_p} B_{\text{dR}} \cong H^i_{\text{log dR}}(X, \mathcal{R}_{\log}(\mathcal{L})), $$

for each $i \geq 0$, where $H^i_{\text{log dR}}(X, \mathcal{R}_{\log}(\mathcal{L}))$ is as in Definition 5.6(1), compatible with the filtrations on both sides.

(4) Suppose $Y$ is another log adic space as in Example 2.12. If $h: Y \to X$ is any morphism (of log adic spaces), then there is a canonical $\text{Gal}(K/k)$-equivariant isomorphism

$$h^*(\mathcal{R}_{\log}(\mathcal{L}), \nabla_{\mathcal{L}}) \sim \mathcal{R}_{\log}(h^*(\mathcal{L})), \nabla_{h^*(\mathcal{L})},$$

compatible with the filtrations on both sides.

As a byproduct, we obtain the log $p$-adic Simpson functor. We refer to \cite{AGT16,Fal05} for more general and thorough treatments.

**Theorem 5.17.** (1) There is a natural functor $\mathcal{H}_{\log}$ from the category of $\mathbb{Q}_p$-local systems $\mathcal{L}$ on $X_{\text{k}\et}$ to the category of $\text{Gal}(K/k)$-equivariant log Higgs bundles

$$\theta: \mathcal{H}_{\log}(\mathcal{L}) \to \mathcal{H}_{\log}(\mathcal{L}) \otimes_{\mathcal{O}_X} \Omega^\log_{X/K}(-1)$$

on $X_{\text{an}}$. Concretely, by Lemma 5.12, $\mathcal{R}^\log_{\text{Higgs}} := \text{Fil}^0 \mathcal{R}_{\log}$ is a functor from the category of $\mathbb{Q}_p$-local systems on $X_{\text{k}\et}$ to the category of $\text{Gal}(K/k)$-equivariant integrable log $t$-connections on $X^+$. Then, by Lemma 5.14, $\mathcal{H}_{\log} := \text{gr}^0 \mathcal{R}_{\log} = \mathcal{R}^+_{\log}/t$ is the desired functor.

(2) Assume that $X$ is proper over $k$ and $K = \overline{k}$. Let $\mathcal{L}$ be a $\mathbb{Z}_p$-local system on $X_{\text{k}\et}$. Then there is a canonical $\text{Gal}(K/k)$-equivariant isomorphism

$$H^i(X_{K,\text{et}}, \mathcal{L}) \otimes_{\mathbb{Z}_p} K \cong H^i_{\text{log Higgs}}(X_{\text{an}}, \mathcal{H}_{\log}(\mathcal{L})), $$

for each $i \geq 0$, where $H^i_{\text{log Higgs}}(X_{\text{an}}, \mathcal{H}_{\log}(\mathcal{L}))$ is as in Definition 5.6(3).

(3) Suppose $Y$ is another log Higgs as in Example 2.12. If $h: Y \to X$ is any morphism (of log adic spaces), then there is a canonical $\text{Gal}(K/k)$-equivariant isomorphism

$$h^*(\mathcal{H}_{\log}(\mathcal{L}), \theta_{\mathcal{L}}) \sim (\mathcal{H}_{\log}(h^*(\mathcal{L})), \theta_{h^*(\mathcal{L})}).$$
We also have an arithmetic log $p$-adic Riemann–Hilbert functor. Consider the natural projection of sites
\[(5.18)\]
\[\mu : X_{\text{prokét}} \to X_{\text{an}}.\]
For any $\mathbb{Q}_p$-local system $\mathbb{L}$ on $X_{\text{két}}$, consider
\[(5.19)\]
\[D_{\text{dR,log}}(\mathbb{L}) := \mu_* (\mathbb{L} \otimes_{\mathbb{Q}_p} \mathcal{O}_{\text{dR,log}}).\]

**Theorem 5.20.**
1. The assignment $\mathbb{L} \mapsto D_{\text{dR,log}}(\mathbb{L})$ defines a functor from the category of $\mathbb{Q}_p$-local systems on $X_{\text{két}}$ to the category of vector bundles on $X_{\text{an}}$ with an integrable log connection
\[\nabla_{\mathbb{L}} : D_{\text{dR,log}}(\mathbb{L}) \to D_{\text{dR,log}}(\mathbb{L}) \otimes \Omega^1_{X},\]
and a decreasing filtration $\text{Fil}^i D_{\text{dR,log}}(\mathbb{L})$ (by coherent subsheaves) satisfying the (usual) Griffiths transversality condition (cf. \[(5.5)\]).

In addition, for each irreducible component $D_0$ (defined as in [Con99]) of the normal crossings divisor $D$ as in Theorem \[(5.16)\] all eigenvalues of the residue $\text{Res}_{D_0}(\nabla)$ are in $\mathbb{Q} \cap [0, 1]$.

2. Assume that $X$ is proper over $k$, that $K = \overline{k}$, and that $\mathbb{L}$ is a $\mathbb{Z}_p$-local system on $X_{\text{két}}$ whose restriction to $(X - D)_{\text{ét}}$ is de Rham (as reviewed in the introduction). Then, for each $i \geq 0$, there is a canonical $\text{Gal}(K/k)$-equivariant isomorphism
\[(5.21)\]
\[H^i(X_{K, \text{két}}, \mathbb{L}) \otimes_{\mathbb{Z}_p} B_{\text{dR}} \cong H^i_{\text{log dR}}(X_{\text{an}}, D_{\text{dR,log}}(\mathbb{L})) \otimes_k B_{\text{dR}}\]
compatible with the filtrations on both sides. Moreover, $\text{gr} D_{\text{dR,log}}(\mathbb{L})$ is a vector bundle on $X$ of rank $\text{rk}_\mathbb{Z}_p(\mathbb{L})$, the (log) Hodge–de Rham spectral sequence for $D_{\text{dR,log}}(\mathbb{L})$ (cf. \[(5.11)\]) degenerates on the $E_1$ page, and there is also a canonical $\text{Gal}(K/k)$-equivariant isomorphism
\[(5.22)\]
\[H^i(X_{K, \text{két}}, \mathbb{L}) \otimes_{\mathbb{Z}_p} K \cong \bigoplus_{a+b=i} H^a_{\text{log Hodge}}(X_{\text{an}}, D_{\text{dR,log}}(\mathbb{L})) \otimes_k K(-a),\]
for each $i \geq 0$, which can be identified with the 0-th graded piece of the isomorphism \[(5.21)\], giving the (log) Hodge–Tate decomposition.

3. Suppose $Y$ is a log adic space as in Example \[(2.12)\] with its log structure defined by a normal crossings divisor $E$. If $h : Y \to X$ is any morphism (of log adic spaces), then there is a canonical $\text{Gal}(K/k)$-equivariant isomorphism
\[h^* (D_{\text{dR,log}}(\mathbb{L})), \nabla_{\mathbb{L}}) \cong (D_{\text{dR,log}}(h^* (\mathbb{L})), \nabla_{h^* (\mathbb{L})}),\]
compatible with the filtrations on both sides. Alternatively, if $f : X \to Y$ is any proper log smooth morphism that restricts to a proper smooth morphism $f|_{X - D} : X - D \to Y - E$, and if $\mathbb{L}$ is a $\mathbb{Z}_p$-local system on $X_{\text{két}}$ such that $\mathbb{L}|_{(X - D)_{\text{ét}}}$ is de Rham and such that $R^i(f|_{X - D})_{\text{ét, *}}(\mathbb{L}|_{(X - D)_{\text{ét}}})$ is a $\mathbb{Z}_p$-local system on $(Y - E)_{\text{ét}}$, for some $i \geq 0$, then $R^i(f|_{X - D})_{\text{ét, *}}(\mathbb{L}|_{(X - D)_{\text{ét}}})$ is de Rham, $R^i f_{\text{két, *}}(\mathbb{L})$ is a $\mathbb{Z}_p$-local system on $Y_{\text{két}}$, and we also have a canonical $\text{Gal}(K/k)$-equivariant isomorphism
\[(5.23)\]
\[(D_{\text{dR,log}}(R^i f_{\text{két, *}}(\mathbb{L})), \nabla_{R^i f_{\text{két, *}}(\mathbb{L})}) \cong (R^i(f\otimes \text{dR})_{\text{ét, *}}(D_{\text{dR,log}}(\mathbb{L})), \nabla_{\mathbb{L}}))_{\text{free}},\]
compatible with the filtrations on both sides, where $R^i(f\otimes \text{dR})_{\text{ét, *}}(D_{\text{dR,log}}(\mathbb{L})), \nabla_{\mathbb{L}})$ is the relative analogue of the log de Rham cohomology defined as in \[(5.7)\], and where the subscript “free” denotes the $\mathcal{O}_Y$-torsion-free quotient.
By combining Corollary 2.75 and Theorem 5.20, we obtain the following:

**Corollary 5.23.** Let $U$ be a smooth rigid analytic variety over $k$. Assume that $U$ admits a proper smooth compactification $U \hookrightarrow X$ such that $X - U$ is a normal crossings divisor. Let $\mathbb{L}$ be a de Rham $\mathbb{Z}_p$-local system on $U_{\text{ét}}$, and consider its extension $\mathbb{L} := j_{\text{ét},*}(\mathbb{L})$ to a $\mathbb{Z}_p$-local system on $X_{\text{ét}}$. Then $H^i(U_{K,\text{ét}}, \mathbb{L})$ is finite dimensional, and there is a canonical $\text{Gal}(K/k)$-equivariant isomorphism

$$H^i(U_{K,\text{ét}}, \mathbb{L}) \otimes_{\mathbb{Z}_p} B_{\text{dR}} \cong H^i_{\text{dR}}(X_\text{an}, D_{\text{dR, log}}(\mathbb{L})) \otimes_k B_{\text{dR}},$$

compatible with the filtrations on both sides. Moreover, the (log) Hodge–de Rham spectral sequence for $D_{\text{dR, log}}(\mathbb{L})$ (cf. [5.11]) degenerates on the $E_1$ page, and there is also a canonical $\text{Gal}(K/k)$-equivariant isomorphism

$$H^i(U_{K,\text{ét}}, \mathbb{L}) \otimes_{\mathbb{Z}_p} K \cong \bigoplus_{a+b=i} \left( H^{a,b}_{\text{Hodge}}(X_\text{an}, D_{\text{dR, log}}(\mathbb{L})) \otimes_k K(-a) \right),$$

which can be identified with the $0$-th graded piece of the previous isomorphism.

Note that, as explained in Remark 2.77, these statements cannot hold for an arbitrary smooth rigid analytic variety $U$.

As mentioned in the introduction, due to the failure of the surjectivity of (1.5), $D_{\text{dR, log}}$ is not a tensor functor in general, and we have similar failures for $\mathcal{R}\mathcal{H}_{\text{log}}$ and $\mathcal{H}_\text{log}$. Nevertheless, we have the following:

**Theorem 5.24.** (1) The functor $\mathcal{R}\mathcal{H}_{\text{log}}$ (resp. $\mathcal{H}_{\text{log}}$) restricts to a tensor functor from the category of $\mathbb{Q}_p$-local systems on $X_{\text{ét}}$ whose restrictions to $(X - D)_{\text{ét}}$ have unipotent geometric monodromy along $D$ to the category of filtered $\text{Gal}(K/k)$-equivariant vector bundles on $X$ equipped with integrable log connections with nilpotent residues along $D$ (resp. the category of $\text{Gal}(K/k)$-equivariant log Higgs bundles on $X_{\text{an}}$).

(2) The functor $D_{\text{dR, log}}$ restricts to a tensor functor from the category of $\mathbb{Q}_p$-local systems on $X_{\text{ét}}$ whose restrictions to $(X - D)_{\text{ét}}$ are de Rham and have unipotent geometric monodromy along $D$ to the category of filtered vector bundles on $X_{\text{an}}$ equipped with integrable log connections with nilpotent residues along $D$.

5.2. Coherence. In this subsection, we prove Theorems 5.16 and 5.17, and show that $D_{\text{dR, log}}(\mathbb{L})$ is a torsion-free reflexive coherent sheaf on $X_{\text{an}}$.

Firstly, by factoring $\mu'$ as a composition

$$X_{\text{pro-}\text{két}}/X_k \cong X_{K,\text{pro-}\text{két}} \to X_{K,\text{ét}} \to X_{K,\text{an}} \to X_{\text{an}},$$

we see that $\mathcal{R}\mathcal{H}_{\text{log}}(\mathbb{L})$ admits a natural $\text{Gal}(K/k)$-action. We need to show that $R\mu'_*((\mathbb{L} \otimes_{\hat{\mathbb{Q}}_p} \text{Fil}^r \mathcal{O}_{\text{dR, log}}))$ is a locally free $O_X \otimes B^+_{\text{dR}}$-module of rank $\text{rk}_{\mathbb{Q}_p}(\mathbb{L})$, for every $r$. Assuming this, it follows that

$$\mathcal{R}\mathcal{H}_{\text{log}}(\mathbb{L}) = R\mu'_*((\mathbb{L} \otimes_{\hat{\mathbb{Q}}_p} \mathcal{O}_{\text{dR, log}}) \otimes \text{Fil}^r \mathcal{O}_{\text{dR, log}})[t^{-1}]$$

is a vector bundle on $X$, equipped with a filtration by locally free $O_X \otimes B^+_{\text{dR}}$-submodules

$$\text{Fil}^r \mathcal{R}\mathcal{H}_{\text{log}}(\mathbb{L}) = \mu'_*((\mathbb{L} \otimes_{\hat{\mathbb{Q}}_p} \text{Fil}^r \mathcal{O}_{\text{dR, log}})).$$

Consider the integrable log connection

$$\nabla : \mathbb{L} \otimes_{\hat{\mathbb{Q}}_p} \mathcal{O}_{\text{dR, log}} \to \mathbb{L} \otimes_{\hat{\mathbb{Q}}_p} \mathcal{O}_{\text{dR, log}} \otimes O_{X, \text{pro-}\text{két}} \Omega^1_X.$$
obtained by tensoring the one on $\mathcal{O}_{\text{dR,log}}$ with $\hat{\mathcal{L}}$. By Proposition 3.50 and the projection formula

$$(5.25) \quad R\mu'_*(\hat{\mathcal{L}} \otimes \hat{\mathcal{F}}_p \mathcal{O}_{\text{dR,log}} \otimes \mathcal{O}_{\text{prokét}} \Omega^\log_X \otimes j^! \mathcal{O}_X) \cong R\mu'_*(\hat{\mathcal{L}} \otimes \hat{\mathcal{F}}_p \mathcal{O}_{\text{dR,log}}) \otimes \mathcal{O}_X \Omega^\log_X,$$

we obtain a log connection

$$\nabla_L : \mathcal{R}\mathcal{H}_{\log}(\mathcal{L}) \to \mathcal{R}\mathcal{H}_{\log}(\mathcal{L}) \otimes \mathcal{O}_X \Omega^\log_X.$$ 

The integrability of $\nabla_L$ and the Griffiths transversality with respect to the above filtration $\text{Fil}^i\mathcal{R}\mathcal{H}_{\log}(\mathcal{L})$ follow from the corresponding properties for the connection $\nabla : \mathcal{O}_{\text{dR,log}} \to \mathcal{O}_{\text{dR,log}} \otimes \mathcal{O}_{\text{prokét}} \Omega^\log_X$ in (3.16).

By the same arguments as in the proofs of [LZ17] Theorems 2.1(i) and 3.8(ii), to show that $R\mu'_*(\hat{\mathcal{L}} \otimes \hat{\mathcal{F}}_p \text{Fil}^i\mathcal{O}_{\text{dR,log}})$ is a locally free $\mathcal{O}_X \otimes \mathcal{O}_{\text{dR,log}}$-module of rank $\text{rk}_{\mathcal{Q}_p}(\mathcal{L})$, for every $r$, it suffices to prove the following:

**Proposition 5.26.** Let $\mathcal{L}$ be a $\mathcal{Q}_p$-local system on $X_{\text{ét}}$. Recall that we introduced $\mathcal{O}_{\text{log}} = \mathcal{g}^0 \mathcal{O}_{\text{dR,log}}$ in (3.40). Then we have the following:

1. $R^i\mu'_*(\hat{\mathcal{L}} \otimes \hat{\mathcal{F}}_p \mathcal{O}_{\text{log}}) = 0$, for all $i > 0$.
2. $\mu'_*(\hat{\mathcal{L}} \otimes \hat{\mathcal{F}}_p \mathcal{O}_{\text{log}})$ is a locally free $\mathcal{g}^0(\mathcal{O}_X \otimes \mathcal{O}_{\text{dR,log}})$-module of rank $\text{rk}_{\mathcal{Q}_p}(\mathcal{L})$.

For simplicity, we may assume that $K = \hat{k}_{\infty}$. We will make use of the fact that the category of finite locally free $\mathcal{g}^0(\mathcal{O}_X \otimes \mathcal{O}_{\text{dR,log}})$-modules is equivalent to the category of finite locally free $\mathcal{g}^0(\mathcal{O}_X \otimes \mathcal{O}_{\text{dR,log}})$-modules via the pushforward under the projection of sites $\lambda : X_{\text{ét}} \to X_{an}$ (by Lemma 5.4(2)). Therefore, it suffices to prove a similar statement for the projection of sites $\nu : X_{\text{prokét}}/X_{\text{ét}} \to X_{\text{ét}}$ (instead of $\mu : X_{\text{prokét}}/X_{\text{ét}} \to X_{\text{ét}}$). Since smooth toric charts exist étale locally on $X$ (see Proposition 2.27 and Example 2.28), we may assume as in Section 3.2 that $X = \text{Spa}(R, R^+)$ is an affinoid log adic space over $k$ equipped with a strictly étale map $X \to \mathbb{D}^n$, which is a composition of rational embeddings and finite étale maps. We shall write $R_K = R \otimes_k K$. By Corollary 3.36 we have

$$\mathcal{O}_{\text{log}}|_{X_{\text{ét}}} \cong \hat{\mathcal{O}}_{X_{\text{prokét}}}[W_1, \ldots, W_n].$$

Let $\mathcal{L} = \hat{\mathcal{L}} \otimes \hat{\mathcal{F}}_p \hat{\mathcal{O}}_{X_{\text{prokét}}}$, which is a locally free $\hat{\mathcal{O}}_{X_{\text{prokét}}}$-module of rank $\text{rk}_{\mathcal{Q}_p}(\mathcal{L})$. Then

$$(\hat{\mathcal{L}} \otimes \hat{\mathcal{F}}_p \mathcal{O}_{\text{log}})|_{X_{\text{ét}}} \cong \mathcal{L}|_{W_1, \ldots, W_n}.$$ 

Note that $R^i\nu'_*(\hat{\mathcal{L}} \otimes \hat{\mathcal{F}}_p \mathcal{O}_{\text{log}})$ is the sheaf associated with the presheaf

$$Y \in X_{\text{ét}} \mapsto H^i(X_{\text{prokét}}/Y_{X_{\text{ét}}}, \hat{\mathcal{L}} \otimes \hat{\mathcal{F}}_p \mathcal{O}_{\text{log}}).$$

Hence, in order to prove Proposition 5.26 it suffices to prove the following two statements:

(a) $H^0(X_{\text{prokét}}/X_{\text{ét}}, \hat{\mathcal{L}} \otimes \hat{\mathcal{F}}_p \mathcal{O}_{\text{log}})$ is a finite projective $R_K$-module of rank $\text{rk}_{\mathcal{Q}_p}(\mathcal{L})$, and $H^i(X_{\text{prokét}}/X_{\text{ét}}, \hat{\mathcal{L}} \otimes \hat{\mathcal{F}}_p \mathcal{O}_{\text{log}}) = 0$ for all $i > 0$.

(b) If $Y = \text{Spa}(S, S^+) \to X$ is a composition of rational embeddings and finite étale morphisms, then

$$H^0(X_{\text{prokét}}/Y_{X_{\text{ét}}}, \hat{\mathcal{L}} \otimes \hat{\mathcal{F}}_p \mathcal{O}_{\text{log}}) \cong H^0(X_{\text{prokét}}/X_{\text{ét}}, \hat{\mathcal{L}} \otimes \hat{\mathcal{F}}_p \mathcal{O}_{\text{log}}) \otimes R_K S_K.$$
Our approach for proving (a) and (b) is similar to the proof of [LZ17, Theorem 2.1]. We will only explain the new ingredients needed in the proof, and refer to [LZ17] for more details. Recall that, in Section 3.2, we have a log affinoid perfectoid covering \( \tilde{X} \to X \) obtained by pulling back \( \tilde{D}^n \to \tilde{D}^n \). Then \( \tilde{X} \to X_{k_{\infty}} \) is a Galois pro-Kummer étale covering with Galois group \( \Gamma_{\text{geom}} \cong (\widehat{\mathbb{Z}}(1))^n \). Consequently, \( \tilde{X} \to X \) is also a Galois pro-Kummer étale covering with Galois group \( \Gamma \), and we have an exact sequence
\[
1 \to \Gamma_{\text{geom}} \to \Gamma \to \text{Gal}(k_{\infty}/k) \to 1.
\]

For any \( Y \) as in (b), we endow it with the induced log structure, and define
\[
\tilde{Y} = Y \times_X \tilde{X} \in X_{\text{prokét}}.
\]

Then \( \tilde{Y} \) is log affinoid perfectoid, and \( \tilde{Y} \to Y_{k_{\infty}} \) is also a Galois pro-Kummer étale covering with Galois group \( \Gamma_{\text{geom}} \). We denote the associated affinoid perfectoid space by \( \tilde{Y}_K = \text{Spa}(\hat{S}_{K,\infty}, \hat{S}_{K,\infty}^\prokét) \).

The following lemmas are proved exactly as [LZ17, Corollary 2.4 and Lemma 2.7] were, by using Corollary 3.36 and Proposition 2.82.

**Lemma 5.28.** Let \( L \) be a Kummer étale \( \mathbb{Q}_p \)-local system on \( X_{\text{két}} \) and \( U \) be log affinoid perfectoid in \( X_{\text{prokét}}/\tilde{X}_K \). For any \(-\infty \leq a \leq b \leq \infty\), and for each \( i > 0 \), we have
\[
H^i(X_{\text{prokét}}/U, \hat{L} \otimes_{\hat{\mathbb{Q}}_p} \mathcal{O}_C[a,b]_{\text{dR,log}}) = 0.
\]

**Lemma 5.29.** For each \( i \geq 0 \), there is a canonical isomorphism
\[
H^i(\Gamma_{\text{geom}}, (\hat{L} \otimes_{\hat{\mathbb{Q}}_p} \mathcal{O}_C_{\text{log}})(\tilde{X})) \cong H^i(X_{\text{prokét}}/Y_K, \hat{L} \otimes_{\hat{\mathbb{Q}}_p} \mathcal{O}_C_{\text{log}}).
\]

For each integer \( m \geq 1 \), we write
\[
X_{K,m} = \text{Spa}(R_{K,m}, R_{K,m}^+) := X_K \times_{\text{Spa}(D_m^n)} (D_m^n)_{\text{k}}.
\]

Then \( \hat{R}_{K,\infty} \) is the \( p \)-adic completion of \( \lim_{\leftarrow} R_{K,m} \). Let \( \Gamma_0 = \Gamma_{\text{geom}} = \hat{\mathbb{Z}}^n \) and \( \Gamma_m = (1 + m\hat{\mathbb{Z}})^n \). By Theorem 4.17 \( \{R_{K,m}\}_{m \geq 0}, \Gamma_m \) is a stable completion system. By Definition 4.1 this implies that \( L_{\infty} = \mathcal{L}(\tilde{X}) \) has a model over \( R_{K,m_0} \) for some \( m_0 \); i.e., a finite projective \( R_{K,m_0} \)-module \( L_{m_0}(X_K) \) of rank \( \text{rk}_{\mathbb{Q}_p}(L) \) with a \( R_{K,m_0} \)-semilinear continuous action of \( \Gamma_{\text{geom}} \) such that
\[
L_{m_0}(X_K) \otimes_{R_{K,m_0}} R_{K,\infty} \cong \mathcal{L}(\tilde{X})
\]
and
\[
H^i(\Gamma_{\text{geom}}, L_{m_0}(X_K)) \to H^i(\Gamma_{\text{geom}}, \mathcal{L}(\tilde{X}))
\]
is an isomorphism for all \( i \geq 0 \).

**Lemma 5.30.** The \( R_K \)-linear representation of \( \Gamma_0 \) on \( L_{m_0}(X_K) \) is quasi-unipotent; i.e., there is a finite index subgroup \( \Gamma'_{\text{geom}} \) of \( \Gamma_{\text{geom}} \) acting unipotently on \( L_{m_0}(X_K) \).

**Proof.** Let \( E_m = R \otimes_{\mathbb{Z}_k} k_m(T_{m_0}^{\mathbb{Z}_m}, \ldots, T_m^{\mathbb{Z}_m}) \) and \( \Gamma_m = \text{Gal}(K/k_m) \), for each \( m \geq m_0 \). By Theorem 4.19 \( \{E_m\}_{m \geq m_0}, \Gamma_m \) is a stable completion system with \( R_{K,m_0} \) equal to the completion of \( \lim_{\leftarrow} E_m \). By Definition 4.1 (with \( L_{\infty} = L_{m_0}(Y_K) \)), there exists (by increasing \( m_0 \) if necessary) an \( E_{m_0} \)-submodule \( L_{k_{m_0}} \) equipped with a continuous \( \Gamma \)-action such that
\[
L_{k_{m_0}} \otimes_{E_{m_0}} R_{K,m_0} \to L_{m_0}(X_K)
\]
is an isomorphism. The rest of the proof is essentially the same as in the proof of [LZ17, Lemma 2.15] (which is a variant of Grothendieck’s proof of the ℓ-adic local monodromy theorem). We reproduce it here since the same idea will be used once again (see the proof of Lemma 5.45 below).

Note that \( \text{Gal}(k_∞/k) \) acts on \( \Gamma_{\text{geom}} \cong (\hat{\mathbb{Z}}(1))^n \) via the cyclotomic character \( \chi : \text{Gal}(k_∞/k) \to \hat{\mathbb{Z}}^\times \). For all \( \gamma \in \Gamma_{\text{geom}} \) sufficiently close to 1, and for all \( \delta \in \Gamma \), we have the identity \( \delta(\log \gamma)^{-1} = (\log \delta)(\log \gamma) \). Since \( \delta \) is a finite extension of \( \mathbb{Q}_p \) by assumption, the image of \( \text{Gal}(k_∞/k) \) under \( \chi \) cannot be a finite subgroup of \( \hat{\mathbb{Z}}^\times \). This forces the coefficients of the characteristic polynomial of \( \log \gamma \) to vanish below the top degree. In other words, \( \log \gamma \) is nilpotent. Hence, the lemma follows. \( \square \)

By Lemma 5.30 we obtain a decomposition

\[
L_{m_0}(X_K) = \bigoplus_{\tau} L_{m_0,\tau}(X_K)
\]

of \( R_K \)-modules, where \( \tau \) are characters of \( \Gamma_{\text{geom}} \) of finite order and

\[
L_{m_0,\tau}(X_K) := \{ x \in L_{m_0}(X_K) \mid (\gamma - \tau(\gamma))^l x = 0 \text{ for } l \gg 0 \}.
\]

Each \( L_{m_0,\tau}(X_K) \) is a finite projective \( S_K \)-module stable under the action of \( \Gamma_{\text{geom}} \). Up to replacing \( L_{m_0}(X_K) \) with \( L_m(X_K) = L_{m_0}(X_K) \otimes_{R_K,m_0} R_{K,m} \) for some sufficiently large \( m \geq m_0 \), and replacing \( m_0 \) with \( m \) accordingly, we may assume that the order of every \( \tau \) divides \( m_0 \). Therefore, for every \( \tau \), there exists a monomial \( T_1^{i_1} \cdots T_n^{i_n} \) in \( R_{K,m_0} \) on which \( \Gamma_0 \) acts via \( \tau \). Write

\[
L(X_K) := L_{m_0,1}(X_K).
\]

As mentioned before, \( L(X_K) \) is a finite projective \( R_K \)-module. In addition, it is straightforward to check that

\[
L(X_K) \otimes_{R_K[K_1^{-1},\ldots,K_n^{-1}]} R_{K,m_0}[T_1^{-1},\ldots,T_n^{-1}] = L_{m_0,1}(X_K)[T_1^{-1},\ldots,T_n^{-1}].
\]

Therefore, the rank of \( L(X_K) \) as a projective \( R_K \)-module is \( \text{rk}_{\mathbb{Q}_p}(L) \).

**Remark 5.33.** However, \( (5.31) \) might not be an isomorphism in general. This is the source of the failure of the surjectivity of (1.5) mentioned in the introduction.

By the property of decomposition systems (in Definition 4.11), we see that the formation of \( L_{m_0}(X_K) \) is compatible with base changes under compositions of rational embeddings and finite étale morphisms \( Y \to X \) (by increasing \( m_0 \) if necessary). The same is true for the formation of the summands \( L_{m_0,\tau}(X_K) \) in the decomposition (5.32). This yields the following:

**Lemma 5.34.** The formation of \( L(X_K) \) is compatible with base changes under compositions of rational embeddings and finite étale morphisms \( Y \to X \).

As in Section 3.2 for \( 1 \leq j \leq n \), let \( \gamma_j \) be the \( j \)-th generator of \( \Gamma_{\text{geom}} \cong (\hat{\mathbb{Z}}(1))^n \). For each \( \tau \neq 1 \), there exists some \( j \) such that the action of \( \gamma_j - 1 \) on \( L_{m_0,\tau}(Y_K) \) is invertible, and so \( H^i(\Gamma_0, L_{m_0,\tau}(Y_K)) = 0 \), for all \( i \geq 0 \). Hence, the natural map

\[
H^i(\Gamma_0, L(Y_K)) \to H^i(\Gamma_0, L_{m_0}(Y_K))
\]

is an isomorphism. With this isomorphism as an input, the following lemma is proved by essentially the same argument as in the proof of [LZ17, Lemma 2.9]:
Lemma 5.35. There is a canonical $\text{Gal}(K/k)$-equivariant isomorphism

$$H^i(X_{\text{prok}^\text{ét}}/X_K, L \otimes \hat{\mathcal{O}}_p \mathcal{O}_{C_{\log}}) \cong \begin{cases} L(X_K), & \text{if } i = 0; \\ 0, & \text{if } i > 0. \end{cases}$$

Since the formation of $L(X_K)$ is compatible with base changes under compositions of rational embeddings and finite étale morphisms $Y \to X$ (by Lemma 5.34), it defines a vector bundle on $X_{\text{ét}}$. This completes the proof of Proposition 5.26 and hence also the proofs of Theorems 5.16(1) and 5.17(1).

Next, we move to the arithmetic situation.

Lemma 5.36. The sheaf $D_{\text{dR,log}}(L)$ is a coherent sheaf on $X_{\text{an}}$.

Proof. Fix $K = \hat{k}_\infty$. Again, to show the coherence of $D_{\text{dR,log}}(\mathbb{L})$, instead of considering the projection of $X_{\text{prok}^\text{ét}}$ to $X_{\text{an}}$, we consider the projection to $X_{\text{ét}}$ instead, and we may assume that $X = \text{Spa}(R, R^+)$ admits a smooth toric chart. Note that this modified $D_{\text{dR,log}}(\mathbb{L})$ is the sheaf associated with the presheaf

$$Y \in X_{\text{ét}} \mapsto H^0(X_{\text{prok}^\text{ét}}/Y, \hat{\mathcal{L}} \otimes \hat{\mathcal{O}}_p \mathcal{O}_{\text{B}_{\text{dR,log}}}) = H^0(\text{Gal}(K/k), \mathcal{R} \mathcal{H}_{\text{log}}(\mathbb{L})(Y)).$$

From the proof of Theorem 5.10[1], we know that

$$\text{gr}^r \mathcal{R} \mathcal{H}_{\text{log}}(L) \cong \mu'_e(\hat{\mathcal{L}} \otimes \hat{\mathcal{O}}_p \mathcal{O}_{C_{\log}})(r).$$

It suffices to prove the following two statements:

1. The $R$-module $H^0(\text{Gal}(K/k), \mu'_e(\hat{\mathcal{L}} \otimes \hat{\mathcal{O}}_p \mathcal{O}_{C_{\log}})(r)(X))$ is finitely generated.
2. If $Y = \text{Spa}(S, S^+) \to X = \text{Spa}(R, R^+)$ is a composition of rational localizations and finite étale morphisms, then

$$H^0(\text{Gal}(K/k), \mu'_e(\hat{\mathcal{L}} \otimes \hat{\mathcal{O}}_p \mathcal{O}_{C_{\log}})(r)(Y)) \cong H^0(\text{Gal}(K/k), \mu'_e(\hat{\mathcal{L}} \otimes \hat{\mathcal{O}}_p \mathcal{O}_{C_{\log}})(r)(X)) \otimes R S.$$

For [1], recall that the $R_{K,\infty}$-module $L(X_K)$ descends to some $R_{k_{\text{an}}}$-module $L_{k_{\text{an}}}(X)$ such that

$$H^i(\text{Gal}(K/k), L_{k_{\text{an}}}(X)) \to H^i(\text{Gal}(K/k), L(X_K))$$

is an isomorphism, for all $i \geq 0$. Hence, by Lemma 5.35

$$H^0(\text{Gal}(K/k), \nu_e(\hat{\mathcal{L}} \otimes \hat{\mathcal{O}}_p \mathcal{O}_{C_{\log}})(r)(X)) \cong H^0(\text{Gal}(K/k), L_{k_{\text{an}}}(X)(r)),$$

which is clearly a finitely generated $R$-module, and vanishes when $|r| \gg 0$.

For [2], note that the group cohomology $H^i(\text{Gal}(K/k), L(X_K)(r))$ can be computed by the complex

$$L(X_K) \xrightarrow{\gamma \circ \mu} L(X_K),$$

where $\triangle$ is the torsion subgroup of $\text{Gal}(K/k) = \text{Gal}(k_\infty/k)$ and $\gamma$ is a topological generator of $\text{Gal}(k_\infty/k)/\triangle$. Therefore, the desired base change follows from the base change property of $L(X_K)$ and the fact that étale morphisms are flat.

Lemma 5.37. The coherent sheaf $D_{\text{dR,log}}(L)$ on $X_{\text{an}}$ is reflexive.

Proof. By [LZ17, Theorem 3.9], $D_{\text{dR,log}}(L)$ is locally free over $X - D$. Since $D_{\text{dR,log}}(L)$ is torsion-free, it is hence locally free outside some locus $Z$ of codimension at least two in $X$. Let $j : X - Z \to X$ denote the canonical open immersion. We claim that $\mathcal{R} \mathcal{H}_{\text{log}}(\mathbb{L}) \cong j_* j^*(\mathcal{R} \mathcal{H}_{\text{log}}(L))$. Since $\mathcal{R} \mathcal{H}_{\text{log}}(\mathbb{L})$ is locally
free, we may work locally and assume that it is isomorphic to \((\mathcal{O}_X \otimes B_{\text{dR}})^n\) for some \(n \geq 0\). By using the filtration on \(\mathcal{O}_X \otimes B_{\text{dR}}\) in Definition 3.44, it suffices to treat the case of \(\mathcal{O}_X\), which follows from [Bar76, Satz 10] or [Kis99, Corollary 2.2.4]. By taking \(\text{Gal}(k_{\text{sc}}/k)\)-invariants, we obtain a similar canonical isomorphism \(D_{\text{dR}, \log}(L) \cong j_* j^*(D_{\text{dR}, \log}(\mathbb{L}))\). Since \(j^* D_{\text{dR}, \log}(\mathbb{L})\) is locally free, and since \(D_{\text{dR}, \log}(\mathbb{L})\) is known to be coherent, it follows that \(D_{\text{dR}, \log}(\mathbb{L})\) is reflexive, by the same argument as in the proof of [Ser06, Proposition 7].

5.3. Calculation of residues. The main goal of this subsection is to prove Theorems 5.16, 5.20[1], and 5.24.

Let us first review the definition of residues for log connections. Let \(X\) be smooth and connected over \(k\), and let \(D\) be a normal crossings divisor. As in Section 2.1, \(X\) is equipped with the log structure defined by \(D\). We first suppose that \(F\) is vector bundle on \(X_{\text{an}}\), equipped with an integrable log connection \(\nabla : F \to F \otimes_{\mathcal{O}_X} \Omega^1_{\log}\).

Let \(Z \subset D\) be an irreducible component (i.e., the image of a connected component of the normalization of \(D\), as in [Con99]). To define the residue \(\text{Res}_Z(\nabla)\) of \(\nabla\) along \(Z\), we may shrink \(X\) and assume that \(Z\) is smooth and connected. Étale locally, we may assume that \(Z\) is affinoid and that there is a smooth toric chart \(X \to \mathbb{D}^n\) such that \(Z = \{T_1 = 0\}\). Let \(i : Z \subset X\) denote the closed immersion. Then there is an \(\mathcal{O}_Z\)-linear endomorphism

\[
\text{Res}_Z(\nabla) := \nabla(T_1 \frac{\partial}{\partial T_1} \mod T_1 : i^*(F) \to i^*(F)),
\]

where \(T_1 \frac{\partial}{\partial T_1}\) denotes the dual of \(\frac{dT_1}{T_1}\). As in the classical situation, this operator depends only on \(Z \subset X\), but not on the choice of local coordinates. Moreover, its formulation is compatible with rational localizations and finite étale base changes (and therefore all étale base changes). Thus, by étale descent (using [KL15, Theorem 8.2.22(d)]), \(\text{Res}_Z(\nabla)\) is a well-defined endomorphism of \(i^*(F)\).

Consider \(Z\) as a smooth rigid analytic variety by itself, which is equipped with the normal crossings divisor \(D' = \bigcup_j (D_j \cap Z)\), where the \(D_j\)'s are irreducible components of \(D\) other than \(Z\). Then \(Z\) admits the structure of a log adic space, defined by \(D'\), as in Example 2.12. Again as in the classical situation, the pullback \(i^*(F)\) is equipped with a log connection \(\nabla' : i^*(F) \to i^*(F) \otimes_{\mathcal{O}_Z} \Omega^1_{\log}\), and the residue \(\text{Res}_Z(\nabla)\) is horizontal with respect to \(\nabla'\). It follows that the characteristic polynomial \(P_Z(x)\) of \(\text{Res}_Z(\nabla)\) has coefficients in the algebraic closure \(k_Z\) of \(k\) in \(\Gamma(Z, \mathcal{O}_Z)\), so that the eigenvalues of \(\text{Res}_Z(\nabla)\) (i.e., the roots of \(P_Z(x)\)) are in \(\overline{k}\).

One easily checks (using Lemma 5.4) that, if we start with an integrable log connection \(\mathcal{E}\) on \(X = (X_{\text{an}}, \mathcal{O}_X \otimes B_{\text{dR}})\), then the above discussions carry through, and the coefficients of the characteristic polynomial \(P_Z(x)\) are in \(B_{\text{dR}}\), up to enlarging the perfectoid field \(K\) used in the definition of \(B_{\text{dR}} = \mathcal{B}_{\text{dR}}(K, \mathcal{O}_K)\) (see (3.46)).

Suppose \(F\) is a torsion-free coherent \(\mathcal{O}_X\)-module equipped with an integrable log connection \(\nabla\). We can still attach to \((\mathcal{F}, \nabla)\) and every irreducible component \(Z \subset D\) a polynomial \(P_Z(x) \in k_Z[x]\). Concretely, let \(U\) be the maximal open subset of \(X\) such that \(\mathcal{F}|_U\) is a vector bundle. Then \(U = X - X_0\), where \(X_0 \subset X\) is an analytic closed subvariety of codimension at least two. In particular, \(X_0\) cannot contain any irreducible component of \(D\). To define the residue, we can replace \(X\) with \(U\) and assume that \(\mathcal{F}\) is a vector bundle, and proceed as above.

Now we start the proof of Theorem 5.16[2]. Since the calculation of residues is a local question, we may assume that \(X = \text{Spa}(R, R^+)\) is affinoid and admits a
smooth toric chart \(X \to \mathbb{D}^n\), and that \(\mathcal{R}_\log(L)\) is free. By Proposition 3.23 and Corollary 3.36
\[
\mathcal{R}_\log(L)(X) = \mu_*^\gamma \left( \mathcal{L} \otimes_{\mathbb{Q}_p} \mathcal{O}_B \right)(X) \\
\cong H^0 \left( \Gamma_{\text{geom}} \left( \mathcal{L} \otimes_{\mathbb{Q}_p} B_{\text{dR}} \right) \right)(X) \{W_1, \ldots, W_n\},
\]
where each \(W_j = t^{-1} X_j\) is defined as in [3.37]. Write \(N_{\infty} = \left( \mathcal{L} \otimes_{\mathbb{Q}_p} B_{\text{dR}} \right)(X)\), which is a module over \(B_{\infty}^+ (R_{\infty}, R_{\infty}^+)\) (see Proposition 3.5). Then we have
\[
\text{Fil}^\gamma \left( \mathcal{L} \otimes_{\mathbb{Q}_p} \mathcal{O}_B \right)(X) \cong N_{\infty} \{W_1, \ldots, W_n\}
\]
where \(W^\Lambda = W_{\Lambda_1}^1 \cdots W_{\Lambda_n}^n\) as in (3.38), and where \(\left( \begin{array}{c} W \\ \Lambda \end{array} \right) := \left( \begin{array}{c} W_1 \\ \vdots \\ W_n \end{array} \right)\), for each \(\Lambda = (\Lambda_1, \ldots, \Lambda_n) \in \mathbb{Z}_0^n\).

For simplicity, we shall further introduce the following notation. Recall that we have chosen in Section 3.2 a topological basis \(\gamma_1, \ldots, \gamma_n\) for \(\Gamma_{\text{geom}} \cong \left( \mathbb{Z}(1) \right)^n\). For each \(\Lambda = (\Lambda_1, \ldots, \Lambda_n)\), we write \((\gamma - 1)^\Lambda\) for \((\gamma - 1)^{\Lambda_1} \cdots (\gamma - 1)^{\Lambda_n}\). Then, \(\gamma_{\infty}\) denote \((\gamma - 1)^{\Lambda_0}\) for all \(\Lambda = (\Lambda_1, \ldots, \Lambda_n) \in \mathbb{Z}_0^n\).

**Lemma 5.39.**

1. If \(\sum c_{\Lambda} \left( \begin{array}{c} W \\ \Lambda \end{array} \right) \in N_{\infty} \{W_1, \ldots, W_n\}\) is \(\Gamma_{\text{geom}}\)-invariant, then \((\gamma - 1)^\Lambda c_0 \to 0\), \(t\)-adically, as \(\Lambda \to \infty\), and \(c_\Lambda = (\gamma - 1)^\Lambda c_0\) for all \(\Lambda \in \mathbb{Z}_0^n\).

2. Let \(N^+ = \{c \in N_{\infty} \mid (\gamma - 1)^\Lambda c \to 0\}, \text{ } t\text{-adically, as } \Lambda \to \infty\}\). Then the map
\[
N_{\infty} \{W_1, \ldots, W_n\} \to N_{\infty}: \text{ } W_i \mapsto 0, \text{ } 1 \leq i \leq n
\]
induces a canonical isomorphism
\[
\eta : \mathcal{R}_\log(L)(X) = (N_{\infty} \{W_1, \ldots, W_n\})^{\Gamma_{\text{geom}}} \cong N^+
\]
with the inverse map given by
\[
\eta^{-1} : N^+ \to (N_{\infty} \{W_1, \ldots, W_n\})^{\Gamma_{\text{geom}}}
\]
\[
c \mapsto \sum_{\Lambda \in \mathbb{Z}_0^n} (\gamma - 1)^\Lambda c \left( \begin{array}{c} W \\ \Lambda \end{array} \right)
\]

3. Let \(N = N^+ \otimes_{B_{\infty}^+} B_{\text{dR}} \cong N^+[t^{-1}]\). Then the above isomorphism \(\eta\) induces a canonical isomorphism \(\mathcal{R}_\log(L)(X) \cong N\), which we still denote by \(\eta\).

**Proof.** We have \(\gamma_{\infty}^{-1} W_j = W_j + \delta_{ij}\). (Note that the \(W_j\) defined in Corollary 3.36 differs from the \(V_j\) defined in the proof of [Sch13 Proposition 6.16] by a sign, and therefore requires the \(\gamma_{\infty}^{-1}\) in our formula rather than the \(\gamma_{\infty}\) as in [Sch13 Lemma 6.17].) This implies that
\[
\gamma_{\infty}^{-1} \left( \begin{array}{c} W_j \\ i \end{array} \right) = \left( \begin{array}{c} W_j + 1 \\ i \end{array} \right) = \left( \begin{array}{c} W_j \\ i \end{array} \right) + \left( \begin{array}{c} W_i \\ j - 1 \end{array} \right).
\]
A straightforward calculation yields

$$(\gamma_i^{-1} - 1) \left( \sum_{\Lambda} c_\Lambda \binom{W}{\Lambda} \right) = \sum_{\Lambda} \left( \gamma_i^{-1} c_{\Lambda + e_i} + \gamma_i^{-1} c_\Lambda - c_\Lambda \right) \binom{W}{\Lambda},$$

where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ has only the $i$-th entry equal to 1. Therefore,

$$c_\Lambda - \gamma_i^{-1} c_\Lambda = \gamma_i^{-1} c_{\Lambda + e_i},$$
or, equivalently,

$$\gamma_i c_\Lambda - c_\Lambda = c_{\Lambda + e_i},$$

for all $i$ and $\Lambda$. In particular, this implies that $(\gamma - 1)^A c_0 = c_\Lambda$, which goes to 0 as $A \to \infty$. This proves (1). Then (2) and (3) also follow. \hfill \Box

By the proof of Theorem 5.17 in Section 5.2, $\mathcal{R}H_{\log}^\gamma(L)(X)$ is a finite projective $R \otimes_k B_{dR}^+$-$\text{-module}$. On the other hand, by Lemma 3.19 (see also (3.30)), we have constructed a natural $\Gamma_{\text{geom}}$-equivariant embedding $R \otimes_k B_{dR}^+ \to \mathcal{R}_dR(\hat{\mathbb{R}}, \hat{\mathbb{R}}^+)$ sending $T_i$ to $[T_i^+]$. Via this embedding, we may regard $N_\infty$ as an $R \otimes_k B_{dR}^+$-$\text{-module}$, and $N^+$ as an $R \otimes_k B_{dR}^+$-$\text{-module}$-submodule of $N_\infty$.

**Lemma 5.41.** The isomorphism (5.40) is an isomorphism of $R \otimes_k B_{dR}^+$-$\text{-modules}$.

**Proof.** This is an immediate consequence of the fact that the map (3.27) is obtained from the map (3.27) by setting $x_1 = \cdots = x_n = 0$. \hfill \Box

Note that the natural action of $\Gamma_{\text{geom}}$ on $N_\infty$ preserves $N^+$, and by transport of structure gives an action of $\Gamma_{\text{geom}}$ on $\mathcal{R}H_{\log}(L)(X)$. This action is closely related to the residues, as we shall see. First recall from Lemma 3.19 that, if we define the $\Gamma_{\text{geom}}$ action of $R \otimes_k B_{dR}$ by requiring that $\gamma_i(T_j) = [e]^{b_j, T_j}$ and that the action becomes trivial after reduction mod $\xi$, then the embedding $R \otimes_k B_{dR} \to \mathcal{R}_dR(\hat{\mathbb{R}}, \hat{\mathbb{R}}^+)$ is $\Gamma_{\text{geom}}$-equivariant. Hence, the action of $\gamma_i$ on $N/T_i N$ is $(A/T_i) \otimes_k B_{dR}$-linear.

**Lemma 5.42.** Under the above isomorphism $\mathcal{R}H_{\log}(L)(X) \cong N$, the residue of the connection

$$\nabla_L : \mathcal{R}H_{\log}(L)(X) \to \mathcal{R}H_{\log}(L)(X) \otimes_{\mathcal{O}_X} \Omega^\log_{X/B_{dR}}$$

along $Z_i = \{ T_i = 0 \}$ is given by

$$t^{-1} \log(\gamma_i) : N/T_i N \to N/T_i N.$$

Note that the definitions of both $t$ and $\gamma_i$ depend on the choice of the compatible system of roots of unit $\{ \xi_n \}_{n \geq 1}$, but $t^{-1} \log(\gamma_i)$ does not.

**Proof of Lemma 5.42.** For an explicit calculation, we need to expand elements of $N_\infty \{ W_1, \ldots, W_n \}$ in the basis $W^{A}$-s, instead of $\binom{W}{A}$-s. If $c_0 \in N$ and

$$\eta^{-1}(c_0) = \sum_{\Lambda} c_\Lambda \binom{W}{\Lambda} = \sum_{\Lambda} b_\Lambda W^{A},$$
then by the definition of residues as in (3.38), by Lemma 5.39 and by (3.41) and (3.42), we obtain
\[
\eta((\text{Res}_{\mathcal{Z}_i}(\nabla)))(\eta^{-1}(c_0))) = t^{-1}b_{e_i} = t^{-1}\sum_{a=1}^{\infty}(-1)^{a-1}\frac{1}{a}c_{ae_i}
\]
\[
= t^{-1}\sum_{a=1}^{\infty}(-1)^{a-1}\frac{1}{a}(\gamma_i - 1)^a c_0
\]
\[
= t^{-1}\log(\gamma_i)(c_0),
\]
as desired. □

To proceed further, we need a technical lemma. We consider the situation as in the paragraph preceding Corollary 3.21. Then we have the closed immersion of affinoid perfectoid spaces \( \imath := \imath_*: \widehat{\mathcal{Z}_i} \to \widehat{X} \), which induces a morphism of topoi
\[
(\widehat{\mathcal{Z}_i})_{\proet} \to (\widehat{X})_{\proet} \cong X_{\prok\et/\widehat{X}},
\]
where the last isomorphism follows from Proposition 2.80. We define the sheaf \( B^+_{\text{dR}, \widehat{X}} \) (resp. \( B^+_{\text{dR}, \widehat{\mathcal{Z}_i}} \)) on \( (\widehat{X})_{\proet} \) (resp. \( (\widehat{\mathcal{Z}_i})_{\proet} \)) as in Definition 3.4. Then the same argument as in the proof of [Sch13, Theorem 6.5] implies that, for every pro-étale \( U = \text{Spa}(S, S^+) \to \widehat{X} \) with \( U \) affinoid perfectoid, \( B^+_{\text{dR}, \widehat{X}}(U) \) is the \( \xi \)-completion of \( W(S^+)[\frac{1}{p}] \). Note that \( B^+_{\text{dR}, \widehat{X}} \cong B^+_{\text{dR}}|_{\widehat{X}} \) via the above isomorphism of topoi.

**Lemma 5.43.** For each \( r \geq 1 \), the functor
\[
\mathcal{L} \mapsto L := \mathcal{L}(\widehat{X})
\]
is an equivalence from the category of locally free \( B^+_{\text{dR}, \widehat{X}}/\xi^r \)-modules of finite rank on \( (\widehat{X})_{\proet} \) to the category of finite projective \( B^+_{\text{dR}, \widehat{X}}(\widehat{X})/\xi^r \)-modules of finite rank.

**Proof.** The argument of the proof of [KL15, Theorem 9.2.15] also applies to the setting here. The quasi-inverse functor is given by
\[
L \mapsto \mathcal{L}(U) := L \otimes_{B^+_{\text{dR}, \widehat{X}}, \xi^r} (B^+_{\text{dR}, \widehat{X}}(U)/\xi^r),
\]
for each \( U \to \widehat{X} \) pro-étale and affinoid perfectoid. □

Now we regard \( \widehat{\mathcal{L}}|_{\widehat{X}} \) as a \( \mathbb{Z}_p \)-local system on \( (\widehat{X})_{\proet} \), still denoted by \( \widehat{\mathcal{L}} \) for simplicity. Then \( \imath^{-1}(\widehat{L}) \) is a local system on \( (\widehat{\mathcal{Z}_i})_{\proet} \).

**Lemma 5.44.** The canonical morphism \( \widehat{\mathcal{L}} \otimes_{\mathbb{Q}_p} \imath_*(\mathbb{B}_{\text{dR}, \widehat{\mathcal{Z}_i}}) \to \imath_*(\imath^{-1}(\widehat{L}) \otimes_{\mathbb{Q}_p} \mathbb{B}_{\text{dR}, \widehat{\mathcal{Z}_i}}) \) induces an isomorphism
\[
N_\infty/(\xi^r, ([T_i^p])_{s \in \mathbb{Q}_p}) \cong (\widehat{\mathcal{L}} \otimes_{\mathbb{Q}_p} (\mathbb{B}^+_{\text{dR}, \widehat{X}}/\xi^r))(\widehat{X}) \otimes (\mathbb{B}^+_{\text{dR}, \widehat{X}}/\xi^r)(\widehat{\mathcal{Z}_i})
\]
\[
\cong (\imath^{-1}(\widehat{L}) \otimes_{\mathbb{Q}_p} \mathbb{B}_{\text{dR}, \widehat{\mathcal{Z}_i}})(\widehat{\mathcal{Z}_i}).
\]

**Proof.** The first isomorphism follows from Corollary 3.21 and the second isomorphism follows from Lemma 5.43. □
Lemma 5.45. If \( \gamma_i v = xv \) for some nonzero \( v \in N_\infty/(\xi^r, (T_i^{\psi})^\wedge_{\mathcal{E}Q^{>0}}) \) and \( x \in B^+_{\text{dR}}/\xi^r \), then \( x = \zeta^y \) for some \( y \in \mathbb{Q} \).

Proof. Note that \( \gamma_i \) acts trivially on \( \hat{\mathbb{Z}}_i \), and \( r^{-1}(\hat{\mathbb{L}}) \) is equipped with an action of \( \gamma_i \). Now the lemma follows from Lemma 5.44 and the fact that the action of \( \gamma_i \) on any stalk of \( L \mid \hat{\mathbb{Z}}_i \) (over a geometric point) is quasi-unipotent. This is because the action of \( \gamma_i \) extends to a continuous action of \( \hat{\mathbb{Z}}(1) \times \text{Gal}(k_{\infty}/k) \) (see (5.27)), and because \( \text{Gal}(k_{\infty}/k) \) acts on \( \hat{\mathbb{Z}}(1) \) via the cyclotomic character \( \chi: \text{Gal}(k_{\infty}/k) \to \hat{\mathbb{Z}}^\times \), so that the same argument as in the proof of Lemma 5.30 applies here.

To prove Theorem 5.16(2), we need to use the decompletion of \( B^+_{\text{dR}}/\xi^r \) established in Section 4.2.3 to descend \( N_\infty/\xi^r = (\hat{\mathbb{L}} \otimes_{\mathbb{Q}} B^+_{\text{dR}}(\hat{\mathbb{X}}))/\xi^r \) to some finite level.

Lemma 5.46. For each \( r \geq 1 \), there is some \( m \geq 3 \) sufficiently large, and a finite projective \( B_{r,m} \)-module \( N_{r,m} \), equipped with a semilinear \( \Gamma \)-action, such that

\[
N_\infty/\xi^r \cong N_{r,m} \otimes_{B_{r,m}} (B^+_{\text{dR}}(\hat{\mathbb{X}}))/\xi^r).
\]

In addition, up to replacing \( N_{r,m} \) with \( N_{r,m} \otimes_{B_{r,m}} B_{r,m'} \) for some sufficiently large multiple \( m' \) of \( m \), we may assume that \( N^+/\xi^r \subset N_{r,m} \).

Proof. The first statement follows from Theorem 4.22. The second statement follows by noting that \( \xi^r N_\infty/\xi^{r+1} N_\infty \cong \mathcal{L}(\hat{\mathbb{X}})(r) \), where \( \mathcal{L} = \hat{\mathbb{L}} \otimes_{\mathbb{Q}} \hat{\mathcal{O}}_{X, m_{\text{max}}} \), as before, and by induction on \( r \), starting from the case \( r = 0 \) treated in Lemma 5.30 and the discussions following it.

Lemma 5.47. If \( \gamma_i v = xv \) for some nonzero \( v \in N^+/\xi^r, T_i \) and \( x \in B^+_{\text{dR}}/\xi^r \), then \( x = \zeta^y[\epsilon^z] \) for some \( y \in \mathbb{Q} \) and \( z \in \mathbb{Q} \cap [0, 1) \).

Proof. By Lemma 5.46, we may assume that \( v \in N^+/\xi^r, T_i \subset N_{r,m}/T_i \) for some \( m \). Note that \( N_{r,m}/T_i^{\mathbb{P}} \subset N_\infty/\xi^r, (T_i^{\psi})^\wedge_{\mathcal{E}Q^{>0}} \). Therefore, by Lemma 5.45, we have either \( x = \zeta^y \) for some \( y \in \mathbb{Q} \), or \( v = T_i^{\mathbb{P}} v' \) for some \( v' \in N_{r,m}/T_i^{\mathbb{P}} \). In the latter case, since \( \gamma_i (T_i^{\mathbb{P}}) = [\epsilon^z] T_i^{\mathbb{P}} \), we have \( T_i^{\mathbb{P}} (\gamma_i v' - [\epsilon^z] xv') = 0 \), so that \( \gamma_i v' = [\epsilon^z] xv' + T_i^{\mathbb{P}}(1 - \frac{1}{\mathbb{P}})w' \), for some \( w' \in N_{r,m}/T_i^{\mathbb{P}} \). Further modulo \( T_i^{\mathbb{P}} \), we see that either \( x = \zeta^y[\epsilon^z] \) for some \( y' \), or \( v' = T_i^{\mathbb{P}} v'' \) for some \( v'' \). The desired statement then follows by repeating this argument (finitely many times).

Finally, we finish the proof of Theorem 5.16(2). Let \( v \in N/T_i N \) be an eigenvector of \( t^{-1} \log(\gamma_i) \) with eigenvalue \( \tilde{\zeta} \in \mathbb{B}_{\text{dR}} \). By multiplying \( v \) by some power of \( t \), we may assume that \( v \in N^+/T_i N^+ \). By Lemma 5.47 and by the assumption that \( (\gamma_i - 1)^t v \to 0 \), \( t \)-adically, as \( l \to \infty \), it is easy to see that

\[
\exp(\tilde{\zeta}) = \zeta^y[\epsilon^z]
\]

with \( y + z \in \mathbb{Z} \) and \( z \in \mathbb{Q} \cap [0, 1) \). Therefore, we may assume that \( y = -z \in \mathbb{Q} \cap [0, 1) \). By Lemma 5.42, the eigenvalues of the residue along \( \{T_i = 0\} \) are of the form

\[
t^{-1} \log(\zeta^{-z}[\epsilon^z]) = z \in \mathbb{Q} \cap [0, 1),
\]

which verifies Theorem 5.16(2), as desired.

Proposition 5.48. For the connection \( \nabla_L : D_{\text{dR}, \log}(\mathbb{L}) \to D_{\text{dR}, \log}(\mathbb{L}) \otimes_{\mathcal{O}_X} \Omega^\log_X \), all eigenvalues of \( \text{Res}(T_i = 0)(\nabla_L) \) are in \( \mathbb{Q} \cap [0, 1) \).
Proof. Note that $D_{\text{dR, log}}(L)(X) \cong \mathcal{R}H_{\text{log}}(L)(\bar{X})^{\text{Gal}(k_{\infty}/k)}$, and the isomorphism $\eta$ in (5.40) is $\text{Gal}(k_{\infty}/k)$-equivariant. Therefore, $D_{\text{dR, log}}(L)(X) \cong N^{\text{Gal}(k_{\infty}/k)}$. In addition, the residue $\text{Res}_{\{T_i = 0\}}(\nabla_L)$ is still given by $t^{-1}\log(\gamma_i)$ as in Lemma 5.42. Then the proposition follows from the above arguments. □

To complete the proof of Theorem 5.20(1), it remains to apply the following proposition to conclude that $D_{\text{dR, log}}(L)$ is indeed a vector bundle.

Proposition 5.49. Let $(F, \nabla)$ be a torsion-free coherent $\mathcal{O}_X$-module equipped with a log connection (with logarithmic singularities along $D$). Suppose the following:

1. $F$ is reflexive (i.e., isomorphic to its bidual).
2. For every $i$, all eigenvalues of $\text{Res}_{\{T_i = 0\}}(\nabla)$ are in $\mathbb{Q} \cap [0, 1)$.

Then $F$ is locally free.

Proof. This follows from the same argument as in the proof of [AB01 Chapter 1, Proposition 4.5]. More precisely, it suffices to note that the completion of the stalk of $\mathcal{E}$ at each classical point of $X$ is free, by [AB01 Chapter 1, Lemma 4.6.1]. □

By Proposition 5.49, to show that $D_{\text{dR, log}}(L)$ is indeed a vector bundle, it suffices to note that the condition 1 is satisfied by Lemma 5.42 and the condition 2 is satisfied by Proposition 5.48. The proof of Theorem 5.20(1) is now complete.

Proposition 5.50. Suppose $F$ (resp. $F'$) is a locally free (resp. torsion-free) coherent $\mathcal{O}_X$-module, with an integrable log connections $\nabla$ (resp. $\nabla'$) as in Definition 5.6(1), whose residues along the irreducible components of $D$ (see Section 5.3) all have eigenvalues in $\mathbb{Q} \cap [0, 1)$. Then any morphism $(F, \nabla) \to (F', \nabla')$ whose restriction to $X - D$ is an isomorphism is necessarily an isomorphism over the whole $X$. The same is true if we replace $\mathcal{O}_X$-modules with $\mathcal{O}_X \otimes B_{\text{dR}}^*$-modules.

Proof. Let $(F'', \nabla'') := ((F', \nabla')^*)^*$, where $F''$ is the double $\mathcal{O}_X$-dual of $F'$, which is by definition a reflexive coherent sheaf over $X$, and where $\nabla''$ is the induced log integrable connection, whose residues along the irreducible components of $D$ also have eigenvalues in $\mathbb{Q} \cap [0, 1)$. Hence, by Proposition 5.49 $F''$ is locally free. Since the restriction of the given morphism $(F, \nabla) \to (F', \nabla')$ to the dense subspace $X - D$ is an isomorphism, we have injective morphisms $(F, \nabla) \to (F', \nabla') \to (F'', \nabla'')$, and it suffices to show that their composition is an isomorphism over $X$. Therefore, we can rename $F''$ by $F'$, and assume that both $F$ and $F'$ are locally free.

Since the question of whether a given morphism is an isomorphism is a local question, we may replace $X$ with its affinoid open subspaces which admit strictly étale morphisms to $\mathbb{D}^d$ as in Example 2.12 which are smooth toric charts as in Definition 2.28 and Remark 2.29 and assume that both $F$ and $F'$ are free of rank $d$. Then, with respect to the chosen bases, the map $F \to F'$ is represented by a matrix $A$ in $\text{M}_d(\mathcal{O}_X(X))$, which is invertible outside $D$. To show that it is invertible over $X$, it is enough to show that the entries of $B = A^{-1}$, which a priori are analytic functions on $X$ meromorphic along $D$, are in fact (everywhere regular) analytic functions on $X$. But this is classical—see, for example, the proof of [AB01 Chapter 1, Proposition 4.7]. Moreover, by Lemma 5.4 the above arguments also apply to log connections on $X'$ (in the sense of Definition 5.6(1)). □

As usual, we define a decreasing filtration on $D_{\text{dR, log}}(L)$ by setting

$$\text{Fil}^n D_{\text{dR, log}}(L) := (\text{Fil}^n \mathcal{R}H_{\text{log}}(L))^\text{Gal}(K/k).$$
Lemma 5.51. We endow $D_{\text{dR, log}}(L) \hat{\otimes}_k B_{\text{dR}}$ with the usual product filtration. Then the canonical morphism

$$D_{\text{dR, log}}(L) \hat{\otimes}_k B_{\text{dR}} \rightarrow \mathcal{R}H_{\text{log}}(L)$$

defined by adjunction is injective (by definition) and strictly compatible with the filtrations on both sides. That is, the induced morphism

$$\text{gr}^r(D_{\text{dR, log}}(L) \hat{\otimes}_k B_{\text{dR}}) \rightarrow \text{gr}^r \mathcal{R}H_{\text{log}}(L)$$

is injective, for each $r$.

Proof. Since $D_{\text{dR, log}}(L) \cong (\mathcal{R}H_{\text{log}}(L))^{\text{Gal}(K/k)}$, the left-hand side of (5.53) can be identified with $\oplus_{a+b=r} \left( (\text{gr}^a(\mathcal{R}H_{\text{log}}(L))^{\text{Gal}(K/k)}) \otimes_k K(b) \right)$, while the right-hand side of (5.53) contains $\oplus_{a+b=r} \left( (\text{gr}^a \mathcal{R}H_{\text{log}}(L))^{\text{Gal}(K/k)} \otimes_k K(b) \right)$ as a subspace, where the sums are direct because of the Gal($K/k$)-actions. Thus, it suffices to note that the canonical morphism $\text{gr}^a((\mathcal{R}H_{\text{log}}(L))^{\text{Gal}(K/k)}) \rightarrow (\text{gr}^a \mathcal{R}H_{\text{log}}(L))^{\text{Gal}(K/k)}$ is injective, for each $a$, essentially by definition.

Corollary 5.54. If $\mathbb{L}|_{(X-D)_{\text{et}}}$ is de Rham, then the morphism (5.52) is an isomorphism of vector bundles on $X$, compatible with the log connections and filtrations on both sides.

Proof. By Lemma 5.51, the question is whether (5.52) is an isomorphism compatible with the log connections on both sides. Since $\mathbb{L}|_{(X-D)_{\text{et}}}$ is de Rham, by [LZ17, Corollary 3.12(ii)], the restriction of (5.52) to $X-D$ is an isomorphism. Hence, we can conclude the proof by Proposition 5.50 and Theorems 5.16(2) and 5.20(1).

Corollary 5.55. If $\mathbb{L}|_{(X-D)_{\text{et}}}$ is de Rham, then $\text{gr} D_{\text{dR, log}}(L)$ is a vector bundle of rank $\text{rk}_{\mathbb{Q}_p}(L)$.

Proof. By Corollary 5.54, we have an isomorphism

$$\oplus_a \left( (\text{gr}^a D_{\text{dR, log}}(L)) \hat{\otimes}_k K(-a) \right) \sim \text{gr}^0 \mathcal{R}H_{\text{log}}(L) \cong \mathcal{H}_{\text{log}}(L).$$

Since $\mathcal{H}_{\text{log}}(L)$ is a vector bundle on $X_K$ by Theorem 5.17(1), this shows that $\text{gr} D_{\text{dR, log}}(L)$ is a vector bundle on $X$ of rank equal to that of $\mathcal{H}_{\text{log}}(L)$, which is in turn equal to $\text{rk}_{\mathbb{Q}_p}(L)$ by the proof of Theorem 5.17(1) in Section 5.2.

Finally, we conclude this subsection with the following:

Proof of Theorem 5.24. Given any $\mathbb{Q}_p$-local system $L$ on $X_{\text{ket}}$ such that $\mathbb{L}|_{(X-D)_{\text{et}}}$ has unipotent geometric monodromy along $D$, the action of $\gamma_i$ as in Lemma 5.42 on any stalk of $L|_{\mathbb{G}_i}$ is unipotent. Consequently, $x = 1$ in Lemmas 5.45 and 5.47 and the residues of $\mathcal{R}H_{\text{log}}(L)$ (by Lemma 5.42) are all nilpotent (i.e., have zero eigenvalues). For such $L$, only the trivial character $\tau$ is needed in the decomposition (5.32). Hence, the morphism (5.31) is an isomorphism (cf. Remark 5.33), which shows that the canonical morphisms

$$\mu^* (\mathcal{H}_{\text{log}}(L)) \otimes_{\mathcal{O}_{X_{\text{proket}}} \mathcal{O}_{\text{Clog}}} \hat{\mathbb{L}} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\text{Clog}}$$

and

$$\mu^* (\mathcal{R}H_{\text{log}}(L)) \otimes_{\mathcal{O}_{X_{\text{proket}}} \mathcal{O}_{\text{BdR, log}}} \hat{\mathbb{L}} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\text{BdR, log}}$$
are also isomorphisms (cf. [LZ17, Theorems 2.1(ii) and 3.8(iii)]). Therefore, we can
argue as in the proofs of [LZ17, Theorems 2.1(iv) and 3.8(i)] that \( \mathcal{H}_{\log} \) and \( \mathcal{R}\mathcal{H}_{\log} \)
restrict to natural tensor functors. This proves part (1) of the theorem.

As for part (2), suppose moreover that \( L \mid (X - D)_{\dR} \) is de Rham. Then the residues of
\( D_{\dR,\log}(L) \) are nilpotent by Proposition 5.48 and the morphism (5.52) is an
isomorphism by Corollary 5.54. It follows that the canonical morphism

\[
\mu^*(D_{\dR,\log}(L)) \otimes_{\mathcal{O}_{X,\text{prokét}}} \hat{\mathcal{O}}_{\dR,\log} \to \hat{L} \otimes_{\hat{\mathcal{O}}} \hat{\mathcal{O}}_{\dR,\log}
\]

(cf. (1.5)) is also an isomorphism. Thus, we can conclude as in [LZ17, Theorem 3.9(v)] that \( D_{\dR,\log} \) also restricts to a natural tensor functor.

\[ \square \]

5.4. Compatibility with pullbacks and pushforwards. In this subsection, we
prove Theorems 5.16[3.], 5.17[3.], and 5.20[3.].

Let us begin with the assertions for pullbacks. Let \( \mathcal{V} \) be defined by \( Y \) as in (5.2).
Let \( h : Y \to X \) be as in the statements of the theorems. Let \( E \) be the normal
crossings divisors defining the log structures on \( Y \), as in Example 2.11 and let \( U := X - D \) and \( V := Y - E \).

Remark 5.56. By the constructions of the functors \( \mathcal{R}\mathcal{H}_{\log}, \mathcal{H}_{\log}, \) and \( D_{\dR,\log} \), the
compatibility assertions for pullbacks in Theorems 5.16[3.], 5.17[3.], and 5.20[3.]
are obvious when \( h : Y \to X \) is an open immersion. Moreover, if \( Y \) is an open subspace
of \( X \) over which the log structure is trivial, then the functors \( \mathcal{R}\mathcal{H}_{\log} \) and \( D_{\dR,\log} \)
coincide over \( \mathcal{V} \) and \( Y \), respectively, with their analogues \( \mathcal{R}\mathcal{H} \) and \( D_{\dR} \) in [LZ17]
Theorems 3.8 and 3.9] (modulo the correction in Remark 3.11).

Lemma 5.57. The morphism \( h : Y \to X \) induces a morphism \( h_\mathcal{V} : \mathcal{V} \to \mathcal{U} \). In
this case, we have \( h^{-1}(D) \subset E \) set-theoretically.

Proof. By the definition of the log structures \( \mathcal{M}_X \) and \( \mathcal{M}_Y \) of \( X \) and \( Y \), respectively,
as in Example 2.11, the map \( h^\sharp : h^{-1}(\mathcal{M}_X) \to \mathcal{M}_Y \) between log structures
is defined only when \( h^{-1}(D) \subset E \) set-theoretically. Hence, the lemma follows. \[ \square \]

Lemma 5.58. The canonical morphism

\[
h^*(\mathcal{H}_{\log}(\mathcal{L})) \to \mathcal{H}_{\log}(h^*(\mathcal{L})),
\]

defined by adjunction is injective.

Proof. Since \( V_K \) is dense in \( Y_K \), and since \( h^*(\mathcal{H}_{\log}(\mathcal{L})) \) is a vector bundle on \( Y_K \)
by Theorem 5.17[1.], the morphism (5.59) is injective because the corresponding
morphisms for \( h_\mathcal{V} : \mathcal{V} \to \mathcal{U} \) is an isomorphism by [LZ17 Theorem 2.1(iii)]. \[ \square \]

Lemma 5.60. The morphisms

\[
h^*(\mathcal{R}\mathcal{H}_{\log}(\mathcal{L})) \to \mathcal{R}\mathcal{H}_{\log}(h^*(\mathcal{L})),
\]

and

\[
h^*(D_{\dR,\log}(\mathcal{L})) \to D_{\dR,\log}(h^*(\mathcal{L}))
\]

are injective and strictly compatible with the filtrations on their both sides. That is,
the induced morphisms

\[
h^*(\operatorname{gr}\mathcal{R}\mathcal{H}_{\log}(\mathcal{L})) \to \operatorname{gr}\mathcal{R}\mathcal{H}_{\log}(h^*(\mathcal{L}))
\]

and

\[
h^*(\operatorname{gr}D_{\dR,\log}(\mathcal{L})) \to \operatorname{gr}D_{\dR,\log}(h^*(\mathcal{L}))
\]
are injective.

Proof. Since $\text{gr}^r \mathcal{R}H_{\log}(L) \cong \mathcal{H}_{\log}(L)(r)$ and $\text{gr}^r \mathcal{R}H_{\log}(h^*(L)) \cong \mathcal{H}_{\log}(h^*(L))(r)$, for all $r$, the injectivity of (5.63) follows from that of (5.59). Then the injectivity of (5.64) follows from that of (5.63) and (5.53), by Lemma 5.51 (and its proof). □

Corollary 5.65. The canonical morphisms (5.61) and (5.62) are isomorphisms compatible with the log connections and filtrations on both sides. Consequently, the canonical morphism (5.59), which can be identified with the 0-th graded piece of (5.61), is an isomorphism compatible with the log Higgs fields on both sides.

Proof. By Lemma 5.60 it suffices to show that (5.61) and (5.62) are isomorphisms compatible with the log connections on both sides. By the same argument as in [Kat71, Section D], it suffices to show that (5.67) is an isomorphism compatible with the log connections on both sides. By Lemma 5.66, it suffices to show that (5.67) is an isomorphism compatible with the log filtrations on both sides. Consequently, the assertion for pullbacks in Theorem 5.20(3). Let $f : U \to U'$ be defined by adjunction is injective and strictly compatible with the filtrations on the both sides. That is, the induced morphism

\[
D_{\text{dR}}(R^i f_{\log, \ast}(L)) \to \left( R^i f_{\log, \ast} D_{\text{dR}}(L) \right)_{\text{free}}
\]

defined by adjunction is injective and strictly compatible with the filtrations on the both sides. Thus, we have finished the proofs of Theorems 5.16(1–2) and 5.20(1), and verified the assertion for pullbacks in Theorem 5.20(3).

Next, let us turn to the assertion for pushforwards in Theorem 5.20(3). Let $f : U \to U'$ be defined by $Y$ as in (5.2), let $E$ be the normal crossings divisors defining the log structures on $Y$, as in Example 2.11, and let $U := X - D$ and $V := Y - E$. Since $f^{-1}(E) \subset D$ by the same argument as in the proof of Lemma 5.57 and since $f : X \to Y$ restricts to a proper smooth morphism $f_U : U \to V$, we must have $D = f^{-1}(E)$ because $U$ is dense in $X$. Let $L$ be a $\text{dR}$ $\mathbb{Z}_p$-local system on $X_{\text{két}}$ such that $\mathcal{R}^i f_U(\mathbb{Z}_p)_{\text{két, \ast}}(L|_{U_{\text{két}}})$ is a $\mathbb{Z}_p$-local system on $V_{\text{két}}$. By Corollary 2.75, the morphism $L \to (U \to X)_{\text{két}, \ast}(L|_{U_{\text{két}}})$ defined by adjunction is an isomorphism, and $\mathcal{R}^i f_{\text{két, \ast}}(L) \cong (V \to Y)_{\text{két}, \ast} \mathcal{R}^i f_U(\mathbb{Z}_p)_{\text{két, \ast}}(L|_{U_{\text{két}}})$ is also a $\mathbb{Z}_p$-local system on $Y_{\text{két}}$.

Lemma 5.66. Under the assumption that $L|_{U_{\text{két}}}$ is de Rham, $\mathcal{R}^i f_U(\mathbb{Z}_p)_{\text{két, \ast}}(L|_{U_{\text{két}}})$ is also de Rham, and the morphism

\[
\begin{aligned}
D_{\text{dR}}(R^i f_{\log, \ast}(L)) &\to \left( R^i f_{\log, \ast} D_{\text{dR}}(L) \right)_{\text{free}} \\
\text{defined by adjunction is injective and strictly compatible with the filtrations on the both sides. That is, the induced morphism}
\end{aligned}
\]

\[
\begin{aligned}
gr^r D_{\text{dR}}(R^i f_{\log, \ast}(L)) &\to \left( \text{gr}^r \left( R^i f_{\log, \ast} D_{\text{dR}}(L) \right) \right)_{\text{free}}
\end{aligned}
\]

is injective, for each $r$.

Proof. Since $L|_{U_{\text{két}}}$ is de Rham, $(R^i f_{\text{két, \ast}}(L))|_{V_{\text{két}}} \cong R^i f_U(\mathbb{Z}_p)_{\text{két, \ast}}(L|_{U_{\text{két}}})$ is also de Rham, by [LZ17] Theorem 3.8(iv) and [Sch13] Theorem 8.8. Therefore, by Corollary 5.55 $\text{gr}^r D_{\text{dR}}(R^i f_{\text{két, \ast}}(L))$ is a vector bundles on $Y$. Since $V$ is dense in $Y$, the morphism (5.68) is injective because the corresponding morphism for $f_U : U \to V$ is an isomorphism, by [Sch13] Theorem 8.8. □

Corollary 5.69. Under the assumption that $L|_{U_{\text{két}}}$ is de Rham, the morphism (5.67) is an isomorphism compatible with the log connections and filtrations on both sides.

Proof. By Lemma 5.66 it suffices to show that (5.67) is an isomorphism compatible with the log connections on both sides. By the same argument as in [Kat71] Section
Theorem 5.16(3), 5.17(2), and 5.20(2). We assume that without requiring \( f^{-1}(D) = E \), and should include the comparison of cohomology in the next subsection as a special case. We leave this as a potential future work.

5.5. Comparison of cohomology. In this subsection, we prove the remaining Theorems 5.16(3), 5.17(2), and 5.20(2). We assume that \( X \) is proper over \( k \), and that \( K = \overline{k} \). (In particular, the rings \( B_{dR}^+ \) and \( B_{dR} \) in Definition 3.44(1) have their usual meaning as Fontaine’s rings.)

\[ \text{Lemma 5.71.} \quad \text{For a } \mathbb{Z}_p\text{-local system } \mathbb{L} \text{ on } X_{K, \text{két}}, \text{ and for each } i \geq 0, \text{ we have a canonical } \text{Gal}(K/k)\text{-equivariant isomorphism of } B_{dR}^+\text{-modules} \]

\[ H^i(X_{K, \text{két}}, \mathbb{L}) \otimes_{\mathbb{Z}_p} B_{dR}^+ \cong H^i(X_{K, \text{prokét}}, \mathbb{L} \otimes_{\mathbb{Z}_p} B_{dR}^+), \]

compatible with the filtrations on both sides, and also (by taking 0-th graded pieces) a canonical \( \text{Gal}(K/k)\text{-equivariant isomorphism of } K\text{-modules} \)

\[ H^i(X_{K, \text{két}}, \mathbb{L}) \otimes_{\mathbb{Z}_p} K \cong H^i(X_{K, \text{prokét}}, \mathbb{L} \otimes_{\mathbb{Z}_p} \mathcal{O}_{X_{K, \text{prokét}}}). \]

**Proof.** The proof is the same as [Sch13, Theorem 8.4], with the input [Sch13, Theorem 5.1] there replaced with Theorem 2.76. □

\[ \text{Lemma 5.72.} \quad \text{Let } \mathbb{L} \text{ be a } \mathbb{Z}_p\text{-local system on } X_{\text{két}}. \text{ For every } i, \text{ we have a canonical } \text{Gal}(K/k)\text{-equivariant isomorphism of } B_{dR}\text{-modules} \]

\[ H^i(X_{K, \text{prokét}}, \mathbb{L} \otimes_{\mathbb{Z}_p} \mathcal{O}_{X_{K, \text{prokét}}}) \cong H^i_{\text{log, dR}}(X, \mathcal{R}\mathcal{H}_{\text{log}}(\mathbb{L})), \]

and also a canonical \( \text{Gal}(K/k)\text{-equivariant isomorphism of } K\text{-modules} \)

\[ H^i(X_{K, \text{prokét}}, \mathbb{L} \otimes_{\mathbb{Z}_p} \mathcal{O}_{X_{K, \text{prokét}}}) \cong H^i_{\text{log, Higgs}}(X_{K, \text{an}}, \mathcal{H}_{\text{log}}(\mathbb{L})). \]

**Proof.** By Corollary 3.39 we have an exact sequence

\[ 0 \to \mathbb{L} \otimes_{\mathbb{Z}_p} \mathcal{O}_{X_{\text{prokét}}} \to \mathbb{L} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\text{prokét}} \mathcal{O}_{\text{log, 1}} \]

\[ \mathcal{O}_{\text{log, 2}} \to \cdots \]

and also (by taking the 0-th graded pieces) an exact sequence

\[ 0 \to \mathbb{L} \otimes_{\mathbb{Z}_p} \mathcal{O}_{X_{\text{prokét}}} \to \mathbb{L} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\text{log, 1}} \mathcal{O}_{\text{log, 2}} \to \cdots \]

on \( X_{K, \text{prokét}} \). For simplicity, we shall denote by \( DR_{\text{log}}(\mathbb{L} \otimes_{\mathbb{Z}_p} \mathcal{O}_{dR}) \) the complex \( (\mathbb{L} \otimes_{\mathbb{Z}_p} \mathcal{O}_{dR} \otimes_{\text{prokét}} \mathcal{O}_{\text{log, } \bullet}, \nabla) \), and by \( Higgs_{\text{log}}(\mathbb{L} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\text{log}}) \) the complex \( (\mathbb{L} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\text{log}} \otimes_{\text{prokét}} \mathcal{O}_{\text{log, } \bullet}(-1), \partial) \) (where the two \( \bullet \) in the latter complex are equal to each other). Then we have

\[ \text{Lemma 5.71.} \quad \text{For a } \mathbb{Z}_p\text{-local system } \mathbb{L} \text{ on } X_{K, \text{két}}, \text{ and for each } i \geq 0, \text{ we have a canonical } \text{Gal}(K/k)\text{-equivariant isomorphism of } B_{dR}^+\text{-modules} \]

\[ H^i(X_{K, \text{két}}, \mathbb{L}) \otimes_{\mathbb{Z}_p} B_{dR}^+ \cong H^i(X_{K, \text{prokét}}, \mathbb{L} \otimes_{\mathbb{Z}_p} B_{dR}^+), \]

compatible with the filtrations on both sides, and also (by taking 0-th graded pieces) a canonical \( \text{Gal}(K/k)\text{-equivariant isomorphism of } K\text{-modules} \)

\[ H^i(X_{K, \text{két}}, \mathbb{L}) \otimes_{\mathbb{Z}_p} K \cong H^i(X_{K, \text{prokét}}, \mathbb{L} \otimes_{\mathbb{Z}_p} \mathcal{O}_{X_{K, \text{prokét}}}). \]

**Proof.** The proof is the same as [Sch13, Theorem 8.4], with the input [Sch13, Theorem 5.1] there replaced with Theorem 2.76. □
and

\[ \text{RT} \left( X_{K, \text{prokét}}, \hat{\mathcal{L}} \otimes_{\mathbb{Z}_p} O_{X_{K, \text{prokét}}} \right) \cong \text{RT} \left( X_{K, \text{prokét}}, \text{Higgs}_{\log} \left( \hat{\mathcal{L}} \otimes_{\mathbb{Z}_p} O_{\mathcal{C}_{\log}} \right) \right) \]

By Theorem 5.16(1) and the projection formula (cf. (5.25)), we have

\[ R\mu'_i \left( \text{DR}_{\log} \left( \hat{\mathcal{L}} \otimes_{\mathbb{Z}_p} O_{\mathcal{B}_{\log}} \right) \right) \cong \text{DR}_{\log} \left( \mathcal{R} \mathcal{H}_{\log}(\mathcal{L}) \right). \]

Similarly, by Proposition 5.26 and the projection formula, we have

\[ R\mu'_i \left( \text{Higgs}_{\log} \left( \hat{\mathcal{L}} \otimes_{\mathbb{Z}_p} O_{\mathcal{C}_{\log}} \right) \right) \cong \text{Higgs}_{\log} \left( \mathcal{H}_{\log}(\mathcal{L}) \right). \]

Therefore, we have

\[ \text{RT} \left( X_{K, \text{prokét}}, \text{DR}_{\log} \left( \hat{\mathcal{L}} \otimes_{\mathbb{Z}_p} O_{\mathcal{B}_{\log}} \right) \right) \cong \text{RT} \left( X, \text{DR}_{\log} \left( \mathcal{R} \mathcal{H}_{\log}(\mathcal{L}) \right) \right) \]

and

\[ \text{RT} \left( X_{K, \text{prokét}}, \text{Higgs}_{\log} \left( \hat{\mathcal{L}} \otimes_{\mathbb{Z}_p} O_{\mathcal{C}_{\log}} \right) \right) \cong \text{RT} \left( X_{K, \text{an}}, \text{Higgs}_{\log} \left( \mathcal{H}_{\log}(\mathcal{L}) \right) \right), \]

and the lemma follows. \( \square \)

Now Theorems 5.16(3) and 5.17(2) follow from Lemmas 5.71 and 5.72. It remains to complete the proof of Theorem 5.20(2). In the remainder of this subsection, we shall assume in addition that \( \mathbb{L}_{| (X - D)_{\text{ad}} } \) is a de Rham local system. Firstly, the isomorphism (5.21) is given by Theorem 5.16(3) and the following:

**Lemma 5.73.** With assumptions as above, there is a canonical isomorphism

\[ H^i_{\text{log} \text{dR}} \left( X, \mathcal{R} \mathcal{H}_{\log}(\mathcal{L}) \right) \cong H^i_{\text{log} \text{dR}} \left( X_{\text{an}}, D_{\text{dR}, \log}(\mathcal{L}) \right) \otimes_k B_{\text{dR}}. \]

**Proof.** Since \( \mathbb{L}_{| (X - D)_{\text{ad}} } \) is de Rham, by Corollary 5.54, we have

\[ \text{RT} \left( X_{\text{an}}, \text{DR}_{\log} \left( \mathcal{R} \mathcal{H}_{\log}(\mathcal{L}) \right) \right) \cong \text{RT} \left( X_{\text{an}}, \text{DR}_{\log} \left( D_{\text{dR}, \log}(\mathcal{L}) \right) \otimes_k B_{\text{dR}} \right). \]

We claim that the last term coincides with

\[ \text{RT} \left( X_{\text{an}}, \text{DR}_{\log} \left( D_{\text{dR}, \log}(\mathcal{L}) \right) \right) \otimes_k B_{\text{dR}}. \]

It suffices to check that we have isomorphisms on the grade pieces. Concretely, it suffices to note that, for any locally free \( O_X \)-module \( \mathcal{F} \) of finite rank, we have

\[ \text{RT} \left( X_{\text{an}}, \mathcal{F} \right) \otimes_k \text{gr}^r B_{\text{dR}} \cong \text{RT} \left( X_{\text{K,an}}, \mathcal{F} \otimes_k \text{gr}^r B_{\text{dR}} \right), \]

where \( \text{gr}^r B_{\text{dR}} \cong K(r) \), because the coherent cohomology of proper adic spaces is finite-dimensional and compatible with arbitrary base field extensions. (See [Kie67a] and [Sch13, Theorem 9.2, and the proof of Lemma 7.13].) \( \square \)

Secondly, \( \text{gr} D_{\text{dR}, \log}(\mathcal{L}) \) is a vector bundle of rank \( \text{rk}_{\mathbb{Z}_p}(\mathcal{L}) \) by Corollary 5.55, and the isomorphism (5.22) is given by Theorem 5.17(2) and the following:

**Lemma 5.75.** With assumptions as above, there is a canonical isomorphism

\[ H^i_{\text{log Higgs}} \left( X_{\text{K,an}}, \mathcal{H}_{\log}(\mathcal{L}) \right) \cong \oplus_{a+b=i} \left( H^{a,b}_{\text{log Hodge}} \left( X_{\text{an}}, D_{\text{dR}, \log}(\mathcal{L}) \right) \otimes_k K(-a) \right). \]
Proof. Since $L_{(X-D)_{dt}}$ is de Rham, by Corollary 5.54 and [5,74], we have

$$H^i_{\log \text{Higgs}}(X_{K,an}, \mathcal{H}_\log(L)) = H^i(X_{K,an}, \text{Higgs}_\log(\mathcal{H}_\log(L)))$$

$$\cong H^i(X_{K,an}, \text{gr}^0 DR_\log(R\mathcal{H}_\log(L)))$$

$$\cong H^i(X_{K,an}, \text{gr}^0 DR_\log(D_{\text{dR},\log}(L)) \otimes_k B_{dR})$$

$$\cong \bigoplus_a H^i(X_{K,an}, \text{gr}^a DR_\log(D_{\text{dR},\log}(L)) \otimes_k K(-a))$$

$$\cong \bigoplus_a \left( H^i(X_{an}, \text{gr}^a DR_\log(D_{\text{dR},\log}(L)) \otimes_k K(-a) \right)$$

$$\cong \bigoplus_{a+b=i} H^{a,b}_{\log \text{Hodge}}(X_{an}, D_{\text{dR},\log}(L)) \otimes_k K(-a).$$

Finally, the (log) Hodge–de Rham spectral sequence for $D_{\text{dR},\log}(L)$ (cf. (5.11)) degenerates on the $E_1$ page because, by the isomorphisms [5.21] and [5.22] established above, we have

$$\dim_k H^i_{\log \text{dR}}(X_{an}, D_{\text{dR},\log}(L)) = \sum_{a+b=i} \dim_k H^{a,b}_{\log \text{Hodge}}(X_{an}, D_{\text{dR},\log}(L)).$$

The proof of Theorem 5.20(3) is now complete.

6. The $p$-adic Riemann–Hilbert Functor

The goal of this section is to prove Theorem 1.1 and record some byproducts in Theorem 6.6. Throughout this section, $X$ is a smooth algebraic variety over a finite extension $k$ of $\mathbb{Q}_p$. Since char($k$) = 0, by Nag62 Hir64 Hir64b, there exists a smooth compactification $\bar{X}$ of $X$ such that the boundary $D = \bar{X} - X$ (with its reduced subscheme structure) is a normal crossings divisor. Let $j : X \hookrightarrow \bar{X}$ denote the canonical open immersion. Let $X_{an}^\prime$, $\bar{X}_{an}$, and $D_{an}$ denote the corresponding analytifications, viewed as adic spaces over Spa($k, \mathcal{O}_k$). By abuse of notation, we shall still denote by $j : X_{an}^\prime \hookrightarrow \bar{X}_{an}$ the corresponding open immersion. Moreover, we view $\bar{X}_{an}$ as a log adic space by equipping it with the log structure $\mathcal{M}$ associated with $D_{an}$, as in Example 2.12.

By [Lut93] Theorem 3.1, the analytification induces an equivalence from the category of étale $\mathbb{Z}_p$-local systems on $X$ to the category of étale $\mathbb{Z}_p$-local systems on $X_{an}$. Let $L$ be an étale local $\mathbb{Z}_p$-local system on $X$, with analytification $L_{an}$. By Corollary 2.75

$$\bar{L}_{an} := j_{\text{ét},*}(L_{an})$$

is a Kummer étale $\mathbb{Z}_p$-local system on $\bar{X}_{an}$. By applying the functor $D_{\text{dR},\log}$ constructed in (5.19), and by Theorem 5.20(1), we obtain a vector bundle $D_{\text{dR},\log}(\bar{L}_{an})$ on $\bar{X}_{an}$ equipped with an integrable log connection and a decreasing filtration satisfying the Griffiths transversality. By GAGA (see Köp74, and also the proof of Sch13 Theorem 9.1), $D_{\text{dR},\log}(\bar{L}_{an})$ is the analytification of a filtered algebraic vector bundle

$$D_{\text{dR},\log}^\text{alg}(\bar{L}_{an})$$

with a log connection satisfying the Griffiths transversality. Consider the restriction

$$D_{\text{dR}}^\text{alg}(L) := (D_{\text{dR},\log}^\text{alg}(\bar{L}_{an}))|_X.$$
This is a vector bundle on $X$ together with an integrable connection and a decreasing filtration satisfying the Griffiths transversality. By [AB01 Chapter 1, Proposition 3.4.2 and Section 4.3], the connection on $D_{\text{alg}}^{\text{dR}}(L)$ has regular singularities at infinity (i.e., along $D$).

We summarize the construction in the following diagram. To simplify the terminology, we shall use the term \textit{filtered connection} (resp. \textit{filtered regular connection}) to mean a filtered vector bundle on $X$ equipped with an integrable connection (resp. an integrable connection with regular singularities) satisfying the Griffiths transversality. Likewise, we shall use the term \textit{filtered log connection} to mean a filtered vector bundle on $\mathcal{X}$ (resp. $\mathcal{X}^{\text{an}}$) equipped with an integrable connection satisfying the Griffiths transversality. In addition, by abuse of terminology, we say that a $\mathbb{Z}_p$-local system on $X_{\text{et}}$ is de Rham if its analytification is, and that a $\mathbb{Z}_p$-local system on $\mathcal{X}^{\text{an}}_{k\text{et}}$ is de Rham if its restriction to $\mathcal{X}^{\text{an}}$ is. Then the construction of the functor $D_{\text{alg}}^{\text{dR}}$ can be summarized by the following commutative diagram:

\[
\begin{array}{ccc}
\{\text{de Rham } \mathbb{Z}_p\text{-local systems on } X_{\text{et}}\} & \xrightarrow{D_{\text{alg}}^{\text{dR}}} & \{\text{filtered regular connections on } X\} \\
\downarrow (\_)^{an} & & \downarrow \gamma \\
\{\text{de Rham } \mathbb{Z}_p\text{-local systems on } X^{\text{an}}_{\text{et}}\} & \xrightarrow{\kappa_{k\text{et}^*}} & \{\text{filtered log connections on } \mathcal{X}\} \\
\downarrow \gamma^{an} & \updownarrow \gamma & \downarrow \gamma^{an} \\
\{\text{de Rham } \mathbb{Z}_p\text{-local systems on } \mathcal{X}^{\text{an}}_{k\text{et}}\} & \xrightarrow{D_{\text{dR},\log}} & \{\text{filtered log connections on } \mathcal{X}^{\text{an}}\}
\end{array}
\]

Note that the de Rham assumptions on the local systems ensure that the associated regular connections or log connections are of the right ranks, and are filtered by vector subbundles rather than coherent subsheaves.

**Lemma 6.4.** The functor $D_{\text{alg}}^{\text{dR}}$ is a tensor functor, and is independent of the choice of the compactification $\mathcal{X}$.

**Proof.** By Proposition 5.48 and [AB01 Chapter 1, Proposition 6.2.2], for all $L$, the exponents of the integrable connection of $D_{\text{alg}}^{\text{dR}}(L)$ consist of only rational numbers, which are not Liouville numbers. Therefore, by [AB01 Chapter 4, Theorem 4.1] or [Bal88] (and by the same arguments as in the proofs of [AB01 Chapter 4, Corollaries 3.6 and 6.8.2]), the functor $D_{\text{alg}}^{\text{dR}}$ is a tensor functor independent of the choice of compactification $\mathcal{X}$ if and only if its further composition with the analytification functor is. But we already know that this composition is a tensor functor $D_{\text{dR}}$ that is defined independent of the choice of $\mathcal{X}$, by [LZ17 Theorem 3.9(v)]. □

It remains to establish the comparison isomorphism in Theorem 1.1. As in Section 5.5, let $K = \hat{k}$, so that the rings $B_{\text{dR}}^+$ and $B_{\text{dR}}$ in Definition 3.44 have their usual meaning as Fontaine’s rings. By [Hub96 Proposition 2.1.4 and Theorem 3.8.1], if $L$ is an étale local $\mathbb{Z}_p$-local systems on $X$, and if $L^{an}$ is its analytification on $X^{an}$, then we have a canonical $\text{Gal}(\overline{k}/k)$-equivariant isomorphism

\[
H^i_{\text{ét}}(X_{\overline{k}}, L) \cong H^i_{\text{ét}}(X^{an}_{\overline{k}}, L^{an}).
\]

\[
(6.5)
\]
By Corollary 2.75 and Theorem 5.20(2), we have a canonical \( \text{Gal}(\mathbb{K}/k) \)-equivariant isomorphism
\[
H^i_{\text{ét}}(X^{\text{an}}_{\mathbb{K}}, L^{\text{an}}) \otimes_{\mathbb{Z}_p} B_{\text{dR}} \cong H^i_{\log \text{dR}}(X^{\text{an}}, D_{\text{dR}, \log}(\mathcal{L}^{\text{an}})) \otimes_k B_{\text{dR}},
\]
compatible with the filtrations on both sides. Finally, it remains to note that
\[
H^i_{\log \text{dR}}(X^{\text{an}}, D_{\text{dR}, \log}(\mathcal{L}^{\text{an}})) \cong H^i_{\log \text{dR}}(X, D_{\text{dR}}(\mathcal{L})),
\]
where the first isomorphism follows from GAGA again (see [Köp74]), and where the second isomorphism follows from Deligne’s comparison result in [Del70, II, 6]. This completes the proof of Theorem 1.1.

By combining (6.5), Corollary 2.75, and GAGA (see [Köp74]) with the other assertions in Theorem 5.20(2), we also obtain the following:

**Theorem 6.6.** In the above setting, the (log) Hodge–de Rham spectral sequence
\[
E_{1}^{a,b} = H_{\log \text{Hodge}}^{a,b}(X, D_{\text{dR}, \log}(\mathcal{L}^{\text{an}})) \Rightarrow H^{a+b}_{\text{dR}}(X, D_{\text{dR}}(\mathcal{L}^{\text{an}}))
\]
degenerates on the \( E_1 \) page, and there is also a canonical \( \text{Gal}(\mathbb{K}/k) \)-equivariant comparison isomorphism
\[
H^i_{\text{ét}}(X_{\mathbb{K}}, L) \otimes_{\mathbb{Q}_p} \hat{k} \cong \oplus_{a+b=i} \left( H^a_{\log \text{Hodge}}(X, D_{\text{dR}, \log}(\mathcal{L}^{\text{an}})) \otimes_{k} \hat{k}(-a) \right),
\]
which can be identified with the 0-th graded piece of the comparison isomorphism in Theorem 1.1.

### 7. Application to Shimura Varieties

In this section, we give a proof of Theorem 1.3, which serves as an evidence of Conjecture 1.2. In addition, by applying our main results in previous sections, we obtain some new results on Shimura varieties, including some results concerning the Hodge–Tate weights of the cohomology of automorphic étale \( p \)-adic local systems.

Unless otherwise specified, the symbols \( \lambda, \mu, \) and \( \nu \) will be reserved for weights and coweights (rather than morphisms of sites).

#### 7.1. The setup.

Let \((G, X)\) be any Shimura datum. That is, \(G\) is a connected reductive \( \mathbb{Q} \)-group, and \(X\) is a hermitian symmetric domain parameterizing a conjugacy class of homomorphisms
\[
h : S := \text{Res}_{\mathbb{C}/\mathbb{R}} G_{m, \mathbb{C}} \to G_{\mathbb{R}},
\]
satisfying a list of axioms (see, for example, [Del70, 2.1.1] and [Mil05, Definition 5.5]). For each neat open compact subgroup \(K\) of \(G(\mathbb{A}_f)\) (see, for example, [Pin89, 0.6] for the definition of neatness here), we denote by \(\text{Sh}_K = \text{Sh}_K(G, X)\) the canonical model of the associated Shimura variety at level \(K\), which is a smooth quasi-projective algebraic variety over a number field \(E \subset \mathbb{C}\), called the reflex field \(E\) of \((G, X)\). Recall that, essentially by definition, the analytification of its base change \(\text{Sh}_{K, \mathbb{C}}\) from \(E\) to \(\mathbb{C}\) is the complex manifold
\[
\text{Sh}_{K, \mathbb{C}}^{\text{an}} \cong G(\mathbb{Q}) \backslash (X \times G(\mathbb{A}_f))/K.
\]
Note that right multiplication by \(g \in G(\mathbb{A}_f)\) induces an isomorphism
\[
[g] : \text{Sh}_{K, \mathbb{C}}^{\text{an}, -1} \cong \text{Sh}_{K, \mathbb{C}}^{\text{an}}.
\]
which algebraizes and descends to an isomorphism $\text{Sh}_{gK^{-1}} \sim \text{Sh}_K$, still denoted by $[g]$. (For such basic facts concerning Shimura varieties, see the survey articles such as [Mil05] and [Lan17], and their further references to the literature.)

As in [Mil90] Chapter III, let us denote by $G^c$ the quotient of $G$ by the maximal $\mathbb{Q}$-anisotropic $\mathbb{R}$-split subtorus of the center of $G$ (as algebraic groups over $\mathbb{Q}$). For any subgroup of $G(\mathbb{A})$ (including those of $G(\mathbb{Q})$, $G(\mathbb{A}_f)$, etc), we shall denote its image in $G^c(\mathbb{A})$ with an additional superscript "c". (We are not introducing a model of $G^c$ over $\mathbb{Z}$.) Therefore, given any open compact subgroup $K$ of $G(\mathbb{A}_f)$, we have an open compact subgroup $K^c$ of $G^c(\mathbb{A}_f)$. Given neat open compact subgroups $K_1$ and $K_2$ such that $K_1$ is a normal subgroup of $K_2$, we obtain a Galois finite étale covering $\text{Sh}_{K_1} \to \text{Sh}_{K_2}$ with Galois group $K_2^f/K_1^f$. Sometimes, it is convenient to consider the projective system $\{\text{Sh}_K\}_K$, which can be viewed as the scheme $\text{Sh} := \lim_{\leftarrow} \text{Sh}_K$ over $E$. Then, the algebraization of (7.3) induces a canonical right action of $G(\mathbb{A}_f)$ on $\{\text{Sh}_K\}_K$. We call these actions (and their various extensions to actions on other objects) Hecke actions of $G(\mathbb{A}_f)$ (sometimes with $G(\mathbb{A}_f)$ omitted).

For later purposes, let $G^{\text{der}}$ denote the derived group of $G$, and let $G^{\text{der},c}$ denote the image of $G^{\text{der}}$ in $G^c$. Let $G^{\text{ad}}$ denote the adjoint quotient of $G$, which is also the common adjoint quotient of $G^{\text{der}}$, $G^c$, and $G^{\text{der},c}$. Note that $G^{\text{der}}$, $G^{\text{der},c}$, and $G^{\text{ad}}$ are connected semisimple algebraic groups over $\mathbb{Q}$, and the canonical homomorphisms $G^{\text{der}} \to G^{\text{der},c} \to G^{\text{ad}}$ are central isogenies.

For any field $F$, let us denote by $\text{Rep}_F(G^c)$ the category of finite-dimensional algebraic representations of $G^c$ over $F$, which we also view as an algebraic representation of $G$ by pullback (whose restriction to the maximal $\mathbb{Q}$-anisotropic $\mathbb{R}$-split subtorus of the center of $G_\mathbb{Q}$ is trivial). Let $\mathbb{Q}$ denote the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, and let $\mathbb{Q}_p$ be an algebraic closure of $\mathbb{Q}_p$, together with a fixed isomorphism $\iota : \mathbb{Q}_p \sim \mathbb{C}$, which induces an injective field homomorphism $\iota^{-1} : \mathbb{Q} \hookrightarrow \mathbb{Q}_p$.

### 7.2. Local systems: complex analytic constructions

For any $V \in \text{Rep}_{\mathbb{Q}}(G^c)$, we define the (Betti) local system

$$p^V_\mathbb{Q} := G(\mathbb{Q})/(X \times V) \times G(\mathbb{A}_f)) / K$$

of $\mathbb{Q}$-vector spaces over $\text{Sh}^\text{an}_{K,\mathbb{C}}$. Essentially by construction, we have the following:

**Proposition 7.4.** The assignment of $p^V_\mathbb{Q}$ to $V$ defines a tensor functor from $\text{Rep}_{\mathbb{Q}}(G^c)$ to the category of $G(\mathbb{A}_f)$-equivariant $\mathbb{Q}$-local systems on $\{\text{Sh}^\text{an}_{K,\mathbb{C}}\}_K$. Moreover, it is functorial with respect to pullbacks under morphisms between Shimura varieties induced by morphisms between Shimura data.

Let us also explain the construction of $p^V_\mathbb{Q}$ more concretely via the representation of the fundamental groups of (connected components) of $\text{Sh}^\text{an}_{K,\mathbb{C}}$, under the classical correspondence between local systems and fundamental group representations. (See, for example, [Voisin03] Corollary 3.10.)

Suppose we have a connected component of $\text{Sh}^\text{an}_{K,\mathbb{C}}$ (see (7.2)) given by

$$\Gamma^+_K, g_0 \setminus X^+ \cong G(\mathbb{Q})_+ \setminus (X^+ \times (g_0 K)) / K$$

(cf. [Del79] 2.1.2 or [Mil05] Lemma 5.13), where $X^+$ is a fixed connected component of $X$ and $g_0 \in G(\mathbb{A}_f)$, and where $G(\mathbb{Q})_+$ is the stabilizer of $X^+$ in $G(\mathbb{Q})$ and

$$\Gamma^+_K, g_0 := G(\mathbb{Q})_+ \cap (g_0 K g_0^{-1})$$
is a neat arithmetic subgroup of $G(\mathbb{Q})$ (see, for example, [Bor69, 17.1], for the definition of neatness here). It follows from the definitions that $\Gamma_{K,g_0}^{+,c}$ is neat when $K$ is. Let $\Gamma_{K,g_0}^{+,c}$ and $\Gamma_{K,g_0}^{+,ad}$ denote the images of $\Gamma_{K,g_0}^+$ in $G^c(\mathbb{Q})$ and $G^{ad}(\mathbb{Q})$, respectively, so that we have surjective homomorphisms

$$\Gamma_{K,g_0}^+ \rightarrow \Gamma_{K,g_0}^{+,c} \rightarrow \Gamma_{K,g_0}^{+,ad}.$$  

**Lemma 7.7.** The subgroup $\Gamma_{K,g_0}^{+,c}$ of $G^c(\mathbb{Q})$ is contained in $G^{der,c}(\mathbb{Q})$, and the second homomorphism in (7.6) is an isomorphism $\Gamma_{K,g_0}^{+,c} \cong \Gamma_{K,g_0}^{+,ad}$.

**Proof.** Since $\ker(G \rightarrow G^c)$ is the maximal $\mathbb{Q}$-anisotropic $\mathbb{R}$-split subtorus of the center of $G$, the quotient $G^c/G^{der,c}$ is a torus isogenous to a product of a split torus and a torus of compact type (i.e., $\mathbb{R}$-anisotropic) over $\mathbb{Q}$. Since all neat arithmetic subgroups of such a torus are trivial, the neat image $\Gamma_{K,g_0}^{+,c}$ of $\Gamma_{K,g_0}$ in $G^c(\mathbb{Q})$ is contained in $G^{der,c}(\mathbb{Q})$. Consequently, the second homomorphism in (7.6) is an isomorphism, because its kernel, being both neat and finite, is trivial. \hfill $\Box$

**Corollary 7.8.** The connected component $\Gamma_{K,g_0}^+ \backslash X^+$ is a connected smooth manifold whose fundamental group (with any prescribed base point of $X^+$) is canonically isomorphic to $\Gamma_{K,g_0}^{+,c}$.

**Proof.** Since $\Gamma_{K,g_0}$ acts on $X^+$ via its image $\Gamma_{K,g_0}^{+,ad}$ in $G^{ad}(\mathbb{Q})$, the corollary follows from Lemma 7.7. \hfill $\Box$

**Remark 7.9.** We shall not write $\Gamma_{K,g_0}^{+,ad}$ again in what follows.

By taking $X^+$ as a universal covering of $\Gamma_{K,g_0}^+ \backslash X^+$, and by fixing the choice of a base point on $X^+$, the pullback of $B_\mathbb{V}$ to $\Gamma_{K,g_0}^+ \backslash X^+$ determines and is determined by the fundamental group representation

$$\rho_{K,g_0}^+(V) : \Gamma_{K,g_0}^+ \rightarrow \text{GL}_\mathbb{Q}(V),$$

which coincides with the restriction of the representation of $G^c$ on $V$. In particular, it is compatible with the change of levels $K' \subset K$.

Moreover, given $g \in g_0 \Gamma(\mathbb{Q}) g_0$, so that $g_0 g = \gamma g_0$ for some $\gamma \in G(\mathbb{Q})_+$, we have $\Gamma_{g K^{-1},g_0}^{+,c} = \gamma \Gamma_{K,g_0}^{+,c} \gamma^{-1}$, and the Hecke action (7.3) induces a morphism

$$\Gamma_{g K^{-1},g_0} \backslash X^+ \cong \Gamma_{K,g_0}^+ \backslash X^+,$$

which is nothing but the isomorphism defined by left multiplication by $\gamma^{-1}$. It follows that the canonical isomorphism $[g]^+(B_\mathbb{V}) \cong B_\mathbb{V}$ of local systems corresponds to the following equality of fundamental group representations

$$\rho_{g K^{-1},g_0}(V) = \gamma (\rho_{K,g_0}^+(V)),$$

where $\gamma (\rho_{K,g_0}^+(V))$ means the representation of $\Gamma_{g K^{-1},g_0}^{+,c} = \gamma \Gamma_{K,g_0}^{+,c} \gamma^{-1}$ defined by conjugating the values of $\rho_{K,g_0}^+(V)$ by $\gamma$ in $\text{GL}(V)$.

Now, by base change along $\mathbb{Q} \subset \mathbb{C}$ via the canonical homomorphism, we obtain the object $V_\mathbb{C} := V \otimes \mathbb{Q} \mathbb{C}$ in $\text{Rep}_\mathbb{C}(G^c)$, as well as the local system

$$p_\mathbb{V}_\mathbb{C} := p_\mathbb{V} \otimes \mathbb{Q} \mathbb{C}$$

of $\mathbb{C}$-vector spaces over $\text{Sh}_K(\mathbb{C})$. It is well known (as stated in [Del70, I, 2.17], which is based on the classical Frobenius theorem) that any local system $p_\mathbb{V}_\mathbb{C}$ as above...
is canonically isomorphic to the local system of horizontal sections of the (complex analytic) integrable connection
\[(\mathrm{dR}V^\mathrm{an}_C, \nabla) := (\Lambda^1 V_C \otimes \mathcal{O}_{\text{Sh}_{K,C}}, 1 \otimes d)\].
Moreover, any \(h\) (as in (7.1)) parameterized by \(X\) induces a homomorphism \(h_C : G_{m,C} \times G_{m,C} \to G_C\), whose restriction to the first factor \(G_{m,C}\) defines the so-called Hodge cocharacter
\[
\mu_h : G_{m,C} \to G_C,
\]
inducing a (decreasing) filtration \(\text{Fil}^\bullet\) on \(\mathrm{dR}V^\mathrm{an}_C\) satisfying the Griffiths transversality condition \(\nabla(\text{Fil}^\bullet) \subset \text{Fil}^{\bullet - 1} \otimes \Omega^1_{\text{Sh}_{K,C}}\). Then we say that \((\mathrm{dR}V^\mathrm{an}_C, \nabla, \text{Fil}^\bullet)\) is a filtered integrable connection.

Let \(\text{Sh}_{K}^{\text{tor}}\) be a toroidal compactification of \(\text{Sh}_K\) (as in [Pit89]), which we assume to be projective and smooth, with the boundary divisor \(D := \text{Sh}_{K}^{\text{tor}} - \text{Sh}_K\) (with its canonical reduced subscheme structure) a normal crossings divisor, whose base change from \(E\) to \(\mathbb{C}\) and whose further complex analytification are denoted by \(\text{Sh}_{K,C}^{\text{tor}}\) and \(\text{Sh}_{K,C}^{\text{tor,an}}\), respectively. As explained in [LS13, Section 6.1], \(\Lambda^1 V_C\) has unipotent monodromy along \(D^\text{an}_C\). Therefore, by [Del70, II, 5] and [Kat71, Sections VI and VII], the integrable connection \((\mathrm{dR}V^\mathrm{can,an}_C, \nabla)\) as above uniquely extends to an integrable log connection
\[(\mathrm{dR}V^\mathrm{can,an}_C, \nabla, \text{Fil}^\bullet)\],
with log poles and nilpotent residues along the boundary divisor \(D^\text{an}_C\). By [Del70, II, 5.2(d)], the assignment of \((\mathrm{dR}V^\mathrm{can,an}_C, \nabla)\) to \(V\) is a tensor functor from \(\text{Rep}_C(\mathbb{G}^\circ)\) to the category of integrable log connections on \(\text{Sh}_{K,C}^{\text{tor,an}}\). Moreover, by [Sch73] (see also the explanation in [CKSS7]), the filtration \(\text{Fil}^\bullet\) on \(\mathrm{dR}V^\mathrm{can}_C\) uniquely extends to a filtration on \(\mathrm{dR}V^\mathrm{can,an}_C\) (by subbundles), which we still denote by \(\text{Fil}^\bullet\). The extended \(\nabla\) and \(\text{Fil}^\bullet\) still satisfy the Griffiths transversality over \(\text{Sh}_{K,C}^{\text{tor,an}}\), because they already do over the open dense subspace \(\text{Sh}_{K,C}^{\text{tor}}\), and therefore the triple \((\mathrm{dR}V^\mathrm{can,an}_C, \nabla, \text{Fil}^\bullet)\) is an analytic filtered log connection (see the paragraph preceding Lemma 6.4 for the terminology). By GAGA (see [Ser59], and also the proof of [Del70, II, 5.9]), this triple canonically algebraizes to an algebraic filtered log connection
\[(\mathrm{dR}V^\mathrm{can}_C, \nabla, \text{Fil}^\bullet)\].
(The complex analytic \(\mathrm{dR}V^\mathrm{can,an}_C\) and its algebraization \(\mathrm{dR}V^\mathrm{can}_C\) agree with the canonical extensions defined differently in [Har89, Section 4], and also [Har90a] and [Mil90].) The restriction of \((\mathrm{dR}V^\mathrm{can}_C, \nabla, \text{Fil}^\bullet)\) to \(\text{Sh}_{K,C}\) then defines an algebraic filtered regular connection
\[(\mathrm{dR}V^\mathrm{can}_C, \nabla, \text{Fil}^\bullet)\]
over \(\text{Sh}_{K,C}\), whose complex analytification is isomorphic to \((\mathrm{dR}V^\mathrm{can}_C, \nabla, \text{Fil}^\bullet)\). We call \((\mathrm{dR}V^\mathrm{can}_C, \nabla)\) the automorphic vector bundle associated with \(V_C\).

**Proposition 7.14.** The assignment of \((\mathrm{dR}V^\mathrm{can}_C, \nabla, \text{Fil}^\bullet)\) to \(V\) defines a tensor functor from \(\text{Rep}_C(\mathbb{G}^\circ)\) to the category of \(G(\mathbb{A}_f)\)-equivariant filtered regular connections on \(\{\text{Sh}_{K,C}\}_K\). Moreover, it is functorial with respect to pullbacks under morphisms between Shimura varieties induced by morphisms between Shimura data.
7.3. **Local systems: p-adic analytic constructions.** Given any \( V \in \text{Rep}_{\mathbb{Q}}(\mathbb{G}^c) \) as above, by base change via \( \iota^{-1}: \overline{\mathbb{Q}}_p \rightarrow \overline{\mathbb{Q}}_p \), we obtain the object \( V_{\overline{\mathbb{Q}}_p} := V \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_p \) in \( \text{Rep}_{\overline{\mathbb{Q}}_p}(\mathbb{G}^c) \). As explained in [LZ17 Section 3] (see also [LZ17 Section 4.2]), given such a finite-dimensional representation \( V_{\overline{\mathbb{Q}}_p} \), there is a canonical automorphic étale local system \( \ell\overline{V}_{\overline{\mathbb{Q}}_p} \) of \( \overline{\mathbb{Q}}_p \)-vector spaces (i.e., lisse \( \overline{\mathbb{Q}}_p \)-étale sheaf) over \( \text{Sh}_K \) (with stalks isomorphic to \( V_{\overline{\mathbb{Q}}_p} \)). In fact, by the very construction of \( \ell\overline{V}_{\overline{\mathbb{Q}}_p} \), for each finite extension \( L \) of \( \mathbb{Q}_p \) in \( \overline{\mathbb{Q}}_p \) such that \( V_{\overline{\mathbb{Q}}_p} \) has a model \( V_L \) over \( L \), we have an étale local system \( \ell\overline{V}_L \) of \( L \)-vector spaces over \( \text{Sh}_K \) (with stalks isomorphic to \( V_L \)) such that

\[
(7.15)\quad \ell\overline{V}_L \otimes_L \overline{\mathbb{Q}}_p \cong \ell\overline{V}_{\overline{\mathbb{Q}}_p}.
\]

In addition, by [AGV73 XI, 4.4] (or by using the canonical homomorphism from the fundamental group to the étale fundamental group), its pullback to \( \text{Sh}_{K,\mathbb{C}} \) induces a local system \( b\overline{V}_{\overline{\mathbb{Q}}_p} \) such that

\[
(7.16)\quad b\overline{V}_{\overline{\mathbb{Q}}_p} \otimes_{\overline{\mathbb{Q}}_p} \mathbb{C} \cong b\overline{V}_\mathbb{C},
\]

where the coefficient ring extension from \( \overline{\mathbb{Q}}_p \) to \( \mathbb{C} \) is defined by the above chosen \( \iota: \overline{\mathbb{Q}}_p \sim \mathbb{C} \). As in the Betti picture (see Proposition 7.4), we have the following:

**Proposition 7.17.** The assignment of \( \ell\overline{V}_{\overline{\mathbb{Q}}_p} \) to \( V \) defines a tensor functor from \( \text{Rep}_{\mathbb{Q}}(\mathbb{G}^c) \) to the category of \( \mathbb{G}(\mathbb{A}_f) \)-equivariant étale \( \overline{\mathbb{Q}}_p \)-local systems on \( \{\text{Sh}_K\}_K \). Moreover, it is functorial with respect to pullbacks under morphisms between Shimura varieties induced by morphisms between Shimura data.

Suppose that \( V_{\overline{\mathbb{Q}}_p} \) has a model \( V_L \) over a finite extension \( L \) of \( \mathbb{Q}_p \) in \( \overline{\mathbb{Q}}_p \). Let \( k \) be a finite extension of the composite of \( E \) and \( L \) in \( \overline{\mathbb{Q}}_p \), where \( E \) is mapped into \( \overline{\mathbb{Q}}_p \) via \( E \rightarrow \overline{\mathbb{Q}}_p \). Note that \( L \otimes_{\mathbb{Q}_p} k \) is a direct product of \( k \) with some extension fields of \( k \) (cf. [Lan13 Lemma 1.1.2.1]), with \( L \) acting on this distinguished factor \( k \) by the canonical homomorphism \( L \rightarrow k \). The corresponding projection map

\[
(7.18)\quad \tau: L \otimes_{\mathbb{Q}_p} k \rightarrow k,
\]

is just the multiplication homomorphism \( a \otimes b \mapsto ab \). Let us denote with an additional subscript “\( k \)” (resp. “\( \overline{\mathbb{Q}}_p \)” ) the base changes of \( \text{Sh}_K \) etc from \( E \) to \( k \) (resp. \( \overline{\mathbb{Q}}_p \)) via the above composition, and with an additional superscript “an” their \( p \)-adic analytifications. We will adopt a similar notation for sheaves.

We can view the étale local system \( \ell\overline{V}_L \) as an étale local system of \( \mathbb{Q}_p \)-vector spaces with compatible \( L \)-actions. By [LZ17 Theorem 1.2], the analytification \( \ell\overline{V}^{\text{an}}_L \) of its pullback to \( \text{Sh}_{K,k} \) is de Rham. Let \( k_{\text{ét}} \overline{V}^{\text{an}}_L \) denote its extension to \( \text{Sh}^{\text{tor,an}}_{K,k} \) as a Kummer étale local system (as in (6.1)). By applying the functor \( D_{\text{dR},\log} \) introduced in (5.19), and by pushing out via the projection \( \tau: L \otimes_{\mathbb{Q}_p} k \rightarrow k \) as in (7.18), we obtain a \( p \)-adic analytic filtered log connection

\[
(7.19)\quad (p_{\text{dR}}\overline{V}^{\text{can,an}}_k := D_{\text{dR},\log}(k_{\text{ét}} \overline{V}^{\text{an}}_L) \otimes_{(L \otimes_{\mathbb{Q}_p} k),\tau} k, \nabla, \text{Fil}^*),
\]

on \( \text{Sh}^{\text{tor,an}}_{K,k} \), which is the analytification of the algebraic filtered log connection

\[
(7.20)\quad (p_{\text{dR}}\overline{V}^{\text{can}}_k := D_{\text{dR},\log}(k_{\text{ét}} \overline{V}^{\text{can}}_L) \otimes_{(L \otimes_{\mathbb{Q}_p} k),\tau} k, \nabla, \text{Fil}^*)
\]
over $\text{Sh}_{K,k}^\text{tor}$, where $D_{\text{dR}}^{\text{alg}}$ is as in (6.2). The restriction of (7.19) to $\text{Sh}_{K,k}^\text{an}$ is the analytic filtered regular connection
\begin{equation}
(p_{\text{dR}}V^\text{can}_k) := D_{\text{dR}}(\alpha V^\text{can}_L) \otimes (L \otimes \varpi_k), k, V, \text{Fil}^*)
\end{equation}
where $D_{\text{dR}}$ is as in [LZ17, Theorem 3.9], which in turn is the analytification of the algebraic filtered regular connection
\begin{equation}
(p_{\text{dR}}V^\text{can}_k) := D_{\text{alg}}(\alpha V^\text{can}_L) \otimes (L \otimes \varpi_k), k, V, \text{Fil}^*)
\end{equation}
obtained by restricting (7.20) to $\text{Sh}_{K,k}$. Here $D_{\text{alg}}$ is as in Theorem 1.1 and (6.3).

The constructions of these filtered log connections and filtered regular connections are compatible with the replacements of $L$ and $k$ with extension fields in $\mathbb{Q}_p$ satisfying the same conditions. Thus, we can canonically assign to each $a_i V^\text{can}_{\mathbb{Q}_p}$ as above an algebraic filtered log connection
\begin{equation}
(p_{\text{dR}}V^\text{can}_{\mathbb{Q}_p}, V, \text{Fil}^*) := (p_{\text{dR}}V^\text{can}_k, V, \text{Fil}^*) \otimes_k \mathbb{Q}_p
\end{equation}
over $\text{Sh}_{K,\mathbb{Q}_p}^\text{tor}$, whose restriction to $\text{Sh}_{K,\mathbb{Q}_p}$ is the algebraic filtered regular connection
\begin{equation}
(p_{\text{dR}}V_{\mathbb{Q}_p}, V, \text{Fil}^*) := (p_{\text{dR}}V_k, V, \text{Fil}^*) \otimes_k \mathbb{Q}_p.
\end{equation}
Both (7.23) and (7.24) are independent of the choices of $L$ and $k$ for a given $V$.

Since $(p_{\text{dR}}V_{\mathbb{Q}_p}, V, \text{Fil}^*)$ is algebraic, we can consider its base change under the chosen isomorphism $\iota: \mathbb{Q}_p \sim \mathbb{C}$, which is a filtered regular connection
\begin{equation}
(p_{\text{dR}}V^\text{can}_{\mathbb{C}}) := (p_{\text{dR}}V^\text{can}_k, V, \text{Fil}^*)
\end{equation}
over $\text{Sh}_{K,\mathbb{C}}$. Since $D_{\text{dR}}^{\text{alg}}$ is a tensor functor that is uniquely determined by its composition $D_{\text{dR}}$ with the analytification functor (see Lemma 6.4 and its proof), by [LZ17, Theorem 3.9(ii)], we obtain the following:

**Proposition 7.26.** The assignments of $(p_{\text{dR}}V^\text{can}_{\mathbb{Q}_p}, V, \text{Fil}^*)$ and $(p_{\text{dR}}V^\text{can}_{\mathbb{C}})$ to $V$ define tensor functors from $\text{Rep}_{\mathbb{Q}_p}(G^c)$ to the categories of $G(\mathbb{A}_f)$-equivariant filtered regular connections on $\{\text{Sh}_{K,\mathbb{Q}_p}\}_K$ and $\{\text{Sh}_{K,\mathbb{C}}\}_K$, respectively, which are compatible with each other via pullback under $\iota: \mathbb{Q}_p \sim \mathbb{C}$. Moreover, they are functorial with respect to pullbacks under morphisms between Shimura varieties induced by morphisms between Shimura data.

Then we can consider the local system of horizontal sections of the complex analytification $(p_{\text{dR}}V^\text{can}_{\mathbb{C}}, \nabla, \text{Fil}^*)$ of (7.25), which is now a local system $p_{\mathbb{B}}V^\text{can}_{\mathbb{C}}$ of $\mathbb{C}$-vector spaces over $\text{Sh}_{K,\mathbb{C}}^\text{an}$. (The filtration $\text{Fil}^*$ plays no role in the formation of the horizontal sections.) Then we also have the following:

**Proposition 7.27.** The assignment of $p_{\mathbb{B}}V^\text{can}_{\mathbb{C}}$ to $V$ defines a tensor functor from $\text{Rep}_{\mathbb{Q}_p}(G^c)$ to the category of $G(\mathbb{A}_f)$-equivariant $\mathbb{C}$-local systems on $\{\text{Sh}_{K,\mathbb{C}}\}_K$. Moreover, it is functorial with respect to pullbacks under morphisms between Shimura varieties induced by morphisms between Shimura data.

Similarly, since $(p_{\text{dR}}V^\text{can}_{\mathbb{Q}_p}, V, \text{Fil}^*)$ is algebraic, we can consider its base change under $\iota: \mathbb{Q}_p \sim \mathbb{C}$, which is a filtered log connection
\begin{equation}
(p_{\text{dR}}V^\text{can}_{\mathbb{C}}) := (p_{\text{dR}}V^\text{can}_k, V, \text{Fil}^*)
\end{equation}
over $\Sh_{K,C}$. Since the functor $D_{dR, \log}^{alg}$ is uniquely determined by its composition $D_{dR, \log}$ with the analytification functor, and since $\check{\alpha}C$ has unipotent geometric monodromy along the boundary (by comparison with $B_{C}$ as in (7.16), by Theorems 5.20 and 5.24, the analogues of Proposition 7.26 for $(p-dR_{\cdR}^{\can}, \nabla, \Fil^*)$ and $(p-dR_{\cdR}^{\can}, \nabla, \Fil^*)$ also hold.

### 7.4. Statements of theorems

It is natural to ask whether the Betti local systems $p_{B}B_{C}$ and $B_{C}$ (resp. the filtered connections $(p-dR_{\cdR}^{\can}, \nabla, \Fil^*)$ and $(dR_{\cdR}^{\can}, \nabla, \Fil^*)$) over $\Sh_{K,C}$ (resp. $\Sh_{K,C}$) are canonically isomorphic to each other, as in the following summarizing diagram:

```
\begin{align*}
\text{classical RH} & \quad \rightsquigarrow \quad \text{coefficient can. base change via} \\
B_{C} & \quad \Longleftrightarrow \quad V \in \Rep(\G^c) \quad \text{via \ \check{\alpha}C} \\
(p-dR_{\cdR}^{\can}, \nabla, \Fil^*) & \quad \Downarrow \quad \text{p-adic (log) RH} \\
(p-dR_{\cdR}^{\can}, \nabla, \Fil^*) & \quad \Downarrow \quad \text{classical RH} \\
\end{align*}
```

The following theorems provide affirmative (and finer) answers:

**Theorem 7.29.** The Betti local systems $p_{B}B_{C}$ and $B_{C}$ over $\Sh_{K,C}$ are canonically isomorphic to each other. More precisely, the two tensor functors in Propositions 7.14 and 7.27 are canonically and $G(\A_f)$-equivariantly isomorphic to each other. In addition, the isomorphisms are compatible with pullbacks under morphisms between Shimura varieties induced by morphisms of Shimura data.

**Theorem 7.30.** The filtered connections $(p-dR_{\cdR}^{\can}, \nabla, \Fil^*)$ and $(dR_{\cdR}^{\can}, \nabla, \Fil^*)$ over $\Sh_{K,C}$ are canonically isomorphic to each other. More precisely, the two tensor functors in Propositions 7.14 and 7.26 are canonically and $G(\A_f)$-equivariantly isomorphic to each other. In addition, the isomorphisms are compatible with pullbacks under morphisms between Shimura varieties induced by morphisms of Shimura data. Consequently, when $V_C$ has a model over a finite extension $E_V$ of $E$, by the theory of canonical models of automorphic vector bundles as in [Mil90] Chapter III, Theorem 5.1, $(p-dR_{\cdR}^{\can}, \nabla)$ also admits a canonical model over $E_V$, given by that of $(dR_{\cdR}^{\can}, \nabla)$. The analogous assertions for the filtered log connections $(p-dR_{\cdR}^{\can}, \nabla, \Fil^*)$ and $(dR_{\cdR}^{\can}, \nabla, \Fil^*)$ are also true.

The proofs of Theorems 7.29 and 7.30 will be given in the remaining subsections. Assuming these two theorems for the moment, since every irreducible algebraic representation of $G^c$ over $\Q_p$ has a model over $\Q_p$, we obtain the following:

**Corollary 7.31.** Theorem 1.3 also holds.

**Remark 7.32.** It follows from Theorems 7.29 and 7.30 that the assignments of $(p-dR_{\cdR}^{\can}, \nabla)$ and $(dR_{\cdR}^{\can}, \nabla)$ to $V$ define canonically isomorphic $G^c$-torsors with integrable connections over $\Sh_{K,C}$, which descend to canonically isomorphic $G^c$-torsors over $\Sh_{K}$, one being the standard principle $G^c$-torsor over $\Sh_{K}$ as in [Mil90] Chapter III, Section 4]. This verifies the conjecture in [LZ17, Remark 4.2(ii)].
Similarly, it follows from Theorem 7.30 that the assignments of \((p\text{-dR} V_C, \nabla, \Fil^*)\) and \((\text{dR} V_C, \nabla, \Fil^*)\) (now also with filtrations) to \(V\) define \(P_C\)-torsors over \(\Sh_{K, C}\), where \(P_C\) is the parabolic subgroup of \(G_C\) defined by some \(\mu_h\) as in (7.13) (cf. [LZ17, Remark 4.1(i)]).

**Remark 7.33.** Theorems 7.29 and 7.30 are not surprising when there are families of motives whose relative Betti, de Rham, and \(p\)-adic étale realizations define the three kinds of local systems \(\mu_{\text{B}} V_C\), \((\text{dR} V_C, \nabla, \Fil^*)\), and \((\text{ét} V_{\mathbb{Q}_p})\). This is the case, for example, when \(\Sh_K\) is a Shimura variety of PEL type, or more generally of Hodge type. (We will take advantage of this in Section 7.6 below.) But Theorem 7.29 also applies to Shimura varieties associated with exceptional groups, over which there are (as yet) no known families of motives defining our three kinds of local systems.

**Remark 7.34.** By considering the de Rham cohomology, log de Rham cohomology, and log Hodge cohomology associated with the above filtered regular connections and filtered log connections as usual (cf. Theorem 5.20(1)), by Theorem 7.30, and by Deligne’s comparison result in [Del70, II, 6], we have the following Hecke-equivariant commutative diagram of canonical isomorphisms and spectral sequences

\[
\begin{array}{ccc}
E_1^{a, b} = H^{a, b}_{\log \text{Hodge}}(\Sh_{\text{tor} K, \mathbb{Q}_p}, p\text{-dR} V_{\mathbb{Q}_p}^\text{can}) & \overset{i}{\sim} & E_1^{a, b} = H^{a, b}_{\log \text{Hodge}}(\Sh_{K, C}, \text{dR} V_{\mathbb{Q}_p}^\text{can}) \\
H^{a+b}_{\text{dR}}(\Sh_{\text{tor} K, \mathbb{Q}_p}, p\text{-dR} V_{\mathbb{Q}_p}^\text{can}) & \overset{i}{\sim} & H^{a+b}_{\text{dR}}(\Sh_{K, C}, \text{dR} V_{\mathbb{Q}_p}^\text{can}) \\
H^{a+b}_{\text{dR}}(\Sh_{K, \mathbb{Q}_p}, \text{dR} V_{\mathbb{Q}_p}^\text{can}) & \overset{i}{\sim} & H^{a+b}_{\text{dR}}(\Sh_{K, C}, \text{dR} V_{\mathbb{Q}_p}^\text{can})
\end{array}
\]

in which the spectral sequence at the left-hand side degenerates at the \(E_1\) page by Theorem 6.6. Note that the Hecke actions on log Hodge and log de Rham cohomology are defined, as usual, up to refining the choices of cone decompositions used in the definition of toroidal compactifications. (Such refinements of cone decompositions induce log étale morphisms between the toroidal compactifications. (Such refinements of cone decompositions induce log étale morphisms between the toroidal compactifications, which induce isomorphisms on the log Hodge and log de Rham cohomology are defined, as usual, up to refining the choices of cone decompositions used in the definition of toroidal compactifications. (Such refinements of cone decompositions induce log étale morphisms between the toroidal compactifications, which induce isomorphisms on the log Hodge and log de Rham cohomology. See [Har90], Section 2) or rather just the arguments there based on the results in [KKMSD73, Chapter I, Section 3].) Consequently, the spectral sequence at the right-hand side also degenerates at the \(E_1\) page. Although this latter degeneration (for cohomology over \(\mathbb{C}\)) was known thanks to Saito’s theory of mixed Hodge modules (or more precisely his direct image theorem; see [Sai90, Theorem 2.14]), our Theorems 6.6 and 7.30 provide a new proof based on \(p\)-adic Hodge theory, not the conventional complex analytic mixed Hodge theory. Moreover, they determine the Hodge–Tate weights (to be explained) of \(H^i(\Sh_{K, \mathbb{Q}_p}, \text{dR} V_{\mathbb{Q}_p}^\text{can})\) in terms of \(\dim_{\mathbb{C}} H^{a, b}_{\log \text{Hodge}}(\Sh_{K, C}, \text{dR} V_{\mathbb{Q}_p}^\text{can})\), for \(a, b \in \mathbb{Z}\) such that \(a + b = i\), and these dimensions can be computed using the dual BGG decomposition and certain relative Lie algebra cohomology—see Section 7.8 below for more details.

7.5. **Proofs of theorems: preliminary reductions.** Let us fix a connected component \(\Gamma_{K, \mathbb{Q}_p}^+ \backslash X^+\) of \(\Sh_{K, \mathbb{C}}^\text{an}\) as in (7.5), which is the analytification of a quasi-projective variety defined over some finite extension \(E^+\) of \(E\) in \(\overline{\mathbb{Q}}\). Let \(h \in X^+\) be
a special point whose corresponding homomorphism \( \overline{\text{7.1}} \) factors through \( T_\mathbb{R} \) for some maximal torus \( T \) of \( G \) over \( \mathbb{Q} \). (Recall that special points exist and are dense in \( X^+ \). See, for example, the proof of \cite[Lemma 13.5]{MR2518490}.) Up to replacing \( E^+ \) with a finite extension in \( \overline{\mathbb{Q}} \), we may assume that the image of \( h \in X^+ \) in \( \Gamma^+_{K,90} \backslash X^+ \) is defined over \( E^+ \) (see \cite[Lemma 13.4]{MR2518490}).

The pullbacks of \( bV_C \) to \( h \in X^+ \) can be canonically identified with \( V_C \) by its very construction. On the other hand, the pullback of \( p_BV_C \) can also be canonically identified with \( V_C \). In fact, in both cases, we have slightly more:

**Proposition 7.35.** The pullbacks of \( bV_C \) and \( p_BV_C \) to \( (G(\mathbb{Q})h) \times G(\mathcal{A}_f) \) are canonically, \( G(\mathbb{Q}) \)-equivariantly (with \( G(\mathbb{Q}) \) acting by diagonal left multiplication on \( (G(\mathbb{Q})h) \times G(\mathcal{A}_f) \) and via \( G(\mathcal{Q}) \) on \( V_C \)), and \( G(\mathcal{A}_f) \)-equivariantly (with \( G(\mathcal{A}_f) \) acting by right multiplication on \( G(\mathcal{A}_f) \) and trivially on \( V_C \)) isomorphic to the trivial local system \( V_C \) over \( (G(\mathbb{Q})h) \times G(\mathcal{A}_f) \).

**Proof.** The assertion for \( bV_C \) follows from its very construction. As for the assertion for \( p_BV_C \), let us first identify the pullback of \( \overline{\text{7.36}} \) to the images of \((h,g)\), for \( g \in G(\mathcal{A}_f) \), by recalling the arguments in the proof of \cite[Lemma 4.8]{LZ17}.

By assumption, \( h : S = \text{Res}_{E/\mathbb{Q}} G_{m,\mathbb{C}} \rightarrow G_\mathbb{R} \) (as in \( \text{7.1} \)) factors through \( T_\mathbb{R} \), and the Hodge cocharacter (as in \( \text{7.13} \)) induces a cocharacter \( \mu_h : G_{m,\mathbb{C}} \rightarrow T_\mathbb{C} \), which is the base change of some cocharacter

\[
\begin{align*}
\mu : G_{m,F} &\rightarrow T_F \\
\end{align*}
\]

defined over some number field \( F \) in \( \overline{\mathbb{Q}} \). Then the composition of \( \mu \) with the norm map from \( T_F \) to \( T \) defines a homomorphism

\[
(7.36) \quad N \mu : \text{Res}_{F/\mathbb{Q}} G_{m,F} \rightarrow T
\]

of tori over \( \mathbb{Q} \), and we have a composition of homomorphisms

\[
F^\times \backslash \mathcal{A}_F^\times \xrightarrow{7.36} \overline{T(\mathbb{Q})} \backslash T(\mathcal{A}) \rightarrow \overline{T(\mathbb{Q})} \backslash T(\mathcal{A}_f),
\]

where \( \overline{T(\mathbb{Q})} \) denotes the closure of \( T(\mathbb{Q}) \) in \( T(\mathcal{A}_f) \), which factors through

\[
(7.37) \quad F^\times \backslash \mathcal{A}_F^\times \xrightarrow{\text{Art}_F} \text{Gal}(\overline{F}/F) \xrightarrow{r(\mu)} \overline{T(\mathbb{Q})} \backslash T(\mathcal{A}_f),
\]

where \( F^{ab} \) is the maximal abelian extension of \( F \) in \( \overline{\mathbb{Q}} \). If \( F_K \) is the subfield of \( F^{ab} \) such that \( \text{Gal}(F^{ab}/F_K) \) is the preimage of \( (K \cap \overline{T(\mathbb{Q})})/(K \cap \overline{T}(\mathcal{A}_f)) \) under \( (7.37) \), then \( F_K \subset E^+ \) by the assumption that the image of \( h \) in \( \Gamma^+_{K,90} \backslash X^+ \) is defined over \( E^+ \), and we have an induced Galois representation

\[
(7.38) \quad r(\mu)^* : \text{Gal}(\overline{\mathbb{Q}}/E^+) \rightarrow (K \cap \overline{T(\mathbb{Q})})/(K \cap \overline{T}(\mathcal{A}_f)).
\]

Since \( T_\mathbb{R} \) stabilizes the special point \( h \), it is \( \mathbb{R} \)-anisotropic modulo the center of \( G \), and hence its maximal \( \mathbb{Q} \)-anisotropic \( \mathbb{R} \)-split subtorus is the same as that of \( G \). Therefore, as explained in the proof of \cite[Lemma 4.5]{LZ17}, the pullback of \( V \) to \( T \) satisfies the requirement that its restriction to \( K \cap \overline{T(\mathbb{Q})} \) is trivial as in \cite[4.4]{LZ17} (with the neatness of \( K \) here implying that the open compact subgroup \( K \) there is sufficiently small). Thus, the composition

\[
T(\mathcal{A}_f) \rightarrow T(\mathbb{Q}_p) \rightarrow G(\mathbb{Q}_p) \rightarrow GL_{\mathbb{Q}_p}(V_{\mathbb{Q}_p}^\vee)
\]
factors through \((K \cap T(\mathbb{Q})) \backslash T(\mathbb{A}_f)\) and induces, by composition with \((7.38)\), a Galois representation
\[
r(\mu, V)^+_{K,p} : \text{Gal}(\mathbb{Q}/E^+) \to \text{GL}_{\mathbb{Q}_p}(V_{\mathbb{Q}_p}^e),
\]
which describes the pullback of \(\iota V_{\mathbb{Q}_p}^e\) to the geometric point above the image of the pullback of \(E^+\) via \(\text{Gal}(\mathbb{Q}/E^+)\). Let \(L\), \(V_L\), \(k\), and \(\tau\) be as in Section 7.3. Without loss of generality, we may assume that \(k\) also contains the image of \(E^+\) in \(\mathbb{Q}_p\) (via the above composition). Then the image of \(r(\mu, V)^+_{K,p}\) is contained in the subgroup \(\text{GL}_{\mathbb{Q}_p}(V_L)\) of \(\text{GL}_{\mathbb{Q}_p}(V_{\mathbb{Q}_p})\), and we can view this representation over \(L\) as a representation over \(\mathbb{Q}_p\) with an additional action of \(L\), as usual. By [LZ17, Lemma 4.4], this representation is potentially crystalline. Since this representation factors through the abelian group \(T(\mathbb{Q}_p)\), by [FM97, §6, Proposition] (which is based on [Ser68, Chapter III, Section A.7, Theorem 3] and [DMOSS2, Chapter IV, Proposition D.1]; see also [Hen82]), it is the \(p\)-adic étale realization of an object in the Tannakian subcategory of motives over \(E^+\) generated by Artin motives and potentially CM abelian varieties. By construction (and by the fact that, over a classical point, \(D_{\text{dr}}\) is the familiar \(D_{\text{DR}}\) in classical \(p\)-adic Hodge theory), the pullback of \(p_{-\text{DR}}V_C^e\) to \(h\) is the pushout of the corresponding de Rham realization via \(L \otimes \mathbb{Q}_p \hookrightarrow \mathbb{Q}_p\) \(\xrightarrow{\iota} \mathbb{Q}_p \xrightarrow{\iota} \mathbb{C}\). Hence, by comparison with the Betti realization, we can canonically identify the pullback of \(p_{-\text{B}}V_C^e\) to \(h\) with \(V_C\).

By exactly the same argument as above, we can also canonically identify the pullback of \(p_{-\text{B}}V_C^e\) to \(\gamma h\) with \(V_C\). When put together as a canonical identification over the whole \((G(\mathbb{Q})h) \times G(\mathbb{A}_f)\), it is \(G(\mathbb{Q})\)-equivariant because the pointwise comparisons are canonically made by comparison with the Betti realizations, and is \(G(\mathbb{A}_f)\)-equivariant because it does not involve the second factor \(G(\mathbb{A}_f)\) at all.

**Proposition 7.39.** Suppose Theorem 7.29 holds. Then Theorem 7.30 also holds.

**Proof.** Under the assumption that Theorem 7.29 holds, \((p_{-\text{DR}}V_C^e, \nabla)\) and \((\text{dr}V_C^e, \nabla)\) are canonically isomorphic to each other over \(\text{Sh}_{K,C}\), because they admit extensions \((p_{-\text{DR}}V_C^{\text{can}}, \nabla)\) and \((\text{dr}V_C^{\text{can}}, \nabla)\) over \(\text{Sh}_{K,C}^{\text{tor}}\) (and hence have regular singularities along the boundary \(D_C = \text{Sh}_{K,C}^{\text{tor}} - \text{Sh}_{K,C}\)). Since the eigenvalues of the residues of \((p_{-\text{DR}}V_C^{\text{can}}, \nabla)\) along \(D_C\) are in \(\mathbb{Q} \cap [0,1]\), and since (as explained in Section 7.2.2) the eigenvalues of the residues of \((\text{dr}V_C^{\text{can}}, \nabla)\) along \(D_C\) are zero and hence in \(\mathbb{Q} \cap [0,1]\), these two extensions of \((p_{-\text{DR}}V_C^{\text{can}}, \nabla)\) and \((\text{dr}V_C^{\text{can}}, \nabla)\) are also canonically isomorphic to each other (by [AB04, Chapter 1, Theorem 4.9]). To verify that the filtrations are respected by such isomorphisms, it suffices to do so at the special points, or just at the arbitrary special point \(h\) we have chosen, because special points are dense in the complex analytic topology (see the proof of [Mil05, Lemma 13.5]).

Consider the Galois representation \(r(\mu, V)^+_{K,p}\) in the proof of Proposition 7.35. By decomposing the representation \(V_{\mathbb{Q}_p}\) of \(T(\mathbb{Q}_p)\) into a direct sum of characters \(T(\mathbb{Q}_p) \to \mathbb{Q}_p^\times\), we obtain a corresponding decomposition of \(r(\mu, V)^+_{K,p}\) into a direct sum of characters \(\text{Gal}(\mathbb{Q}/E^+) \to \mathbb{Q}_p^\times\). By construction, the restriction of these characters of \(\text{Gal}(\mathbb{Q}/E^+)\) to the decomposition group at the place \(v\) of \(E^+\) given by the composition \(E^+ \xrightarrow{\text{can}} \mathbb{Q} \xrightarrow{\iota} \mathbb{Q}_p\) are locally algebraic (in the sense of [Ser68].
Chapter III, Section 1.1, Definition]), because they are induced (up to a sign convention) by the composition of the local Artin map, the cocharacter $\overline{\mu}_p \to T(\overline{\mu}_p)$ given by the base change of $\mu$ under the same $E^+ \to \overline{\mu}_p$, and the corresponding characters of $T(\overline{\mu}_p)$. Thus, the Hodge filtrations on the pullbacks of $\rho_{K,g}^+(V)$ and $\rho_{K,g}^+(V)$ to $h$ are both determined by the Hodge cocharacter $\mu_h : G_{m,C} \to G_C$.

The remainder of Theorem 7.30 then follows from the descent argument in the proofs of [Mil90] Chapter III, Theorem 5.1, and Chapter V, Theorem 6.2.

Thanks to Proposition 7.39, it only remains to prove Theorem 7.29 from now on. As explained before (cf. (7.10)), by taking $X^+$ to be the universal covering of $\Gamma_{K,g_0}^+ \backslash X^+$ and by taking the chosen special point $h \in X^+$ as a base point, the pullbacks of the local systems $p_B V_{C}$ and $B V_{C}$ to $\Gamma_{K,g_0}^+ \backslash X^+$ determine and are determined by the fundamental group representations

$$\rho_{K,g_0}^+(V) : \Gamma_{K,g_0}^+ \to GL_C(V_C)$$

and

$$\rho_{K,g_0}^+(V) : \Gamma_{K,g_0}^+ \to GL_C(V_C),$$

respectively, by canonically identifying the pullbacks of the local systems $p_B V_{C}$ and $B V_{C}$ to $\Gamma_{K,g_0}^+ \backslash X^+$ using Proposition 7.35. Then it suffices to show that $\rho_{K,g_0}^+(V)$ and $\rho_{K,g_0}^+(V)$ coincide as representations of $\Gamma_{K,g_0}^+$. In this case, they are isomorphic via the identity morphism on $V_C$, and therefore the choice of such an isomorphism is functorial in $V$ and compatible with tensor products and duals, because the assignment of $\rho_{K,g_0}^+(V)$ to $V$ is. Moreover, such isomorphisms are compatible with Hecke actions because of Proposition 7.35, and with morphisms between Shimura varieties induced by morphisms of Shimura data, because all constructions involved are.

Lemma 7.40. It suffices to show that $\rho_{K,g_0}^+(V)$ extends to an algebraic representation of $G_{der,c}^+$ that coincides with the representation $V_C|_{G_{der,c}^+}$ of $G_{der,c}^+$ on $V_C$.

Proof. This is because $\Gamma_{K,g_0}^+$ is contained in $G_{der,c}^+(Q)$, and the representation $\rho_{K,g_0}^+(V)$ depends only on the restriction $V_C|_{G_{der,c}^+}$ of $V_C$ to $G_{der,c}^+$.

Remark 7.41. Such an extension is necessarily unique, because arithmetic subgroups of semisimple groups without noncompact $\mathbb{Q}$-simple factors are Zariski dense, by the Borel density theorem; see [Bor60] Lemma 1.4 and Corollary 4.3 and [BHC62] Theorem 7.8.

Lemma 7.42. It suffices to prove Lemma 7.40 in the special case where $G_{der}$ and $G_{der,c}^+$ are $\mathbb{Q}$-simple and simply-connected as algebraic groups over $\mathbb{Q}$.

Proof. Let us start with an arbitrary $G_{der}$ that is not necessarily simply-connected. By [Del79] Lemma 2.5.5], there exists a connected Shimura datum with the semisimple algebraic group over $\mathbb{Q}$ being the simply-connected covering $\tilde{G}$ of $G_{der}$. Moreover, by the very definition of connected Shimura data, we have a decomposition $\tilde{G} \cong \prod_{i \in I} \tilde{G}_i$ of $\tilde{G}$ into a product of its $\mathbb{Q}$-simple factors such that each $\tilde{G}_i$ is part of a connected Shimura datum. Suppose the analogue of Lemma 7.40 for $\tilde{G}_i$ is known for each $i \in I$, and suppose $\Gamma$ is any arithmetic subgroup of $G(Q)$ of the form $\Gamma = \prod_{i \in I} \tilde{\Gamma}_i$ for some neat arithmetic subgroups $\tilde{\Gamma}_i$ of $\tilde{G}_i(Q)$ such that its
image $\Gamma$ in $G^{\text{der}, c}(\mathbb{Q})$ is a normal subgroup of $\Gamma_{K,0}^{+,(c)}$ (of finite index). Since $\widetilde{\Gamma}$ is neat, it maps isomorphically into $G^{\text{der}}(\mathbb{Q})$, $G^{\text{der},c}(\mathbb{Q})$, and $G^{\text{ad}}(\mathbb{Q})$, because $G^{\text{ad}}$ is the common adjoint quotient of $G$, $G^{\text{der}}$, and $\widetilde{G}$. Then the restriction of $\rho_{K,0}^{+,(p)}$ to $\widetilde{\Gamma}$ lifts to a representation of $\Gamma$, which (by assumption) extends to an algebraic representation of $G$ that coincides with the pullback of $V_C$. This algebraic representation is trivial on $\ker(\widetilde{G} \to G^{\text{der}, c})$ and hence descends back to the algebraic representation $V_{C|G^{\text{der}, c}}$ of $G^{\text{der}, c}$.

For simplicity of notation, let $\sigma : \Gamma_{K,0}^{+,(c)} \to \text{GL}_C(V_C)$ denote the homomorphism given by the restriction of $V_{C|G^{\text{der}, c}}$, let $\tau := \rho_{K,0}^{+,(p)}(V)$, and let $\delta : \Gamma_{K,0}^{+,(c)} \to \text{GL}_C(V_C)$ be the map defined by

$$\delta(\gamma) := \sigma(\gamma)^{-1} \tau(\gamma),$$

for all $\gamma$. For any $\gamma \in \Gamma_{K,0}^{+,(c)}$ and $\gamma' \in \Gamma$, we have $\gamma' \gamma^{-1} \in \Gamma$, and hence

$$\delta(\gamma) \sigma(\gamma') \delta(\gamma)^{-1} = \sigma(\gamma)^{-1} \tau(\gamma) \tau(\gamma') \tau(\gamma')^{-1} \sigma(\gamma) = \sigma(\gamma)^{-1} \tau(\gamma' \gamma^{-1}) \sigma(\gamma) = \sigma(\gamma)^{-1} \sigma(\gamma' \gamma^{-1}) \sigma(\gamma) = \sigma(\gamma'),$$

which shows that $\delta(\Gamma_{K,0}^{+,(c)})$ commutes with $\sigma(\Gamma)$. But since $\sigma$ is the restriction of the algebraic representation $V_{C|G^{\text{der}, c}}$, and since $\Gamma$ is Zariski dense in $G^{\text{der}, c}$ by the Borel density theorem, as in Remark 7.41, it follows that $\delta(\Gamma_{K,0}^{+,(c)})$ commutes with the whole $\sigma(\Gamma_{K,0}^{+,(c)})$. Then $\delta : \Gamma_{K,0}^{+,(c)} \to \text{GL}_C(V_C)$ is a group homomorphism, because

$$\delta(\gamma') = \sigma(\gamma')^{-1} \tau(\gamma') = \sigma(\gamma')^{-1} \tau(\gamma) \tau(\gamma') = \sigma(\gamma')^{-1} \delta(\gamma) \sigma(\gamma') \delta(\gamma) = \delta(\gamma) \delta(\gamma'),$$

for all $\gamma, \gamma' \in \Gamma_{K,0}^{+,(c)}$, which factors through the finite quotient $\Gamma_{K,0}^{+,(c)}/\Gamma$ by construction. (We learned these arguments from the proof of [Mar91], Chapter VII, p. 224, Lemma 5.1, although we have slightly modified them so that we do not need the additional assumptions there.)

By Proposition 7.27, without loss of generality, we may assume that $V$ and hence $V_C$ are irreducible. By Schur's lemma (see, for example, [Kna02], Proposition 5.1 and Corollary 5.2, which is applicable because the base field of $V_C$ is $\mathbb{C}$), the finite image of $\delta$ lies in the roots of unity in $\mathbb{C}^\times$, which is then trivial because $\Gamma_{K,0}^{+,(c)}$ and hence $\Gamma_{K,0}^{+,(c)}$ are neat. Thus, Lemma 7.40 also holds for $G^{\text{der}, c}$, as desired. \hfill $\Box$

Consequently, in what follows, we may and we shall assume that $G^{\text{der}, c}$ is simply-connected as an algebraic group over $\mathbb{Q}$, so that $G^{\text{der}} \cong G^{\text{der}, c}$ also is.

### 7.6. Cases of real rank one, or of abelian type

In this subsection, we assume that $G^{\text{der}, c}$ is $\mathbb{Q}$-simple and simply-connected as an algebraic group over $\mathbb{Q}$ (so that $G^{\text{der}} \cong G^{\text{der}, c}$), and that $\text{rk}_R(G^{\text{der}}) \leq 1$.

**Lemma 7.43.** Under the above assumptions, the Shimura datum $(G, X)$ is necessarily of abelian type (see, for example, [Lan17], Definition 5.2.2.1). Up to replacing $G$ with another group with the same derived group $G^{\text{der}}$, we may assume that $(G, X)$ is of Hodge type (see, for example, [Lan17], Definition 5.2.1.1).

**Proof.** Let $G^{\text{der}}(\mathbb{R})_{\text{nc}}$ denote the product of all noncompact simple factors of $G^{\text{der}}(\mathbb{R})$ (as a real Lie group). According to the classification of Hermitian symmetric domains (see [Hel01], Chapter X, Section 6); see also the summaries in [Mil05], Section
dual abelian scheme, whose over \( Sh \) such that, for all \( i \)

\[ (7.44) \]

(See the classification in [Kna02] Chapter VI, Theorem 6.105, (6.107), (6.108),
and (6.110) for the ranks and for the accidental isomorphisms in low ranks.) In
each of these cases, there is a unique Hermitian symmetric domain associated with
\( G \) as above is necessarily of abelian type, according to the classification in [Del79] 2.3 (see also the summary in [Lan17],
Section 5.2.2]). By the very definition of a Shimura datum of abelian type, there
exist a Shimura datum \((G, X_1)\) of Hodge type and an isogeny \( G_{1}^{\text{der}} \to G^{\text{der}} \). Since
\( G^{\text{der}} \) is simply-connected by assumption, the isogeny \( G_{1}^{\text{der}} \to G^{\text{der}} \) is necessarily an
isomorphism, as desired.

Consequently, for our purpose, we may assume that the Shimura datum \((G, X)\) is
of Hodge type. Note that \( G \cong G^c \) in this case. Moreover, there exists some faithful
representation \( V_0 \) of \( G \cong G^c \) over \( \overline{\mathbb{Q}} \), together with an alternating pairing

\[ (7.44) \]

(\( -1 \)) denotes the formal Tate twist, which are defined by some Siegel embed-
ding in the definition of a Shimura datum of Hodge type, together with an abelian
scheme \( f : A \to Sh_K \) with a polarization \( \lambda : A \to A^\vee \) over \( Sh_K \), where \( A^\vee \) is the
dual abelian scheme, whose \( m \)-fold self-fiber product we denote by \( f^m : A^m \to Sh_K \),
such that, for all \( i \geq 0 \), we have the following canonical isomorphisms:

\[ R^i_{\text{et}, s}(\overline{\mathbb{Q}}) \cong \wedge^i (\kappa_0 V_0^m) \]

over \( Sh_{K, \mathbb{C}} \);

\[ R^i_{\text{et}, s}(\overline{\mathbb{Q}}_p) \cong \wedge^i (\kappa_0 V_0^m) \]

over \( Sh_K \), where \( V_0_{\mathbb{Q}_p} := V_0 \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_p \) and

\[ (7.44) \]

(\( R^i_{\text{et}, s}(\Omega^m_{A^m/\mathbb{Q}_p}) \otimes_E \mathbb{C}, \nabla, Fil^* \) \( \cong (\wedge^i (\kappa_0 V_0^m), \nabla, Fil^* \)

over \( Sh_{K, \mathbb{C}} \), where \( V_0_{\mathbb{C}} := V_0 \otimes_{\mathbb{Q}} \mathbb{C} \), where the connection \( \nabla \) and the filtration \( Fil^* \)
at the left-hand side are given by the Gauss–Manin connection and the relative
Hodge filtration, respectively (see, for example, [Kat72] Sections 1.2–1.4). The
polarization \( \lambda \) then compatibly induces (as in [DP94] 1.5) the alternating pairings

\[ b\nabla_0 \times b\nabla_0 \to b\overline{\mathbb{Q}}(-1), \]

\[ (7.46) \]

\( \kappa_0 V_0_{\mathbb{Q}_p} \times \kappa_0 V_0_{\mathbb{Q}_p} \to \kappa_0 \overline{\mathbb{Q}}_p(-1), \]

and

\[ (7.47) \]

defined by \( (7.44) \), with \( (-1) \) denoting the Tate twists in the respective categories.

By Lieberman’s trick (see, for example, [LS12] Section 3.2], for each integer
\( m \geq 0 \), there exists an endomorphism \( \varepsilon_m \) of the abelian scheme \( f^m : A^m \to Sh_K \)
such that, if we abusively denote by \( \varepsilon^*_m \), the images of the induced endomorphisms, then we have the canonical isomorphisms

\[
\varepsilon^*_m \left( R^{\text{et}}_{\ast} \left( \Omega^*_{m/\text{Sh}_K} \right) \right) \cong B_{\mu}^{\otimes m},
\]

\[
\varepsilon^*_m \left( R^{\text{et}}_{\ast} \left( \Omega^*_{p} \right) \right) \cong \text{ét} V^{\otimes m}_{0/\mathbb{Q}_p},
\]

and

\[
\varepsilon^*_m \left( R^{\text{et}}_{\ast} \left( \Omega^*_{\ast} \right) \right) \cong \text{ét} V^{\otimes m}_{0/\mathbb{Q}_p},
\]

(7.48) \( \varepsilon^*_m \left( R^{\text{et}}_{\ast} \left( \Omega^*_{m/\text{Sh}_K} \right) \right) \otimes_{E} \mathbb{C}, \nabla, \text{Fil}^\bullet \). \]

**Lemma 7.49.** We also have canonical isomorphisms

(7.50) \( R^{\ast}_{\ast} \left( \Omega^*_{m/\text{Sh}_K} \right) \otimes \mathbb{C}, \nabla, \text{Fil}^\bullet \cong \left( \Lambda^i \left( p^{\text{dR}} V^{\otimes m}_{0/\mathbb{C}} \right), \nabla, \text{Fil}^\bullet \right), \]

and

(7.51) \( R^{\ast}_{p} \left( \Omega^*_{m/\text{Sh}_K} \right) \otimes \mathbb{C}, \nabla, \text{Fil}^\bullet \cong \left( p^{\text{dR}} V^{\otimes m}_{0/\mathbb{C}} \otimes \mathbb{C}, \nabla, \text{Fil}^\bullet \right), \]

for all \( i \geq 0 \) and \( m \geq 0 \). Moreover, the polarization \( \lambda \) induces the alternating pairing

(7.52) \( p^{\text{dR}} V^{\otimes m}_{0/\mathbb{C}} \times p^{\text{dR}} V^{\otimes m}_{0/\mathbb{C}} \rightarrow p^{\text{dR}} \mathbb{C}(-1) \]

defined by the base change of (7.44).

**Proof.** The first statement is because \( p \)-adic analytification induces a fully faithful functor from the category of algebraic filtered connections to the category of \( p \)-adic analytic filtered connections, by [AB01 Chapter 4, Corollary 6.8.2] (which is applicable because \( \text{Sh}_K \) is defined over \( E \)); because the \( p \)-adic analytification of \( R^{\ast}_{\ast} \left( \Omega^*_{m/\text{Sh}_K} \right) \otimes_k \mathbb{C}, \nabla, \text{Fil}^\bullet \) is canonically isomorphic to \( D_{\text{dR}} \left( R^{\ast}_{k,\text{ét}}, \left( \mathbb{Q}_p \right) \right), \nabla, \text{Fil}^\bullet \) for any finite extension \( k \) of the composite of \( E \) and \( \mathbb{Q}_p \) in \( \mathbb{Q}_p \), by [Sch13 Theorems 8.8 and 9.1]; and because such an isomorphism is functorial in the schemes involved, and hence is compatible with Lieberman’s trick. The second statement is because (7.46) is induced by (7.44). \( \square \)

**Corollary 7.53.** We have canonical isomorphisms

(7.54) \( \left( \Lambda^i \left( p^{\text{dR}} V^{\otimes m}_{0/\mathbb{C}} \right), \nabla, \text{Fil}^\bullet \right) \)

and

(7.55) \( \left( p^{\text{dR}} V^{\otimes m}_{0/\mathbb{C}}, \nabla, \text{Fil}^\bullet \right) \)

for all \( i \geq 0 \) and \( m \geq 0 \). Accordingly, we have canonical isomorphisms

(7.56) \( \Lambda^i \left( B_{\mu}^{\otimes m} \right) \cong \Lambda^i \left( p^{\text{dR}} V^{\otimes m}_{0/\mathbb{C}} \right) \)

and

(7.57) \( B_{\mu}^{\otimes m} \cong p^{\text{dR}} V^{\otimes m}_{0/\mathbb{C}} \).

These are compatibly induced by their special cases when \( i = 1 \) and \( m = 1 \), and are compatible with the canonical pairings (7.47) and (7.52) (both of which can be defined by the polarization \( \lambda \)). Moreover, the pullback of the isomorphisms (7.56) and (7.57) to the image of the special point \( h \in X^+ \) in \( \Gamma^+_{K_0,0} X^+ \) (see the beginning of Section 7.5), which is defined over a finite extension \( E^+ \) of \( E \) in \( \mathbb{Q} \), is given by the identity morphisms on \( \Lambda^i \left( V^{\otimes m}_{0/\mathbb{C}} \right) \) and \( V^{\otimes m}_{0/\mathbb{C}} \), respectively.
Proof. The isomorphism \(7.54\) is the composition \(7.50 \circ 7.45^{-1}\). The isomorphism \(7.55\) is the composition \(7.51 \circ 7.48^{-1}\). Then the isomorphisms \(7.56\) and \(7.57\) follow from \(7.54\) and \(7.55\), respectively, by taking horizontal sections. These isomorphisms are compatibly induced by their special cases when \(i = 1\) and \(m = 1\), because the relative cohomology of abelian schemes are exterior powers of the first relative cohomology. Finally, the pullbacks of these isomorphisms to the image of \(h\) are the identity morphisms, because the comparison isomorphisms among the Betti, étale, and de Rham cohomology of an abelian variety defined over \(E^+\) are all compatible with each other. \(\square\)

Lemma 7.58. For each irreducible representation \(V\) of \(G\) over \(\overline{\mathbb{Q}}\), there exist integers \(m_v \geq 0\) and \(t_v\) (depending noncanonically on \(V\)) such that \(V\) is a direct summand of \(V_0^\otimes m_v (-t_v)\), where \((-t_v)\) denotes the formal Tate twist (which has no effect when restricted to the subgroup \(G^\text{der}\) of \(G\)), so that \(V = s\_V(V_0^\otimes m_v (-t_v))\) for some Hodge tensor \(s\_V \in V_0^\otimes\) (i.e., a tensor of weight \((0, 0)\) with respect to the induced Hodge structure on \(V_0^\otimes\) ; cf. [DMOSS2] Chapter I, Proposition 3.4).

Proof. See [LS18] Proposition 3.2], whose argument is based on [DMOSS2] Chapter I, Proposition 3.1(a)]. \(\square\)

Remark 7.59. Since \((G, X)\) is a Shimura datum of Hodge type, the reciprocal of the restriction of the homomorphisms \(h\) parameterized by \(X\) (see (7.1)) to the subtorus \(G_{m, \mathbb{R}}\) of \(S\) descends to a homomorphism \(w : G_{m, \mathbb{Q}} \to G\), which induces a homomorphism \(G_{m, \mathbb{Q}} \to G_{\mathbb{Q}}\), which makes \(V_0\) a representation of weight \(-1\), and the pairing \((7.44)\) makes \(\mathbb{Q}(-1)\) a representation of weight \(-2\). Then it follows that an irreducible representation \(V\) of \(G\) over \(\mathbb{Q}\) can be direct summands of both \(V_0^\otimes m (-t)\) and \(V_0^\otimes m’(-t’)\) exactly when \(m + 2t = m' + 2t'\).

Corollary 7.60. For each irreducible representation \(V\) of \(G\) over \(\overline{\mathbb{Q}}\), there exist some integers \(m_v \geq 0\) and \(t_v\) (depending noncanonically on \(V\)) such that the local systems \(b\_V\_C\) and \(p\_B\_V\_C\) are direct summands of \(b\_V\_0\_C^\otimes m_v (-t_v)\) and \(p\_B\_V\_0\_C^\otimes m_v (-t_v)\), respectively. Consequently, there is a morphism

\[7.61\]
\[b\_V\_C \to p\_B\_V\_C\]

defined by the composition \(b\_V\_C \to b\_V\_0\_C^\otimes m_v (-t_v) \overset{7.57}{\to} p\_B\_V\_0\_C^\otimes m_v (-t_v) \to p\_B\_V\_C\), in which the first and last morphisms are defined by the inclusions from and projections to direct summands. This morphism is independent of the choice of \(m_v\) and \(t_v\).

Proof. The first statement follows from Lemma 7.58. The second and third statements follow from the first, and from Remark 7.59 and the compatibility in Corollary 7.53 between the canonical pairings \((7.47)\) and \((7.52)\).

\(\square\)

Proposition 7.62. The above morphism \(7.61\) is an isomorphism over the connected component \(\Gamma^+_{K, 0} \setminus X^+\) of \(\Sha_{K, \mathbb{C}}\) that induces an identity morphism between the two representations \(\rho_{\_K, 0}^{\_s, (p)}(V)\) and \(\rho_{\_K, 0}^{\_s, (p)}(V)\).

Proof. By Corollary 7.53 it suffices to show that \(7.61\) is an isomorphism after pulled back to the image of the special point \(h \in X^+\) in \(\Gamma^+_{K, 0} \setminus X^+\), which is defined over a finite extension \(E^+\) of \(E\) in \(\mathbb{Q}\). Since the constructions of the comparison isomorphisms thus far are functorial and compatible with pullbacks to special points,
the pullback of (7.61) is induced by the simpler comparison isomorphisms for the
cohomology of an abelian variety, and it suffices to note that the Hodge tensor \( s_V \)
in Lemma [7.58] are respected by such comparison isomorphisms, because Hodge
cycles on abelian varieties over number fields are absolute Hodge (see [DMOS82
Chapter I, Main Theorem 2.11]) and de Rham (see [Bla94, Theorem 0.3]). □

Let us reformulate Proposition [7.62] as follows:

**Proposition 7.63.** Assume that \( G^{\text{der},c} \) is \( \mathbb{Q} \)-simple and simply-connected as an
algebraic group over \( \mathbb{Q} \), and that \( \text{rk}_R(G^{\text{der}}_R) \leq 1 \), or more generally that \( (G,X) \) is
of abelian type, regardless of the value of \( \text{rk}_R(G^{\text{der}}_R) \). Then \( \rho_{K,g_0}^+(V) \) coincides with
\( \rho_{K,g_0}^+(V) \), so that it is also induced by an algebraic representation of \( G^{\text{der},c} \)
that coincides with \( V_{C}(G^{\text{der},c}, \text{cf. Lemma } 7.40 \text{ and Remark } 7.41) \).

**7.7. Cases of real rank at least two.** In this subsection, we assume that \( G^{\text{der},c} \)
is \( \mathbb{Q} \)-simple and simply-connected as an algebraic group over \( \mathbb{Q} \) (so that \( G^{\text{der}} \cong
G^{\text{der},c} \)), and that \( \text{rk}_R(G^{\text{der}}_R) \geq 2 \). We shall make use of the following special case of
Margulis’s superrigidity theorem:

**Theorem 7.64** (Margulis). Let \( H \) be a \( \mathbb{Q} \)-simple simply-connected connected
algebraic group over \( \mathbb{Q} \), and let \( \Gamma \) be an arithmetic subgroup of \( H(\mathbb{Q}) \). Suppose that
\( \text{rk}_R(H(\mathbb{R})) \geq 2 \). Then, given any representation \( \rho : \Gamma \to \text{GL}_m(\mathbb{C}) \), there exists a
finite index normal subgroup \( \Gamma_0 \) of \( \Gamma \) such that \( \rho|_{\Gamma_0} \) extends to a (unique)
group homomorphism \( \tilde{\rho} : H(\mathbb{C}) \to \text{GL}_m(\mathbb{C}) \) that is induced by an algebraic group homomorphism
\( H(\mathbb{C}) \to \text{GL}_m(\mathbb{C}) \), and such that \( \tilde{\rho}(\gamma) = \rho(|\gamma|) \tilde{\rho}(\gamma) \), for all \( \gamma \in \Gamma \), for some
representation \( \delta : \Gamma_0 \to \text{GL}_m(\mathbb{C}) \) whose image commutes with \( \rho(H(\mathbb{C})) \).

**Proof.** This follows from [Mar91] Chapter VIII, p. 258, Theorem (B), part (iii)]
with \( S = \{\infty\}, \Lambda = \Gamma, K = \mathbb{Q}, \text{ and } \ell = \mathbb{C} \) (in the notation there). □

By applying Theorem 7.64 with \( H = G^{\text{der},c}, \Gamma = \Gamma_{K,g_0}^{+,c} \), \( \rho = \rho_{K,g_0}^+(V) \) as
in Section 7.5 we see that \( \rho_{K,g_0}^+(V) : \Gamma_{K,g_0}^{+,c} \to \text{GL}_C(V_C) \) extends to an algebraic representation
\( \bar{\rho}_{K,g_0}^+(V) : G^{\text{der},c}_C \to \text{GL}_C(V_C) \)
up to a finite error given by some representation
\( \delta_{K,g_0}(V) : \Gamma_{K,g_0}^{+,c}/\Gamma_{K,g_0}^{+,c} \to \text{GL}_C(V_C) \)
whose image in \( \text{GL}_C(V_C) \) commutes with \( \bar{\rho}_{K,g_0}^+(V)(G^{\text{der},c}(\mathbb{C})) \). Let us record what
we have obtained as a triple
\[
(\rho_{K,g_0}^+(V), \Gamma_{K,g_0}^{+,c}/\Gamma_{K,g_0}^{+,c}, \delta_{K,g_0}(V)),
\]
which depends on \( K \) and \( g_0 \), or rather the arithmetic subgroup \( \Gamma_{K,g_0}^{+,c} \) they define.

If \( K_1 \subset K_2 \) are neat open compact subgroups of \( G(\mathbb{A}_f) \), then \( \Gamma_{K_1,g_0}^{+,c} \subset \Gamma_{K_2,g_0}^{+,c} \)
and \( \rho_{K_1,g_0}^+(V) = \rho_{K_2,g_0}^+(V)|_{\Gamma_{K_1,g_0}^{+,c}} \), and therefore
\[
(\rho_{K_1,g_0}^+(V), \Gamma_{K_1,g_0}^{+,c}/\Gamma_{K_1,g_0}^{+,c}, \delta_{K_1,g_0}(V)),
\]
where \( \Gamma_{K_1,g_0}^{+,c}/\Gamma_{K_2,g_0}^{+,c} \) is a neat arithmetic subgroup of \( G^{\text{der},c}(\mathbb{Q}) \) because it is of
finite index in \( \Gamma_{K_1,g_0}^{+,c} \) (because \( \Gamma_{K_1,g_0}^{+,c}/\Gamma_{K_1,g_0}^{+,c} \subset \Gamma_{K_2,g_0}^{+,c}/\Gamma_{K_2,g_0}^{+,c} \)). For
\( i = 1, 2 \), let \( (\bar{\rho}_{K_i,g_0}^+(V), \Gamma_{K_i,g_0}^{+,c}/\Gamma_{K_i,g_0}^{+,c}, \delta_{K_i,g_0}(V)) \) be the triple associated with \( \rho_{K_i,g_0}^+(V) \),
as in (7.65). Then it follows from (7.66) and the Borel density theorem (see Remark 7.41) that $\tilde{\rho}_{K_{1,90}}^+ (V) = \tilde{\rho}_{K_{2,90}}^+ (V)$. Since $K_1$ and $K_2$ are arbitrary, there is a well-defined assignment (to $V$) of an algebraic representation

$$\tilde{\rho}_{90}^+(p)(V) : G_{C,\text{der}}^c \to \text{GL}_C(V_C)$$

such that $\tilde{\rho}_{K,90}^+(p)(V) = \tilde{\rho}_{90}^+(p)(V)$ for all neat open compact subgroups $K$ of $G(A_f)$.

By the above construction, and by Proposition 7.27, the assignment of $\tilde{\rho}_{90}^+(p)(V)$ to $V \in \text{Rep}_C(G^c)$ defines a tensor functor from $\text{Rep}_C(G^c)$ to $\text{Rep}_C(G_{C,\text{der}}^c)$, and hence induces (as in [DMOSS2 Chapter II, Corollary 2.9]) a group homomorphism $G_{C,\text{der}}^c \to G_C$. Since $G_{C,\text{der}}^c$ is semisimple, this homomorphism factors through

$$G_{C,\text{der}}^c \to G_{C}^c.$$

For each $V \in \text{Rep}_C(G^c)$, let us denote by

$$\pi(V) : G_{C}^c \to \text{GL}_C(V_C)$$

the algebraic representation given by the restriction $V_{C,\text{der}}^c$.

For each $\gamma \in G_{C,\text{der}}^c(\mathbb{Q})$, the Hecke action of $g = g_0^{-1}\gamma g_0 \in G(A_f)$ (see (7.3)) induces an isomorphism $\Gamma_{gK_0,90}^+ \times^+ \xrightarrow{\sim} \Gamma_{K,90}^+ \times^+$ defined by left multiplication by $\gamma^{-1}$, compatible with the isomorphism $\Gamma_{gK_0,90}^+ \xrightarrow{\sim} \Gamma_{K,90}^+$ induced by conjugation by $\gamma^{-1}$. By Proposition 7.35 and (7.12), we have the compatibility

$$\tilde{\rho}_{90}^+(p)(V)(\gamma \gamma')^{-1} = (\tilde{\rho}_{90}^+(p)(V)(\gamma))(\tilde{\rho}_{90}^+(p)(V)(\gamma'))^{-1}$$

$$= (\pi(V)(\gamma))(\tilde{\rho}_{90}^+(p)(V)(\gamma'))^{-1},$$

for all $\gamma' \in \Gamma_{K,90,0}^c \cap \gamma^{-1} \Gamma_{gK_0,90,0}^c$, where $g = g_0^{-1}\gamma g_0$.

**Lemma 7.69.** Suppose the representation $\tilde{\rho}_{90}^+(p)(V)$ is irreducible when $V$ is. Then the following are true for all $V \in \text{Rep}_C(G^c)$:

1. We have $\tilde{\rho}_{90}^+(p)(V) = \pi(V)$ as algebraic representations of $G_{C,\text{der}}^c$.
2. The representation $\delta_{K,90}(V) : \Gamma_{K,90}^c / \Gamma_{K,90,0}^c \to \text{GL}_C(V_C)$ is trivial.

**Proof.** By Proposition 7.27, we may assume that $V$ is irreducible, so that $\tilde{\rho}_{90}^+(p)(V)$ also is, by assumption. Let us temporarily write $\tilde{\rho} := \tilde{\rho}_{90}^+(p)(V)$ and $\pi := \pi(V)$, and measure their difference by introducing the algebraic morphism

$$\epsilon : G_{C,\text{der}}^c \to \text{GL}_C(V_C)$$

(which is not shown to be a group homomorphism yet) such that $\epsilon(g) = (\tilde{\rho}(g) - \pi(g))^{-1}$ for all $g \in G_{C,\text{der}}(\mathbb{C})$. By (7.68), we have

$$\tilde{\rho}(\gamma') = \epsilon(\gamma)\tilde{\rho}(\gamma')\epsilon(\gamma)^{-1}$$

for all $\gamma' \in \Gamma' := \Gamma_{K,90,0}^c \cap \gamma^{-1} \Gamma_{gK_0,90,0}^c$. Since $\Gamma'$ is a neat arithmetic subgroup of $G_{C,\text{der}}^c(\mathbb{Q})$, it follows from the Borel density theorem (again, see Remark 7.41) that (7.70) holds for all $\gamma' \in G_{C,\text{der}}(\mathbb{C})$. Then $\epsilon$ is a group homomorphism, because

$$\epsilon(\gamma \gamma') = (\epsilon(\gamma)\epsilon(\gamma'))^{-1} = (\epsilon(\gamma)^{-1}\tilde{\rho}(\gamma)^{-1}\pi(\gamma)\pi(\gamma'))^{-1}$$

$$= \tilde{\rho}(\gamma)^{-1}\epsilon(\gamma)\pi(\gamma') = \epsilon(\gamma)\epsilon(\gamma)^{-1} \epsilon(\gamma)\epsilon(\gamma)^{-1} \epsilon(\gamma)\epsilon(\gamma)^{-1} = \epsilon(\gamma)\epsilon(\gamma)^{-1},$$

...
for all $\gamma, \gamma' \in G_{\text{der},c}(\mathbb{Q})$, and because $G_{\text{der},c}(\mathbb{Q})$ is Zariski dense in $G_{\text{der},c}$ (by [Spr98 Corollary 13.3.10], or still by the Borel density theorem). Moreover, by Schur’s lemma, $\epsilon$ factors through an algebraic group homomorphism $G_{\text{der},c} \to G_{\text{der},c}$, which must be trivial because $G_{\text{der},c}$ is semisimple. Thus, $\epsilon$ is trivial, and part (2) follows.

As for part (2), we can prove it using Schur’s lemma and the neatness of $\Gamma_{K, \nu, \rho}$, as in the last paragraph of the proof of Lemma 7.42.

\begin{lemma}
The above homomorphism (7.67) is an automorphism. In particular, the representation $\tilde{\rho}_{g_0}^+(p)(V)$ is indeed irreducible when $V$ is.
\end{lemma}

\begin{proof}
Since $G_{\text{der},c}$ is semisimple and simply-connected, it suffices to show that the Lie algebra of the kernel of (7.67), which is a priori a product of $\mathbb{C}$-simple factors of the Lie algebra of $G_{\text{der},c}$, is trivial. Therefore, it suffices to show that (7.67) has nontrivial restrictions to all $\mathbb{C}$-simple factors of $G_{\text{der},c}$, and it suffices to find some $V \in \text{Rep}_\mathbb{C}(G^{\epsilon})$ such that $\tilde{\rho}_{g_0}^+(p)(V)$ is nontrivial on all simple $\mathbb{C}$-simple factors.

As explained in [Bor84, MIL83], based on a construction due to Piatetski-Shapiro, there exist morphisms $\varphi_1 : (G, X) \to (G_1, X_1)$ and $\varphi_2 : (G_2, X_2) \to (G_1, X_1)$ between Shimura data such that the following hold:

- $G_1^{\text{der},c}$ is $\mathbb{Q}$-simple, and we have $G_1^{\text{der},c} \cong G_1^{\text{der},c} \to \text{Res}_{F/\mathbb{Q}} G_F^{\text{der},c}$ for some totally real number field $F$, identifying $G_1^{\text{der},c}$ as a direct factor of $G_1^{\text{der},c}$.
- All $\mathbb{C}$-simple factors of $G_2^{\text{der},c}$ are of type $A_1$. In this case, as in the proof of Lemma 7.43, according to the classification in [Del79, 2.3] (see also the summary in [Lan17 Section 5.2.2]), the Shimura datum $(G_3, X_2)$ is of abelian type, for which Proposition 7.68 and hence Theorem 7.29 hold. Moreover, the homomorphism $G_2^{\text{der},c} \to G_1^{\text{der},c}$ induced by $\varphi_2$ embeds distinct $\mathbb{C}$-simple factors of $G_2^{\text{der},c}$ into distinct $\mathbb{C}$-simple factors of $G_1^{\text{der},c}$.

Therefore, there exists some $V_1 \in \text{Rep}_\mathbb{C}(G_1^{\epsilon})$ such that its pullback $V_2 \in \text{Rep}_\mathbb{C}(G_2^{\epsilon})$ is nontrivial on all $\mathbb{C}$-simple factors of $G_2^{\text{der},c}$. By Proposition 7.26, $p_2^+ V_{2,c}$ and $p_2 B V_{2,c}$ are canonically isomorphic to pullbacks of $p_2 B V_{1,c}$, and we already know that the fundamental group representations associated with $p_2 B V_{2,c} \cong B V_{2,c}$ are given by the restrictions of $V_{2,c}$. Let $\rho_{g_0}^+(p)(V_1)$ be associated with $p_2 B V_{2,c}$ as in the case of $\rho_{g_0}^+(p)(V)$. By [Mar91] Chapter I, Section 3, Lemma 3.13, the pullbacks of arithmetic subgroups of $G_1^{\text{der},c}(\mathbb{Q})$ to $G_1^{\text{der},c}(\mathbb{Q})$ and $G_2^{\text{der},c}(\mathbb{Q})$ contain arithmetic subgroups. Therefore, by the Borel density theorem (see Remark 7.41), the pullback of $\rho_{g_0}^+(p)(V_1)$ to $G_2^{\text{der},c}$ is nontrivial on all $\mathbb{C}$-simple factors of $G_2^{\text{der},c}$, and hence $\rho_{g_0}^+(p)(V_1)$ is nontrivial on all $\mathbb{C}$-simple factors of $G_1^{\text{der},c}$. By the Borel density theorem again, $\rho_{g_0}^+(p)(V)$ is isomorphic to the pullback of $\rho_{g_0}^+(p)(V_1)$, which is then nontrivial on all simple $\mathbb{C}$-simple factors of $G_2^{\text{der},c}$, as desired.

By combining Lemmas 7.69 and 7.71, we obtain the following:

\begin{proposition}
Under the assumptions that $G^{\text{der},c}$ is $\mathbb{Q}$-simple and simply-connected as an algebraic group over $\mathbb{Q}$, and that $\text{rk}_{E}(\tilde{G}^{\text{der},c}) \geq 2$, it is true that $\rho_{g_0}^+(p)(V)$ extends to an algebraic representation of $G^{\text{der},c}$ that coincides with $V_{c|\text{Rep}_{\mathbb{C}}}$ (cf. Lemma 7.40 and Remark 7.41).
\end{proposition}
7.8. Dual BGG decompositions and Hodge–Tate weights. In this subsection, we introduce the dual BGG decomposition for the log Hodge cohomology of $dR V^c_{\mathbb{C}}$ over $\text{Sh}^\text{tor}_{\mathbb{C}}$, which is useful for computing the Hodge–Tate weights of Hecke-invariant subspaces of the étale cohomology of $\text{ét}_V^{\mu}$, using Theorems 6.9 and 7.30 as explained in Remark 7.34. We shall begin by reviewing the so-called dual BGG complexes introduced by Faltings in [Fal83]. (The abbreviation BGG refers to Bernstein, Gelfand, and Gelfand, because of their seminal work [BGG75].)

Let us fix the choice of some $\mu_h$ as in (7.13) which is induced by some homomorphism $G_{m,\overline{Q}} \to G_{\overline{Q}}$, which we abusively denote by the same symbols. Let $P$ (resp. $P^c$) denote the parabolic subgroup of $G_{\overline{Q}}$ (resp. $G_{\overline{Q}}^c$) defined by the choice of $\mu_h$ (cf. Remark 7.32). Let $M$ (resp. $M^c$) denote the Levi subgroup of $P$ (resp. $P^c$) given by the centralizer of the image of $\mu_h$. As in the case of $G$ and $G^c$, for any field $F$ over $\overline{Q}$, let us denote by $\text{Rep}_F(P^c)$ (resp. $\text{Rep}_F(M^c)$) the category of finite-dimensional algebraic representations of $P^c$ (resp. $M^c$) over $F$, which we view as an algebraic representation of $P$ (resp. $M$) by pullback. We shall also view the representations of $M^c$ (resp. $M$) as representations of $P^c$ (resp. $P$) by pullback via the canonical homomorphism $P^c \to M^c$ (resp. $P \to M$).

As explained in [Har89] Section 3 (or [Lan16] Section 2.2), there is a tensor functor assigning to each $W \in \text{Rep}_F(P^c)$ a vector bundle $\text{coh} W^c_{\mathbb{C}}$ over $\text{Sh}^\text{tor}_{\mathbb{C}}$, which is canonically isomorphic to $dR V^c_{\mathbb{C}}$ when $W_{\mathbb{C}} \cong V_{\mathbb{C}}$ for some $V \in \text{Rep}_F(G^c)$. We call $\text{coh} W^c_{\mathbb{C}}$ the automorphic vector bundle associated with $W_{\mathbb{C}}$. Moreover, as explained in [Har89] Section 4, this tensor functor canonically extends to a tensor functor assigning to each $W \in \text{Rep}_F(P^c)$ a vector bundle $\text{coh} W^c_{\mathbb{C}}$ over $\text{Sh}^\text{tor}_{\mathbb{C}}$, called the canonical extension of $\text{coh} W_{\mathbb{C}}$ which is canonically isomorphic to $dR V^c_{\mathbb{C}}$ when $W_{\mathbb{C}} \cong V_{\mathbb{C}}$ for some $V \in \text{Rep}_F(G^c)$. For $W \in \text{Rep}_F(M^c)$, this $\text{coh} W^c_{\mathbb{C}}$ is canonical isomorphic to the canonical extensions defined as in [Mum77] Main Theorem 3.1.

Let us fix the choice of a maximal torus $T^c$ of $M^c$, which is also a maximal torus of $G^c$. With this choice, let us denote by $\Phi_{G_{\overline{Q}}^c}$, $\Phi_{M^c}$, etc the roots of $G_{\overline{Q}}^c$, $M^c$, etc, respectively; and by $X_{G_{\overline{Q}}^c}$, $X_{M^c}$, etc the weights of $G_{\overline{Q}}^c$, $M^c$, etc, respectively. Then we have naturally $\Phi_{M^c} \subset \Phi_{G_{\overline{Q}}^c}$ and $X_{G_{\overline{Q}}^c} = X_{M^c}$. Let us denote by $H$ the coweight induced by $\mu_h$. Let us also make compatible choices of positive roots $\Phi_{G_{\overline{Q}}^c}^+$ and $\Phi_{M^c}^+$, and of dominant weights $X_{G_{\overline{Q}}^c}^+$ and $X_{M^c}^+$, so that $\Phi_{M^c}^+ \subset \Phi_{G_{\overline{Q}}^c}^+$ and $X_{G_{\overline{Q}}^c}^+ \subset X_{M^c}^+$. For an irreducible representation $V$ of highest weight $\lambda \in X_{G_{\overline{Q}}^c}^+$, we write $V = V_{\lambda}$, $V_{\mathbb{C}} = V_{\lambda,\mathbb{C}}$, $dR V_{\mathbb{C}} = dR V_{\nu,\mathbb{C}}$, etc. Similarly, for an irreducible representation $W$ of highest weight $\nu \in X_{M^c}^+$, we write $W = W_{\nu}$, $W_{\mathbb{C}} = W_{\nu,\mathbb{C}}$, $\text{coh} W_{\mathbb{C}} = \text{coh} W_{\nu,\mathbb{C}}$, etc. Let $\rho_{G_{\overline{Q}}^c} := \frac{1}{2} \sum_{\lambda \in \Phi_{G_{\overline{Q}}^c}^+} \lambda$ and $\rho_{M^c} := \frac{1}{2} \sum_{\nu \in \Phi_{M^c}^+} \nu$ denote the usual half-sums of positive roots, and let $\rho_{M^c}^+ := \rho_{G_{\overline{Q}}^c} - \rho_{M^c}$. Let $W_{G_{\overline{Q}}^c}$ and $W_{M^c}$ denote the Weyl groups of $G_{\overline{Q}}^c$ and $M^c$ with respect to the common maximal torus $T^c$, which allows us to identify $W_{M^c}$ as a subgroup of $W_{G_{\overline{Q}}^c}$. Given any element $w$ in the above Weyl groups, we shall denote its length by $l(w)$. In addition to the natural action of $W_{G_{\overline{Q}}^c}$ on $X_{G_{\overline{Q}}^c}$, there is also the dot action $w \cdot \lambda = w(\lambda + \rho_{G_{\overline{Q}}^c}) - \rho_{G_{\overline{Q}}^c}$, for all $w \in W_{G_{\overline{Q}}^c}$ and $\lambda \in X_{G_{\overline{Q}}^c}$.
Let \( W^{M^r} \) denote the subset of \( W_{G^\mathbb{C}} \) consisting of elements mapping \( X^+_{G^\mathbb{C}} \) into \( X^+_{M^r} \), which are the minimal length representatives of \( W_{M^r} \) in \( W_{G^\mathbb{C}} \).

As in Definition 7.73, consider the log de Rham complex
\[
DR_{\log}(d\text{r}V_C) := (\text{d}rV_C^\text{can} \otimes_{\text{Sh}_{K,C}} Q_{\text{Sh}_{K,C}}^\text{log}, \nabla),
\]
where \( Q_{\text{Sh}_{K,C}}^\text{log} := Q_{\text{Sh}_{K,C}}(\log D_C) \) as usual, and consider the log Hodge cohomology
\[
(7.73) \quad H^{a,b}_{\log \text{Hodge}}(\text{Sh}_{K,C}^\text{tor} \cdot d\text{r}V_C^\text{can}) = H^{a+b}(\text{Sh}_{K,C}^\text{tor} \cdot \text{gr}^a DR_{\log}(d\text{r}V_C)).
\]
While it is difficult to compute hypercohomology in general, the miracle is that \( \text{gr}^a DR_{\log}(d\text{r}V_C) \) has a quasi-isomorphic direct summand, called the graded dual BGG complex, whose differentials are zero and whose terms are direct sums of \( \text{coh}W_{w,a,C}^\text{can} \) for some representations \( W \) determined explicitly by \( V \). Then the hypercohomology of this graded dual BGG complex is just a direct sum of usual coherent cohomology of \( \text{coh}W_{w,a,C}^\text{can} \) up to degree shifting. More precisely, we have the following:

**Theorem 7.74 (dual BGG complexes; Faltings).** There is a canonical filtered quasi-isomorphic direct summand \( \text{BGG}^\text{log}(d\text{r}V_C) \) of \( DR_{\log}(d\text{r}V_C) \) (in the category of complexes of abelian sheaves over \( \text{Sh}_{K,C}^\text{tor} \) whose terms are coherent sheaves and whose differentials are differential operators) satisfying the following properties:

1. The formation of \( \text{BGG}^\text{log}(d\text{r}V_C) \) is functorial and exact in \( V_C \).
2. The differentials on \( \text{gr} \text{BGG}^\text{log}(d\text{r}V_C) \) are all zero.
3. Suppose \( V \cong V^\lambda \) for some \( \lambda \in X^+_G \). Then, for each \( i \geq 0 \) and each \( a \in \mathbb{Z} \),
   
   \[
   (7.75) \quad \text{gr}^a \text{BGG}_{\log}^\text{log}(d\text{r}V_C) \cong \oplus_{w \in W^{M^r}} \{ l(w) = i, (w, \lambda)(H) = -a \} \left( \text{coh}W_{w,a,C}^\text{can} \right).
   \]

**Proof.** See [Fal83, Sections 3 and 7, L.C.0, Chapter VI, Section 5], and [LLP, Theorem 5.9]. (Although these references were written in less general settings, the methods of the constructions still generalize to our setting here.) \( \square \)

**Remark 7.76.** The various automorphic vector bundles \( d\text{r}V^\lambda_{\Lambda,C} \), \( \text{Fil}^* (d\text{r}V^\lambda_{\Lambda,C}) \), and \( \text{coh}W_{w,\lambda,\Lambda} \) (and their canonical extensions) in Theorem 7.74 have models over \( \mathbb{Q} \) or even over a finite extension \( E' \) of \( E \) (depending on \( \lambda \)) over which \( V^\lambda \), \( \text{Fil}^* V^\lambda \), and \( W_{w,\lambda,\Lambda} \) all have models. (The case of \( d\text{r}V^\lambda_{\Lambda,C} \) and its canonical extension follows from [Mih90, Chapter III, Theorem 5.1, and Chapter V, Theorem 6.2] as before, and the cases of \( \text{Fil}^* d\text{r}V^\lambda_{\Lambda,C} \), \( \text{coh} W^\lambda_{w,\Lambda,\Lambda} \), and their canonical extensions follow from similar descent arguments. Note that \( \text{Fil}^* (d\text{r}V^\lambda_{\Lambda,C}) \) did appear in Theorem 7.74 when we said there is a canonical filtered quasi-isomorphic direct summand \( \text{BGG}^\text{log}(d\text{r}V_C) \) of \( DR_{\log}(d\text{r}V_C) \).

Then the statements of Theorem 7.74 remain true if we replace \( \mathbb{C} \) with \( E' \), by the same descent argument as in [LLP, Section 6].

**Corollary 7.77.** Suppose \( V \cong V^\lambda \) for some \( \lambda \in X^+_G \). Then, given any \( a, b \in \mathbb{Z} \) such that \( a + b \geq 0 \), we have the dual BGG decomposition
\[
(7.78) \quad H^{a,b}_{\log \text{Hodge}}(\text{Sh}_{K,C}^\text{tor} \cdot d\text{r}V_C^\text{can}) \cong \oplus_{w \in W^{M^r}, (w, \lambda)(H) = -a} H^{a+b-l(w)}(\text{Sh}_{K,C}^\text{tor} \cdot (\text{coh}W^\lambda_{w,\Lambda,\Lambda})^\text{can}).
\]

**Proof.** By Theorem 7.74, the complex \( \text{gr}^a DR_{\log}(d\text{r}V_C) \) is quasi-isomorphic to the complex \( \text{gr}^a \text{BGG}^\text{log}(d\text{r}V_C) \), and the differentials of the latter are all zero. Hence, the desired isomorphism (7.78) follows from (7.73) and (7.75). \( \square \)
Corollary 7.79. Suppose $V \cong V^\vee_\lambda$ for some $\lambda \in X^\vee_{\mathcal{C}^\vee}$. Then the (log) Hodge–de Rham spectral sequence at the right-hand side of the commutative diagram in Remark 7.34 induces the dual BGG spectral sequence

$$E_1^{a,b} = \oplus_{w \in W^{MC}, (w,\lambda)} H^{a+b-l(w)}(\text{Sh}_{K,C}^{\text{tor}}, (\text{coh} W^\vee_{w,\lambda,C})^{\text{can}})$$

which degenerates at the $E_1$ page, and induces a dual BGG decomposition

$$H^{a+b}_{dR}(\text{Sh}_{K,C}, dR V_C) \cong \oplus_{w \in W^{MC}} H^{a+b-l(w)}(\text{Sh}_{K,C}^{\text{tor}}, (\text{coh} W^\vee_{w,\lambda,C})^{\text{can}}).$$

Proof. This follows from Corollary 7.77 and the explanation in Remark 7.34. \qed

Remark 7.82. The analogues of Theorem 7.74 and Corollaries 7.77 and 7.79 for the subcanonical extensions, which differ from the canonical extensions by tensoring the locally sheaves with the $\mathcal{O}_{\text{Sh}_{K,C}^{\text{tor}}}$-ideal defining $D_C$, are also true. As for Theorem 7.74 and Corollary 7.77, the case of subcanonical extensions follows from the case of canonical extensions essentially by definition. As for Corollary 7.79, the degeneration of spectral sequences can be proved by comparing dimensions, and hence the case of subcanonical extensions follow from the case of canonical extensions by Poincaré and Serre dualities.

Remark 7.83. Faltings first introduced the dual BGG spectral sequence associated with the stupid ("béte") filtration in [Fal83] Section 4, p. 76, and Section 7, Theorem 11, whose degeneration at the $E_1$ page would nevertheless (by comparison of total dimensions) imply the degeneration of the dual BGG spectral sequence in (7.80) associated with the Hodge filtration. The degeneracy at the $E_1$ page was first proved by Faltings himself in the compact case (see [Fal83] Section 4, Theorem 4), later in the case of Siegel modular varieties by Faltings–Chai (see [FC90] Chapter VI, Theorem 5.5) by reducing to the case of trivial coefficients over some toroidal compactifications of self-fiber products of universal abelian schemes (and this method can be generalized to the case of all PEL-type and Hodge-type Shimura varieties using [Lan12a], [Lan12b], and [LS18] Section 4.5]), and in general by Harris–Zucker (see [HZ01] Corollary 4.2.3) using Saito’s theory of mixed Hodge modules (see [Sai90] Theorem 2.14). Our proof of Corollary 7.79 which can be alternatively based on our Theorems 6.6 and 7.30 is therefore a new one.

In the remainder of this subsection, as alluded to in Remark 7.34, we would like to describe the Hodge–Tate weights of $H^i(\text{Sh}_{K,\mathcal{O}_p}, \pi V_L)$ in terms of the dimensions of the dual BGG pieces at the right-hand side of (7.81).

We first need to provide a definition for the Hodge–Tate weights of the cohomology of étale local systems over the infinite extension $\mathcal{O}_p$ of $\mathbb{Q}_p$. As in Section 7.3, let $L$ be a finite extension of $\mathbb{Q}_p$ in $\mathcal{O}_p$ such that $V_{\mathcal{O}_p}$ has a model $V_L$ over $L$, and let $\mathfrak{e} V_L$ be as in (7.15). Let $k$ be a finite extension of the composite of $E$ and $L$ in $\mathcal{O}_p$, so that we have $\mathcal{O}_p \rightarrow \mathcal{O}_p \leftarrow \mathfrak{e} V_L$, and let $\tau : L \otimes_{\mathbb{Q}_p} k \rightarrow k$ be as in (7.18).

Let $(dR V_C^{\mathfrak{e} V_L}, \nabla, \text{Fil}^*)$ denote the pullback of $(dR V_C^{\mathfrak{e} V_L}, \nabla, \text{Fil}^*)$ under $\tau^{-1} : \mathcal{O}_p \rightarrow \mathcal{O}_p$, which is canonically isomorphic to $(p_{dR} V_C, \nabla, \text{Fil}^*)$ by Theorem 7.30.

By Theorem 1.1, $H^i(\text{Sh}_{K,\mathcal{O}_p}, \pi V_L)$ is a $dR$ representation of $\text{Gal}(\mathcal{O}_p/k)$, and we have a canonical $\text{Gal}(\mathcal{O}_p/k)$-equivariant Hecke-equivariant isomorphism

$$H^i(\text{Sh}_{K,\mathcal{O}_p}, \pi V_L) \otimes_{\mathbb{Q}_p} B_{dR} \cong H^i_{dR}(\text{Sh}_{K,k}, D_{dR,\log}(\pi V_L)) \otimes_k B_{dR},$$
which is compatible with the filtrations on both sides. By pushing out (7.84) via the projection \( \tau \) in (7.18), and by the construction of \( p_{dR}V_{\overline{\pi}_p} \) (see (7.22) and (7.24)), we obtain the following:

**Theorem 7.85.** There is a canonical \( \text{Gal}(\overline{\mathbb{Q}}_p/k) \)-equivariant Hecke-equivariant de Rham comparison isomorphism

\[
H^i(\text{Sh}_{K,\overline{\pi}_p}^{\text{an}}, \text{et}V_{\overline{\pi}_p}) \otimes_{\overline{\pi}_p} B_{dR} \cong H^i_{dR}(\text{Sh}_{K,\overline{\pi}_p}^{\text{an}}, \text{et}V_{\overline{\pi}_p}) \otimes_{\overline{\pi}_p} B_{dR},
\]

which is compatible with the filtrations on both sides.

Next, let us consider the log Hodge–Tate comparison. Let \( \mathbb{C}_p \) denote the \( p \)-adic completion of \( \overline{\mathbb{Q}}_p \) as usual. Let \( \text{Sh}_{K,k}^{\text{tor}} \) (resp. \( \text{Sh}_{K,p}^{\text{tor}} \)) denote the \( p \)-adic analytyification of \( \text{Sh}_{K,k}^{\text{an}} \) (resp. \( \text{Sh}_{K,p}^{\text{an}} \)) by \( \text{Hub93} \) Proposition 2.1.4 and Theorem 3.8.1 (see [6.5]) and Corollary 2.75, we have a canonical \( \text{Gal}(\overline{\mathbb{Q}}_p/k) \)-equivariant Hecke-equivariant isomorphism

\[
H^i(\text{Sh}_{K,\overline{\pi}_p}^{\text{tor,an}}, \text{et}V_{\overline{\pi}_p}) \cong H^i(\text{Sh}_{K,\overline{\pi}_p}^{\text{tor,an}}, \text{et}V_{\overline{\pi}_p}),
\]

where \( \text{et}V_{\overline{\pi}_p}^{\text{an}} \) extends \( \text{et}V_{\overline{\pi}_p}^{\text{an}} \) as in (6.1). By Theorem 5.20(2), and by GAGA (see [Kop14], and also the proof of [Sch13] Theorem 9.1), we have a canonical \( \text{Gal}(\overline{\mathbb{Q}}_p/k) \)-equivariant Hecke-equivariant isomorphism

\[
H^i(\text{Sh}_{K,\overline{\pi}_p}^{\text{tor,an}}, \text{et}V_{L}) \otimes_{\overline{\pi}_p} \mathbb{C}_p
\]

\[
\cong \otimes_{a+b=i} \left( H^a_{\log \text{Hodge}}(\text{Sh}_{K,\overline{\pi}_p}^{\text{tor}}, \text{et}V_{\overline{\pi}_p}^{\text{an}}) \otimes_k \mathbb{C}_p(-a) \right)
\]

(cf. (5.22)). By (7.87), by pushing out (7.88) via the projection \( \tau \) in (7.18), and by the constructions of \( p_{dR}V_{\overline{\pi}_p}^{\text{can}} \) (see (7.20) and (7.23)), we obtain a canonical \( \text{Gal}(\overline{\mathbb{Q}}_p/k) \)-equivariant Hecke-equivariant isomorphism

\[
H^i(\text{Sh}_{K,\overline{\pi}_p}^{\text{tor,an}}, \text{et}V_{\overline{\pi}_p}) \otimes_{\overline{\pi}_p} \mathbb{C}_p
\]

\[
\cong \otimes_{a+b=i} \left( H^a_{\log \text{Hodge}}(\text{Sh}_{K,\overline{\pi}_p}^{\text{tor}}, \text{et}V_{\overline{\pi}_p}^{\text{can}}) \otimes_{\overline{\pi}_p} \mathbb{C}_p(-a) \right)
\]

Note that this last isomorphism is independent of the choice of \( L \), and can be identified with the 0-th graded piece of (7.86) (cf. the proof of Theorem 6.6).

**Definition 7.90.** We consider this isomorphism (7.89) the **Hodge–Tate decomposition** for \( H^i(\text{Sh}_{K,\overline{\pi}_p}^{\text{tor,an}}, \text{et}V_{\overline{\pi}_p}) \), and abusively define the multiset of **Hodge–Tate weights** of \( H^i(\text{Sh}_{K,\overline{\pi}_p}^{\text{tor,an}}, \text{et}V_{\overline{\pi}_p}) \) to be the multiset of integers in which each \( a \in \mathbb{Z} \) has multiplicity \( \dim_{\overline{\pi}_p} H^a_{\log \text{Hodge}}(\text{Sh}_{K,\overline{\pi}_p}^{\text{tor}}, \text{et}V_{\overline{\pi}_p}^{\text{can}}) \). We naturally extend the definition to \( \overline{\mathbb{Q}}_p \)-subspaces of \( H^i(\text{Sh}_{K,\overline{\pi}_p}^{\text{tor,an}}, \text{et}V_{\overline{\pi}_p}) \) cut out by \( \overline{\mathbb{Q}}_p \)-valued Hecke operators by replacing \( H^a_{\log \text{Hodge}}(\text{Sh}_{K,\overline{\pi}_p}^{\text{tor}}, \text{et}V_{\overline{\pi}_p}^{\text{can}}) \) with their corresponding \( \overline{\mathbb{Q}}_p \)-subspaces cut out by the same \( \overline{\mathbb{Q}}_p \)-valued Hecke operators.

**Theorem 7.91.** Suppose \( V \cong V^\lambda_X \) for some \( \lambda \in X^+_G \). For any \( W \) in \( \text{Rep}(\mathbb{Q}^c) \), let \( \text{coh}W_{\overline{\pi}_p}^{\text{can}} \) and \( \text{coh}W_{\text{et}}^{\text{can}} \) be the pullbacks of \( \text{coh}W_C \) and \( \text{coh}W_{\text{et}}^{\text{can}} \), respectively, under \( \iota^{-1} : \mathbb{C} \to \overline{\mathbb{Q}}_p \). Then we have a canonical \( \text{Gal}(\overline{\mathbb{Q}}_p/k) \)-equivariant Hecke-equivariant
(7.92) \[ H^i(\text{Sh}_{K,p}, \eta \mathcal{V}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong \bigoplus_{w \in W_M^w} H^{i-l(w)}(\text{Sh}_{K,p}^{\text{tor}}, (\text{coh} \mathcal{W}^\vee_{w,\lambda} \mathbb{Q}_p)_{\text{can}}) \otimes_{\mathbb{Q}_p} \mathbb{C}_p((w, \lambda)(H)), \]

which is the dual BGG version of the Hodge–Tate decomposition (7.89). Moreover, the multiset of Hodge–Tate weights of any Hecke-invariant $\mathbb{C}$-subspace of $H^i(\text{Sh}_{K,p}, \eta \mathcal{V}_p)$ cut out by some $\mathbb{Q}_p$-valued Hecke operator (as in Definition 7.90) contains each $a \in \mathbb{Z}$ with multiplicity given by the $\mathbb{C}$-dimension of the corresponding Hecke-invariant $\mathbb{C}$-subspace of

\[ \bigoplus_{w \in W^w} (w, \lambda)(H) = a \quad H^{i-l(w)}(\text{Sh}_{K,C}^{\text{tor}}, (\text{coh} \mathcal{W}^\vee_{w,\lambda} C)_{\text{can}}) \]

cut out by the pullback of the same $\mathbb{Q}_p$-valued Hecke operator under $\iota : \mathbb{Q}_p \to \mathbb{C}$.

**Proof.** This follows from (7.89) and Corollary 7.77 and from the fact (which we have implicitly used several times) that the formation of coherent hypercohomology of schemes is compatible with arbitrary base field extensions. \( \square \)

**Remark 7.94.** All previously known special cases of (7.92) (see, for example, [FC90, Theorem 6.2] and [HT01, Section III.2]) were proved using the (log) Hodge–Tate comparison for the cohomology with trivial coefficients of some families of abelian varieties (and their smooth toroidal compactifications, in the noncompact case). The novelty in Theorem 7.91 is that we can deal with nontrivial coefficients that are not at all related to families of abelian varieties.

**Remark 7.95.** As in Remark 7.82, by Poincaré and Serre dualities, we also have the analogues of (7.89), Definition 7.90, and Theorem 7.91 for subcanonical extensions.

**Remark 7.96.** As in [Har90a, Example 4.6] and [HT01, Proposition III.2.1], we can often compute $H^{i-l(w)}(\text{Sh}_{K,C}^{\text{tor}}, (\text{coh} \mathcal{W}^\vee_{w,\lambda} C)_{\text{can}})$ (as a $\mathbb{C}$-vector space) and its Hecke-invariant $\mathbb{C}$-subspaces cut out by $\mathbb{C}$-valued Hecke operators in terms of relative Lie algebra cohomology.

**Remark 7.97.** In the special (but still common) case where $\mathcal{V}_{\mathbb{Q}_p}$ has a model $\mathcal{V}_Q$ over $\mathbb{Q}_p$, we can take $L = \mathbb{Q}_p$ in the above, and the choice of $\iota : \mathbb{Q}_p \to \mathbb{C}$ corresponds to the choice of places $v$ of $E$ above $p$. Then $H^i(\text{Sh}_{K,\mathbb{Q}_p}, \eta \mathcal{V}_{\mathbb{Q}_p})$ is a de Rham representation of $\text{Gal}(\overline{\mathbb{Q}}_p/k)$, and the de Rham comparison isomorphism (7.86) can be rewritten as

\[ H^i(\text{Sh}_{K,\mathbb{Q}_p}, \eta \mathcal{V}_{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} B_{dR}) \cong H^i_{dR}(\text{Sh}_{K,k}, \eta \mathcal{V}_{k} \otimes_k B_{dR}) \]

(cf. (7.84)). Moreover, the assertion in Theorem 7.91 that the Hodge–Tate weights of $H^i(\text{Sh}_{K,\mathbb{Q}_p}, \eta \mathcal{V}_{\mathbb{Q}_p})$ (as in Definition 7.90) depend only on the $\mathbb{C}$-dimension of (7.93), but not on the choice of $v$, implies that the (usual) Hodge–Tate weights of $H^i(\text{Sh}_{K,\mathbb{Q}_p}, \eta \mathcal{V}_{\mathbb{Q}_p})$ (as a representation of $\text{Gal}(\overline{\mathbb{Q}}_p/k)$) are also independent of the choice of $v$.

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