LOGARITHMIC ADIC SPACES: SOME FOUNDATIONAL RESULTS

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Abstract. We develop a theory of log adic spaces by combining the theories of adic spaces and log schemes, and study the Kummer étale and pro-Kummer étale topology for such spaces. We also establish the primitive comparison theorem in this context, and deduce from it some related cohomological finiteness or vanishing results.

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1. Introduction

There are two main goals of this paper. Firstly, we would like to adapt many fundamental notions and features of the theory of log geometry for schemes, as in [Kat89b, Kat91, Kat89a, Ill02, Ogu18], to the theory of adic spaces, as in [Hub94, Hub96]. For example, we would like to introduce the notion of log adic spaces, which allow us to study the de Rham and étale cohomology of nonproper adic spaces by introducing the log de Rham and Kummer étale cohomology of proper adic spaces equipped with suitable log structures. Secondly, we would like to adapt many foundational techniques in recent developments of \( p \)-adic geometry, as in [Sch12, KL15, Sch13a, Sch16, SW17], to the context of log geometry. For example, we would like to introduce the pro-Kummer étale site, and show that log affinoid perfectoid objects form a basis for such a site, under suitable assumptions. In particular, we would like to establish the \textit{primitive comparison theorem} and some related cohomological finiteness or vanishing results in this context.

Although a general formalism of log topoi has been introduced in [GR19, Sec. 12.1], there are nevertheless several special features (such as the integral structure sheaves) or pathological issues (such as the lack of fiber products in general, or the necessary lack of noetherian property when working with perfectoid spaces) in the theory of adic spaces, which resulted in some complications in our adaption of many “well-known arguments”; and we have chosen to spell out the modifications of such arguments in some detail, for the sake of clarity. Moreover, this paper is intended to serve as the foundation for our development of a \( p \)-adic analogue of the Riemann–Hilbert correspondence in [DLLZ] (and forthcoming works such as [LLZ]). Therefore, in addition to the above-mentioned goals, we have also included some foundational treatment of quasi-unipotent nearby cycles, following (and reformulating) Beilinson’s ideas in [Bei87].

Here is an outline of this paper.

In Section 2, we introduce log adic spaces and study their basic properties. In Section 2.1, we review some basic terminologies of monoids. In Section 2.2, we introduce the definition and some basic notions of log adic spaces, and study some important examples. In Section 2.3, we study the important notion of \textit{charts} in the context of log adic spaces, which are useful for defining the categories of coherent, fine, and fs log adic spaces, and for constructing fiber products in these categories.

In Section 3, we study log smooth morphisms of log adic spaces, and their associated sheaves of log differentials. In Section 3.1, we introduce the notion of log smooth and log étale morphisms of fs log adic spaces, and show the existence of smooth toric charts for smooth fs log adic spaces. In Section 3.2, we develop a theory of log differentials for homomorphisms of Huber rings and morphisms of coherent log adic spaces, and compare it with the theory in Section 3.1.

6.2. Primitive comparison theorem
6.3. \( p \)-adic local systems
6.4. Quasi-unipotent nearby cycles
Appendix A. Kiehl’s property for coherent sheaves
References
In Section 4, we study the Kummer étale topology of locally noetherian fs log adic spaces. In Section 4.1, we introduce the Kummer étale site and study its basic properties. In Section 4.2, we establish an analogue of Abhyankar’s lemma for rigid analytic varieties, and record some related general facts. In Section 4.3, we study the structure sheaves and analytic coherent sheaves on the Kummer étale site, and show that their higher cohomology vanishes on affinoids. In Section 4.4, we show that Kummer étale surjective morphisms satisfy effective descent in the category of finite Kummer étale covers, and show that the Kummer étale fundamental group can be defined with desired properties. In Section 4.5, we study certain direct and inverse images of abelian sheaves on Kummer étale sites. In Section 4.6, we establish some purity results for torsion Kummer étale local systems.

In Section 5, we study the pro-Kummer étale topology of locally noetherian fs log adic spaces. In Section 5.1, we introduce the pro-Kummer étale site, by using the theory developed in Section 4 and following Scholze’s ideas in [Sch13a, Sch16]. In Section 5.2, we study certain direct and inverse images of abelian sheaves on pro-Kummer étale sites. In Section 5.3, we introduce the notion of log affinoid perfectoid objects, and show that they form a basis for the pro-Kummer étale topology, for locally noetherian fs log adic spaces over Spa($\mathbb{Q}_p, \mathbb{Z}_p$). In Section 5.4, we introduce the completed structure sheaves and their integral and tilted variants on the pro-Kummer étale site, and prove various almost vanishing results for them.

In Section 6, we study the Kummer étale cohomology of fs log adic spaces log smooth over a nonarchimedean base field $k$. In Section 6.1, we start with some preparations using the log affinoid perfectoid objects defined using some associated toric charts. In Section 6.2, we establish the primitive comparison theorem, generalizing the strategy in [Sch13a Sec. 5], and deduce form it some finiteness results for the cohomology of torsion Kummer étale local systems. In Section 6.3, we introduce the notions of $\mathbb{Z}_p^\ast$, $\mathbb{Q}_p^\ast$, $\mathbb{Z}_p^\ast$, and $\hat{\mathbb{Q}}_p$-local systems, and record some finiteness results (as consequences of earlier results). In Section 6.4, as an application of the theory thus developed, we reformulate Beilinson’s ideas in [Bei87] and define the unipotent and quasi-unipotent nearby cycles in the rigid analytic setting.

In Appendix A, we state a version of Tate’s sheaf property and Kiehl’s gluing property for the analytic and étale sites of adic spaces that are either locally noetherian or analytic and étale sheafy. This includes, in particular, a proof of Kiehl’s property for coherent sheaves on (possibly nonanalytic) noetherian adic spaces which (as far as we know) is not yet available in the literature.

**Notation and conventions.** By default, all monoids are assumed to be commutative, and the monoid operations are written additively (rather than multiplicatively), unless otherwise specified. For a monoid $P$, let $P^{gp}$ denote its group completion. For any commutative ring $R$ with unit and any monoid $P$, we denote by $R[P]$ the monoid algebra over $R$ associated with $P$. The image of $a \in P$ in $R[P]$ will often be denoted by $e^a$.

Group cohomology will always mean continuous group cohomology.

For each site $C$, we denote by $\text{Sh}(C)$ (resp. $\text{Sh}_{Ab}(C)$) the category of sheaves (resp. abelian sheaves) on $C$, although we shall denote the associated topos by $C^\sim$ instead.

We shall follow [SW17 Sec. 2–7] for the general definitions and results of Huber rings and pairs, adic spaces, and perfectoid spaces. Unless otherwise specified, all Huber rings or pairs will be assumed to be complete.
We say that an adic space is **locally noetherian** if it is locally isomorphic to \( \text{Spa}(R, R^+) \), where either \( R \) is analytic (see [SW17, Rem. 2.2.7 and Prop. 4.3.1]) and **strongly noetherian**—i.e., the rings

\[
R(T_1, \ldots, T_n) = \left\{ \sum_{i_1, \ldots, i_n \geq 0} a_{i_1, \ldots, i_n} T_1^{i_1} \cdots T_n^{i_n} \in R[[T_1, \ldots, T_n]] : a_{i_1, \ldots, i_n} \to 0 \right\}
\]

are noetherian, for all \( n \geq 0 \); or \( R \) is (complete, by our convention, and) finitely generated over a noetherian ring of definition. We say that an adic space is **noetherian** if it is locally noetherian and qcqs (i.e., quasi-compact and quasi-separated).

We shall follow [Hub96, Def. 1.2.1] for the definition for morphisms of locally noetherian adic spaces to be **locally of finite type** (lift for short). A useful fact is that a fiber product \( Y \times_X Z \) of locally noetherian adic spaces exist when the first morphism \( Y \to X \) is lft, in which case its base change (i.e., the second projection) \( Y \times_X Z \to Z \) is also lft (see [Hub96, (1.1.1), Prop. 1.2.2, and Cor. 1.2.3]).

An **affinoid field** \((k, k^+)\) is a Huber pair in which \( k \) is a (possibly trivial) nonarchimedean local field (i.e., a field complete with respect to a nonarchimedean multiplicative norm \( \lvert \cdot \rvert : k \to \mathbb{R}_{\geq 0} \)), and \( k^+ \) is an open valuation subring of \( \mathcal{O}_k := \{ x \in k : \lvert x \rvert \leq 1 \} \) (see [SW17, Def. 4.2.4]). When \( k \) is a nontrivial nonarchimedean field (i.e., a field that is complete with respect to a nontrivial nonarchimedean multiplicative norm), we shall regard rigid analytic varieties over \( k \) as adic spaces over \((k, \mathcal{O}_k)\), by virtue of [Hub96, (1.1.11)].

We shall follow [Hub96, Sec. 1.6 and 1.7] for the definition and basic properties of adic spaces. More generally, we say that a homomorphism \((R, R^+) \to (S, S^+)\) of Huber pairs is **finite étale** if \( R \to S \) is finite étale as a ring homomorphism, and if \( S^+ \) is the integral closure of \( R^+ \) in \( S \); and that a morphism \( f : Y \to X \) of adic spaces is **finite étale** if, for each \( x \in X \), there exists an open affinoid neighborhood \( U \) of \( x \) in \( X \) such that \( V = f^{-1}(U) \) is affinoid, and if the induced homomorphism of Huber pairs \((\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \to (\mathcal{O}_Y(V), \mathcal{O}_Y^+(V))\) is finite étale. We say that a morphism \( f : Y \to X \) of adic spaces is **étale** if, for each \( y \in Y \), there exists open neighborhood \( V \) of \( y \) in \( Y \) such that the restriction of \( f \) to \( V \) factors as the composition of an open immersion, a finite étale morphism, and another open immersion.

Given any adic space \( X \), we denote by \( X_{\text{ét}} \) the category of adic spaces étale over \( X \). If fiber products exist in \( X_{\text{ét}} \), then \( X_{\text{ét}} \) acquires a structure of a site. We say that adic space \( X \) is **étale sheafy** if \( X_{\text{ét}} \) admits a basis \( \mathcal{B} \) consisting of affinoid adic spaces such that, for any morphism \( Y' \to Y \) in \( X_{\text{ét}} \) that is either finite étale or a rational localization, if \( Y \in \mathcal{B} \), then \( Y' \in \mathcal{B} \) as well. Étale sheafiness is known when \( X \) is either locally noetherian (see the above and [Hub96, (1.1.1) and Sec. 1.7]) or a perfectoid space (see [SW17, Sec. 7]). The presheaf \( \mathcal{O}_{X_{\text{ét}}} \) is a sheaf when \( X \) is either locally noetherian or analytic and étale sheafy (see Corollary [A.1.1]).

A **geometric point** of an adic space \( X \) is a morphism \( \eta : \xi = \text{Spa}(l, l^+) \to X \), where \( l \) is a separably closed nonarchimedean field. For simplicity, we shall write \( \xi \to X \), or even \( \xi \), when the context is clear. The image of the unique closed point \( \xi_0 \) of \( \xi \) under \( \eta : \xi \to X \) is called the **support** of the geometric point. Given any \( x \in X \), we have a geometric point \( \pi = \text{Spa}(\pi(x), \pi(x)^+) \) above \( x \) (i.e., \( x \) is the support of \( \pi \)), as in [Hub96, (2.5.2)], where \( \pi(x) \) is the completion of a separable closure of the residue field \( \kappa(x) \) of \( \mathcal{O}_{X,x} \). An **étale neighborhood** of \( \eta \) is a lifting of \( \eta \) to a composition \( \xi \to U \xrightarrow{\phi} X \) in which \( \phi \) is étale. For any sheaf \( F \) on \( X_{\text{ét}} \), the **stalk**
of \( \mathcal{F} \) at \( \eta \) is \( \mathcal{F}_\xi := \Gamma(\xi, \eta^{-1}(\mathcal{F})) \cong \varinjlim \mathcal{F}(V) \), where the direct limit runs through all étale neighborhoods \( V \) of \( \xi \). (Recall that, by [Hub96] Prop. 2.5.5, when \( X \) is locally noetherian, geometric points form a conservative family for \( X_{\text{ét}} \).

An adic space \( X = \text{Spa}(R, R^+) \) is strictly local if \( R \) is a strictly local ring and if \( X \) contains a unique closed point \( x \) such that the support of the valuation \( \cdot(x) \) is the maximal ideal of \( R \). We shall denote by \( X(\xi) = \text{Spa}(\mathcal{O}_{X, \xi}, \mathcal{O}_{X, \xi}^+) \) the strict localization of a geometric point \( \xi \to X \) of a locally noetherian adic space \( X \), as in [Hub96] (2.5.9) and Lem. 2.5.10]. By the explicit description of the completion of \( (\mathcal{O}_{X, \xi}, \mathcal{O}_{X, \xi}^+) \) as in [Hub96] Prop. 2.5.13], \( X(\xi) \) is a noetherian adic space, which is canonically isomorphic to \( \xi \) when the support of \( \xi \) is analytic.

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2. Log adic spaces

2.1. Recollection on monoids. In this subsection, we recollect some basics in the theory of monoids. This is mainly to introduce the terminologies and fix the notation. For more details, we refer the readers to [Ogu18].

Definition 2.1.1. (1) A monoid \( P \) is called finitely generated if there exists a surjective homomorphism \( \mathbb{Z}_{\geq 0}^n \to P \) for some \( n \).

(2) A monoid \( P \) is called integral if the natural homomorphism \( P \to P^{gp} \) is injective.

(3) A monoid \( P \) is called fine if it is integral and finitely generated.

(4) A monoid \( P \) is called saturated if it is integral and, for every \( a \in P^{gp} \) such that \( na \in P \) for some integer \( n \geq 1 \), we have \( a \in P \). A monoid that is both fine and saturated is called an fs monoid.

(5) For any monoid \( P \), we denote by \( P^\ast \) the subgroup of invertible elements in \( P \), and write \( \overline{P} := P/P^\ast \). A monoid \( P \) is called sharp if \( P^\ast = \{0\} \).

(6) An sharp fs monoid is called a toric monoid.

Remark 2.1.2. Inductive and projective limits exist in the category of monoids (see [Ogu18] Sec. I.1.1). In particular, for a homomorphism of monoids \( u : P \to Q \), we have \( \ker(u) = u^{-1}(0) \), and \( \coker(u) \) is determined by the conditions that \( Q \to \coker(u) \) is surjective and that two elements \( q_1, q_2 \in Q \) have the same image in \( \coker(u) \) if and only if there exist \( p_1, p_2 \in P \) such that \( u(p_1) + q_1 = u(p_2) + q_2 \). In general, the induced map \( Q/\ker(u) \to \text{im}(u) \) is surjective, but not necessarily injective. Therefore, the category of monoids is not abelian. Nevertheless, if \( P \) is a submonoid of \( Q \), and if \( u : P \to Q \) is the canonical inclusion, then we shall denote \( \coker(u) \) by \( Q/P \). Note that \( Q/P \) can be zero even when \( P \neq Q \).

Definition 2.1.3. Given any two homomorphisms of monoids \( u_1 : P \to Q_1 \) and \( u_2 : P \to Q_2 \), the amalgamated sum \( Q_1 \oplus_P Q_2 \) is the coequalizer of \( P \rightrightarrows Q_1 \oplus Q_2 \), with the two homomorphisms given by \((u_1, 0)\) and \((0, u_2)\), respectively.

Lemma 2.1.4. In Definition 2.1.3 suppose moreover that any of \( P, Q_1 \), or \( Q_2 \) is a group. Then the natural map \( Q_1/P \to (Q_1 \oplus_P Q_2)/Q_2 \) is an isomorphism.
Proof. The surjectivity is clear. As for the injectivity, by assumption and by [Ogu18] Prop. I.1.1.5, two elements \((q_1, q_2), (q'_1, q'_2) \in Q_1 \oplus Q_2\) have the same image in \(Q_1 \oplus p Q_2\) if and only if there exist \(a, b \in P\) such that \(q_1 + u_1(a) = q'_1 + u_1(b)\) and \(q_2 + u_2(b) = q'_2 + u_2(a)\). Therefore, for \(q_1, q'_1 \in Q_1\), if they have the same image in \((Q_1 \oplus p Q_2)/Q_2\)—i.e., there exist \(q_2, q'_2 \in Q_2\) such that \((q_1, q_2)\) and \((q'_1, q'_2)\) have the same image in \(Q_1 \oplus p Q_2\) —then there exist \(a, b \in P\) such that \(q_1 + u_1(a) = q'_1 + u_1(b)\). It follows that \(q_1\) and \(q'_1\) have the same image in \(Q_1/P\), as desired. □

Definition 2.1.5. For any monoid \(P\), let \(P^{\text{int}}\) denote the image of the canonical homomorphism \(P \to P^{\text{gp}}\). For an integral monoid \(P\), let \(P^{\text{sat}} = \{a \in P^{\text{gp}} : n a \in P, \text{ for some } n \geq 1\}\). For a general monoid \(P\) that is not necessarily integral, we write \(P^{\text{sat}}\) for \((P^{\text{int}})^{\text{sat}}\).

Remark 2.1.6. The functor \(P \to P^{\text{int}}\) is the left adjoint of the inclusion from the category of integral monoids into the category of all monoids. Similarly, \(P \to P^{\text{sat}}\) is the left adjoint of the inclusion from the category of saturated monoids into the category of integral monoids.

Lemma 2.1.7. Let \(P \to Q_1\) and \(P \to Q_2\) be homomorphisms of monoids. Then \((Q_1 \oplus p Q_2)^{\text{int}}\) can be naturally identified with the image of \(Q_1 \oplus p Q_2\) in \(Q_1^{\text{gp}} \oplus p\text{sat} Q_2^{\text{gp}}\). Moreover, if \(P, Q_1,\) and \(Q_2\) are integral and if any of these monoids is a group, then \(Q_1 \oplus p Q_2\) is also integral.

Proof. See [Ogu18] Prop. I.1.3.4]. □

Lemma 2.1.8. The quotient of an integral (resp. a saturated) monoid by a submonoid is also integral (resp. saturated). In particular, for any fs monoid \(P\), the quotient \(\overline{P} = P/P^*\) is a toric monoid.

Proof. Let \(Q\) be a submonoid of an integral monoid \(P\). Then \(P/Q\) is also integral, by [Ogu18] Prop. I.1.3.3]. Suppose moreover that \(P\) is saturated. For \(\overline{a} \in (P/Q)^{\text{sat}}\), suppose that \(n \overline{a} \in P/Q\) for some \(n \geq 1\). That is, there exist \(b \in P\) and \(q_1, q_2 \in Q\) such that \(n a = b + (q_1 - q_2)\) in \(P^{\text{gp}}\). Then \(n (a + q_2) = b + q_1 + (n - 1)q_2\) and hence \(a + q_2 \in P\), yielding \(\overline{a} \in P/Q\). Thus, \(P/Q = (P/Q)^{\text{sat}}\) is also saturated. □

Lemma 2.1.9. Let \(P\) be an integral monoid, and \(u : P \to Q\) a surjective homomorphism onto a toric monoid \(Q\). Suppose that \(\ker(u^{\text{gp}}) \subset P\). Then \(u\) admits a (noncanonical) section. In particular, for any fs monoid \(P\), the canonical homomorphism \(P \to \overline{P}\) admits a (noncanonical) section.

Proof. For \(a \in Q^{\text{gp}}\), if \(n a = 0\) for some \(n \geq 1\), then \(a = 0\), as \(Q\) is saturated and sharp. Hence, \(Q^{\text{gp}}\) is torsionfree, \(Q^{\text{gp}} \cong \mathbb{Z}^r\) for some \(r\), and the projection \(u^{\text{gp}} : P^{\text{gp}} \to Q^{\text{gp}}\) admits a section \(s : Q^{\text{gp}} \to P^{\text{gp}}\). It remains to show that \(s(Q) \subset P\). For each \(q \in Q\), choose any \(\overline{q} \in P\) lifting \(q\). Then \(s(q) - \overline{q} \in P^{\text{gp}}\) lies in \(\ker(u^{\text{gp}}) = P\), and therefore \(s(q) \in \overline{q} + P \subset P\), as desired. □

Construction 2.1.10. Let \(P\) be a monoid, and \(S\) a subset of \(P\). There exists a monoid \(S^{-1}P\) together with an homomorphism \(\lambda : P \to S^{-1}P\) sending elements of \(S\) to invertible elements of \(S^{-1}P\) satisfying the universal property that any homomorphism of monoids \(u : P \to Q\) with the property that \(u(S) \subset Q^*\) uniquely factors through \(S^{-1}P\). The monoid \(S^{-1}P\) is called the localization of \(P\) with respect to \(S\). Concretely, let \(T\) denote the submonoid of \(P\) generated by \(S\). Then, as a set, \(S^{-1}P\) consists of equivalence classes of pairs \((a, t) \in P \times T\), where two
such pairs \((a,t)\) and \((a',t')\) are considered equivalent if there exists some \(t'' \in T\) such that \(a + t' + t'' = a' + t + t''\). The monoid structure of this set is given by 
\[(a,t) + (a',t') = (a + a', t + t').\] The homomorphism \(\lambda\) is given by \(\lambda(a) = (a,0)\).

**Remark 2.1.11.** The localization of an integral (resp. saturated) monoid is still integral (resp. saturated).

**Remark 2.1.12.** Let \(P \rightarrow Q_1\) and \(P \rightarrow Q_2\) be homomorphisms of monoids, and let \(S\) be a subset of \(P\). Let \(S_1\), \(S_2\), and \(S_3\) denote the images of \(S\) in \(Q_1\), \(Q_2\), and \(Q_1 \oplus_p Q_2\), respectively. Then the natural homomorphism \(Q_1 \oplus_p Q_2 \rightarrow (S_1^{-1}Q_1) \oplus_{S^{-1}p} (S_1^{-1}Q_2)\) factors through an isomorphism \(S_1^{-1}(Q_1 \oplus_p Q_2) \sim (S_1^{-1}Q_1) \oplus_{S^{-1}p} (S_1^{-1}Q_2)\), by the universal properties of the objects.

**Definition 2.1.13.** Let \(u : P \rightarrow Q\) be a homomorphism of monoids.

1. We say it is **local** if \(P^* = u^{-1}(Q^*)\).
2. We say it is **sharp** if the induced homomorphism \(P^* \rightarrow Q^*\) is an isomorphism.
3. We say it is **strict** if the induced homomorphism \(\overline{P} \rightarrow \overline{Q}\) is an isomorphism.
4. We say it is **exact** if the induced homomorphism \(P \rightarrow P^{gp} \times_{Q^{gp}} Q\) is an isomorphism. (When \(P\) and \(Q\) are integral and canonically identified as submonoids of \(P^{gp}\) and \(Q^{gp}\), respectively, we simply need \(P = (u^{gp})^{-1}(Q)\).)

### 2.2 Log adic spaces

In this subsection, we give the definition of log adic spaces, introduce some basic notions, and study some important examples.

**Convention 2.2.1.** From now on, we shall only work with adic spaces that are étale sheafy. (They include locally noetherian adic spaces and perfectoid spaces.)

**Definition 2.2.2.** Let \(X\) be an (étale sheafy) adic space.

1. A **pre-log structure** on \(X\) is a pair \((\mathcal{M}_X, \alpha)\), where \(\mathcal{M}_X\) is a sheaf of monoids on \(\mathcal{X}_{\text{ét}}\) and \(\alpha : \mathcal{M}_X \rightarrow \mathcal{O}_{\mathcal{X}_{\text{ét}}}\) is a morphism of sheaves of monoids, called the structure morphism. (Here \(\mathcal{O}_{\mathcal{X}_{\text{ét}}}\) is equipped with the natural multiplicative monoid structure.)
2. Let \((\mathcal{M}, \alpha)\) and \((\mathcal{N}, \beta)\) be pre-log structures on \(X\). A morphism from \((\mathcal{M}, \alpha)\) to \((\mathcal{N}, \beta)\) is a morphism \(\mathcal{M} \rightarrow \mathcal{N}\) of sheaves of monoids that is compatible with the structure morphisms.
3. A pre-log structure \((\mathcal{M}_X, \alpha)\) on \(X\) is called a **log structure** if \(\alpha^{-1}(\mathcal{O}_{\mathcal{X}_{\text{ét}}}^\times) \rightarrow \mathcal{O}_{\mathcal{X}_{\text{ét}}}^\times\) is an isomorphism. In this case, we call the triple \((X, \mathcal{M}_X, \alpha)\) a **log adic space**. We shall simply write \((X, \mathcal{M}_X)\) or \(X\) when the context is clear.
4. We say that a sheaf of monoids \(\mathcal{M}\) on \(\mathcal{X}_{\text{ét}}\) is **integral** (resp. saturated) if it is a sheaf of integral (resp. saturated) monoids. A pre-log structure \((\mathcal{M}_X, \alpha)\) on \(X\) is called **integral** (resp. **saturated**) if \(\mathcal{M}_X\) is. We say that a log adic space \((X, \mathcal{M}_X, \alpha)\) is **integral** (resp. **saturated**) if \(\mathcal{M}_X\) is.
5. For a log structure \((\mathcal{M}_X, \alpha)\) on \(X\), we set \(\overline{\mathcal{M}_X} := \mathcal{M}_X / \alpha^{-1}(\mathcal{O}_{\mathcal{X}_{\text{ét}}}^\times)\), called the **characteristic** of the log structure.
6. For a pre-log structure \((\mathcal{M}_X, \alpha)\) on \(X\), we have the **associated log structure** \((^a\mathcal{M}_X, ^a\alpha)\), where \(^a\mathcal{M}_X\) is the pushout of \(\mathcal{O}_{\mathcal{X}_{\text{ét}}}^\times \leftarrow \alpha^{-1}(\mathcal{O}_{\mathcal{X}_{\text{ét}}}^\times) \rightarrow \mathcal{M}_X\) in the category of sheaves of monoids on \(\mathcal{X}_{\text{ét}}\), and where \(^a\alpha : ^a\mathcal{M}_X \rightarrow \mathcal{O}_{\mathcal{X}_{\text{ét}}}^\times\) is canonically induced by the natural morphism \(\mathcal{O}_{\mathcal{X}_{\text{ét}}}^\times \rightarrow \mathcal{O}_{\mathcal{X}_{\text{ét}}}\) and the structure morphism \(\alpha : \mathcal{M}_X \rightarrow \mathcal{O}_{\mathcal{X}_{\text{ét}}}\) (cf. [GR19, Sec. 12.1.6]). Again, we shall simply write \(^a\mathcal{M}_X\) when the context is clear.
Remark 2.2.4. A morphism $f : (Y, \mathcal{M}_Y, \alpha_Y) \to (X, \mathcal{M}_X, \alpha_X)$ of log adic spaces is a morphism $f : Y \to X$ of adic spaces together with a morphism of sheaves of monoids $f^\sharp : f^{-1}(\mathcal{M}_X) \to \mathcal{M}_Y$ compatible with $f^{-1}(\alpha_X) : f^{-1}(\mathcal{M}_X) \to f^{-1}(\mathcal{O}_{\mathcal{M}_X})$ and $\alpha_Y : \mathcal{M}_Y \to \mathcal{O}_{\mathcal{Y}_\eta}$. In this case, we have the log structure $f^*(\mathcal{M}_X)$ on $Y$ associated with the pre-log structure $f^{-1}(\mathcal{M}_X) \to f^{-1}(\mathcal{O}_{\mathcal{M}_X}) \to \mathcal{O}_{\mathcal{Y}_\eta}$. The morphism $f$ is called strict if the induced morphism $f^*(\mathcal{M}_X) \to \mathcal{M}_Y$ is an isomorphism.

Example 2.2.7. Every $\eta$-adic log structure on $X$ is the left adjoint of the natural inclusion functor from the category of log adic spaces to the category of pre-log structures on $X$.

As explained in [GR19, Sec. 12.1.6], the functor of taking associated log structures from the category of pre-log structures to the category of log structures on $X$ is the left adjoint of the natural inclusion functor from the category of log structures to the category of pre-log structures on $X$.

Remark 2.2.8. For a log adic space $(X, \mathcal{M}_X, \alpha)$ and a geometric point $\eta$ of $X$, it follows that $\mathcal{M}_{X,\eta} = \alpha^{-1}(\mathcal{O}_{X,\eta}) \sim \mathcal{O}_{X,\eta}$ (i.e., the homomorphism $\mathcal{M}_{X,\eta} \to \mathcal{O}_{X,\eta}$ is local and sharp). Consequently, $\overline{\mathcal{M}}_{X,\eta} \cong \mathcal{M}_{X,\eta}/\alpha^{-1}(\mathcal{O}_{X,\eta}^\times)$ is a sharp monoid, and $\overline{\mathcal{M}}_{X,\eta}^{\text{gp}} \cong \mathcal{M}_{X,\eta}^{\text{gp}}/\alpha^{-1}(\mathcal{O}_{X,\eta}^\times)$. In particular, $\mathcal{M}_{X,\eta} \to \overline{\mathcal{M}}_{X,\eta}$ is exact.

Remark 2.2.9. Let $f : (Y, \mathcal{M}_Y, \alpha_Y) \to (X, \mathcal{M}_X, \alpha_X)$ be a morphism of log adic spaces. At each geometric point $\eta$ of $Y$, since $f^\sharp_{\eta} : \mathcal{O}_{X, f(\eta)} \to \mathcal{O}_{Y, \eta}$ is local, and since $f^\sharp_{\eta} : \mathcal{M}_{X, f(\eta)} \cong f^{-1}(\mathcal{M}_X)_{\eta} \to \mathcal{M}_{Y, \eta}$ is compatible with $f^\sharp_{\eta} : \mathcal{O}_{X, f(\eta)} \cong f^{-1}(\mathcal{O}_X)_{\eta} \to \mathcal{O}_{Y, \eta}$ by definition, we see that $f^\sharp_{\eta} : \mathcal{M}_{X, f(\eta)} \to \mathcal{M}_{Y, \eta}$ is local as in Definition 2.1.13. By Lemma 2.1.4, $(f^*(\mathcal{M}_X))_{\eta} \cong \overline{\mathcal{M}}_{X, f(\eta)}$. Therefore, $f$ is strict if and only if $\mathcal{M}_{X, f(\eta)} \cong \overline{\mathcal{M}}_{X, f(\eta)}$, i.e., $f^\sharp_{\eta} : \mathcal{M}_{X, f(\eta)} \to \mathcal{M}_{Y, \eta}$ is strict, for every $\eta$.

Lemma 2.2.6. A sheaf of monoids $\mathcal{M}$ on an adic space $X_{\text{et}}$ is integral (resp. saturated) if and only if the stalk $\mathcal{M}_\eta$ is integral (resp. saturated) for every geometric point $\eta$ of $X$. In particular, a log adic space $(X, \mathcal{M}_X, \alpha)$ is integral (resp. saturated) if and only if $\mathcal{M}_{X, \eta}$ is integral (resp. saturated) for every geometric point $\eta$ of $X$.

Proof. This follows from [GR19, Lem. 12.1.18(ii)].

Here are some basic examples of log adic spaces.

Example 2.2.7. Every $\text{étale sheafy}$ adic space $X$ has a natural log structure with $\mathcal{M}_X = \mathcal{O}_{X_{\text{et}}}$ and $\alpha$ the identity map. We call it the trivial log structure on $X$.

Example 2.2.8. A log point is a log adic space whose underlying adic space is $\text{Spa}(l, l^+)$, where $l$ is a nonarchimedean local field. We remark that the underlying topological space may not be a single point.

Example 2.2.9. In Example 2.2.8, if $l$ is separably closed, then the étale topos of $\text{Spa}(l, l^+)$ is equivalent to the category of sets (see [Hub96, Cor. 1.7.3 and Prop. 2.3.10, and the paragraph after (2.5.2)]). In this case, a log structure on $\text{Spa}(l, l^+)$ is given by a homomorphism of monoids $\alpha : M \to l$ inducing an isomorphism...
\[ \alpha^{-1}(I^\times) \cong I^\times \]. For simplicity, by abuse of notation, we shall sometimes introduce a log point by writing \( s = (\text{Spa}(I, I^+), M) \). Also, we shall simply denote by \( s \) the underlying adic space \( \text{Spa}(I, I^+) \), when the context is clear.

**Example 2.2.10.** Let \((X, \mathcal{M}_X, \alpha_X)\) be a perfectoid log adic space; i.e., a log adic space whose underlying adic space \( X \) is a perfectoid space. Let \( \mathcal{M}_{X^+} := \lim_{\leftarrow r} \mathcal{M}_X \), where the transition maps are given by sending a section to its \( p \)-th multiple. Let \( X^+ \) be the tilt of \( X \). Then there is a natural morphism of sheaves of monoids \( \alpha_{X^+} : \mathcal{M}_{X^+} \to \mathcal{O}_{X^+_\mathbb{A}} \) making \((X^+, \mathcal{M}_{X^+}, \alpha_{X^+})\) a perfectoid log adic space, called the tilt of \((X, \mathcal{M}_X, \alpha_X)\).

We would like to study log adic spaces of the form \( \text{Spa}(R[P], R^+[P]) \), whenever \((R, R^+)\) is a Huber pair.

**Lemma 2.2.11.** Suppose that \((R, R^+)\) is a Huber pair with a ring of definition \( R_0 \subset R \), which is adic with respect to a finitely generated ideal \( I \). Equip \( R[P] \) with the topology determined by the ring of definition \( R_0[P] \) such that \( \{I^n R_0[P]\}_{n \geq 0} \) forms a basis of open neighborhoods of \( 0 \). Then \((R[P], R^+[P])\) is also a Huber pair.

**Proof.** Note that \( R^+[P] \) is open in \( R[P] \) because \( R^+ \) is open in \( R \). We only need to check that \( R^+[P] \) is integrally closed in \( R[P] \). By writing \( P \) as the direct limit of its finitely generated submonoids, we may assume that \( P \) is finitely generated. But this case is standard (see, for example, [BG09, Thm. 4.42]). \( \square \)

**Remark 2.2.12.** Let \((R<P>, R^+(P))\) denote the completion of \((R[P], R^+[P])\). Since taking completions does not alter the associated adic spectrum, we can identify \( \text{Spa}(R[P], R^+[P]) \) with \( \text{Spa}(R<P>, R^+(P)) \) whenever it is convenient.

**Lemma 2.2.13.** Let \( P \) be a finitely generated monoid. Suppose that \( R \) is either (1) analytic and strongly noetherian; or (2) complete, by our convention, and finitely generated over a noetherian ring of definition. Then so is \( R(P) \) (which is complete by definition). Consequently, \( \text{Spa}(R(P), R^+(P)) \) is a noetherian adic space when \( \text{Spa}(R, R^+) \) is, and \( \text{Spa}(R(P), R^+(P)) \) is étale sheafy. Moreover, the formation of the canonical morphism \( \text{Spa}(R(P), R^+(P)) \to \text{Spa}(R, R^+) \) is compatible with rational localizations in \( \text{Spa}(R, R^+) \).

**Proof.** Suppose that \( R \) is analytic and strongly noetherian. Since \( P \) is finitely generated, there is some surjection \( \mathbb{Z}_{\geq 0}^r \to P \), which induces a continuous surjection \( R(T_1, \ldots, T_r) \cong R(\mathbb{Z}_{\geq 0}^r) \to R<P> \). In this case, \( R<P><T_1, \ldots, T_r> \) is a quotient of \( R(\mathbb{Z}_{\geq 0}^r, T_1, \ldots, T_n) \cong R(T_1, \ldots, T_{r+n}) \), for each \( n \geq 0 \), which is noetherian as \( R \) is strongly noetherian. Hence, \( R(P) \) is also analytic and strongly noetherian.

Alternatively, suppose that \( R \) is generated by some \( u_1, \ldots, u_n \) over a noetherian ring of definition \( R_0 \), with an ideal of definition \( I \subset R_0 \). Since \( R_0[P] \) is noetherian as \( P \) is finitely generated, its \( IR_0[P] \)-adic completion \( R_0<P> \) is also noetherian. Then the image of \( R_0<P> \) is a noetherian ring of definition of \( R<P> \), over which \( R<P> \) is generated by the images of \( u_1, \ldots, u_n \), as desired.

In both cases, the formation of the canonical morphism \( \text{Spa}(R<P>, R^+(P)) \to \text{Spa}(R, R^+) \) is clearly compatible with rational localizations in \( \text{Spa}(R, R^+) \). \( \square \)

In a different direction, we would like to show that, under certain condition on \( P \), if \((R, R^+)\) is a perfectoid affinoid algebra, then \((R<P>, R^+(P))\) also is. In this case, \((R<P>, R^+(P))\) is étale sheafy.
Definition 2.2.14. For any \( n \in \mathbb{Z}_{\geq 1} \), a monoid \( P \) is called \( n \)-divisible (resp. uniquely \( n \)-divisible) if the \( n \)-th multiple map \([n] : P \to P\) is surjective (resp. bijective).

Lemma 2.2.15. Suppose that \((R, R^+)\) is a perfectoid Huber pair. If \( P \) is uniquely \( p \)-divisible, then \((R(P), R^+(P))\) is a perfectoid Huber pair as well. Moreover, the formation of the canonical morphism \( \text{Spa}(R(P), R^+(P)) \to \text{Spa}(R, R^+)\) is compatible with rational localizations in \( \text{Spa}(R, R^+)\).

Proof. If \( pR = 0 \), then the unique \( p \)-divisibility of \( P \) implies that \( R^+[P] \) is perfect, and so is its completion \( R^+(P) \). Moreover, it is clear that \((R(P))^\circ = R^+(P)\) in \( R(P)\). Hence, \( R(P)\) is uniform, and \((R(P), R^+(P))\) is a perfectoid Huber pair.

In general, let \((R^\circ, R^{\circ +})\) be the tilt of \((R, R^+)\). Let \( \varpi \in R \) be a pseudo-uniformizer of \( R \) satisfying \( \varpi^p|p \) in \( R^\circ \) and admitting a sequence of \( p \)-th power roots \( \varpi, \varpi^\frac{1}{p}, \ldots \in R^\circ \) is a pseudo-uniformizer of \( R^\circ \), as in [SW17 Lem. 6.2.2]. Let \( \xi \) be a generator of \( \ker(\theta : W(R^{\circ +}) \to R^+) \), which can be written as \( \xi = p + [\varpi]a \) for some \( a \in W(R^+) \), by [SW17 Lem. 6.2.8]. By the first paragraph above and the tilting equivalence (see [SW17 Thm. 6.2.11]), it suffices to show that \( R^+(P) \cong W(R^{\circ +}(P))/\langle \xi \rangle \). For this purpose, note that there is a natural homomorphism \( \theta' : W(R^{\circ +}(P)) \to R^+(P) \) induced by the surjective homomorphism \( R^+(P) \to (R^{\circ +}/\varpi^1)[P] \cong (R^+/[\varpi])[P] \) and the universal property of Witt vectors, and \( \theta' \) is surjective because both its source and target are complete. Since \( W(R^{\circ +}(P))/\langle \xi, [\varpi] \rangle \cong (R^{\circ +}/\varpi^1)[P] \cong (R^+/[\varpi])[P] \), the homomorphism \( \theta' \) induces \( W(R^{\circ +}(P))/\langle \xi, [\varpi] \rangle \cong (R^+/[\varpi^n])[P] \), for each \( n \geq 1 \). As \( W(R^{\circ +}(P))/\xi \) is \([\varpi]\)-adically complete and separated, \( \ker(\theta') \) is generated by \( \xi \), as desired.

Finally, as before, the formation of the canonical morphism \( \text{Spa}(R(P), R^+(P)) \to \text{Spa}(R, R^+)\) is clearly compatible with rational localizations in \( \text{Spa}(R, R^+)\). \( \square \)

Remark 2.2.16. In Lemma 2.2.15, perfectoid Huber pairs are Tate by our convention following [SW17 Sec. 6], but the statement of the lemma remains true for more general analytic perfectoid Huber pairs as in [Ked19], by using [Ked19 Lem. 2.7.9] instead of [SW17 Thm. 6.2.11].

Definition 2.2.17. When \( X = \text{Spa}(R[P], R^+[P]) \cong \text{Spa}(R(P), R^+(P)) \) is étale sheafy, we denote by \( P_X \) the constant sheaf on \( X_{\text{ét}} \) defined by \( P \). Then the natural homomorphism \( P \to R(P) \) of monoids induces a pre-log structure \( P_X \to \mathcal{O}_{X_{\text{ét}}} \) on \( X \), whose associated log structure we simply denote by \( P_{\log} \).

Convention 2.2.18. From now on, when \( \text{Spa}(R[P], R^+[P]) \) or \( \text{Spa}(R(P), R^+(P)) \) is étale sheafy and regarded as a log adic space, we shall endow it with the log structure \( P_{\log} \) as in Definition 2.2.17 unless otherwise specified.

Let us continue with some more examples of log adic spaces.

Example 2.2.19. Given any locally noetherian adic space \( Y \) with trivial log structure as in Example 2.2.7 and given any finitely generated monoid \( P \), by gluing the morphisms \( \text{Spa}(R(P), R^+(P)) \to \text{Spa}(R, R^+) \) as in Lemma 2.2.13 over the noetherian affinoid open \( \text{Spa}(R, R^+) \) in \( Y \), and by equipping \( \text{Spa}(R(P), R^+(P)) \) with the structure of a log adic space as in Definition 2.2.17 and Convention 2.2.18 we obtain a morphism \( X \to Y \) of log adic spaces, which we shall denote by \( Y(P) \to Y \). In this case, we shall also denote the log structure on \( X = Y(P) \) by \( P_{\log} \).
Example 2.2.20. If $P$ is a toric monoid as in Definition 2.1.16, then we say that $X = \text{Spa}(k(P), k^+(P))$ is an affinoid toric log adic space. This is a special case of Example 2.2.19 with $Y = \text{Spa}(k, k^+)$, and is closely related to the theory of toroidal embeddings and toric varieties (see, e.g., [KKMS73] and [PM93]). Roughly speaking, such affinoid toric log adic spaces provide affinoid open subspaces of the rigid analytification of toric varieties, which are then also useful for studying local properties of more general varieties or rigid analytic varieties which are locally modeled on toric varieties. Note that the underlying spaces of affinoid toric log adic spaces are always normal, by [BGR84] Sec. 7.3.2, Prop. 8, [GD67] IV-2, 7.8.3.1, and [Hoc72] Thm. 1 (cf. [Kas93] Thm. 4.1).

Example 2.2.21. A special case of Example 2.2.20 is when $P \cong \mathbb{Z}_{\geq 0}^n$ for some integer $n \geq 0$. In this case, we obtain

$$X = \text{Spa}(k(P), k^+(P)) \cong \mathbb{D}^n := \text{Spa}(k(T_1, \ldots, T_n), k^+(T_1, \ldots, T_n)),$$

the $n$-dimensional unit disc, with the log structure on $\mathbb{D}^n$ associated with the pre-log structure given by $\mathbb{Z}_{\geq 0}^n \rightarrow k(T_1, \ldots, T_n) : (a_1, \ldots, a_n) \mapsto T_1^{a_1} \cdots T_n^{a_n}$.

The following proposition provides many more examples of log adic spaces coming from locally noetherian log formal schemes.

Proposition 2.2.22. The fully faithful functor from the category of locally noetherian formal schemes to the category of locally noetherian adic spaces defined locally by $\text{Spf}(A) \rightarrow \text{Spa}(A, A)$ (as in [Hub94] Sec. 4.1) canonically extends to a fully faithful functor from the category of locally noetherian log formal schemes (as in [GR19] Sec. 12.1) to the category of locally noetherian log adic spaces.

Proof. Given any locally noetherian log formal scheme $(\mathcal{X}, \mathcal{M}_X)$ with associated adic space $X$, we have a canonical morphism of sites $\lambda : X_{\text{et}} \rightarrow \mathcal{X}_{\text{et}}$, as in [Hub94] Lem. 3.5.1. By construction, we have a canonical morphism $\lambda^{-1}(\mathcal{O}_{X_M}) \rightarrow \mathcal{O}_{\mathcal{X}_M}$. Let $\mathcal{M}_X$ be the log structure of $X$ associated with the pre-log structure $\lambda^{-1}(\mathcal{M}_X) \rightarrow \lambda^{-1}(\mathcal{O}_{X_M}) \rightarrow \mathcal{O}_{X_M} \rightarrow \mathcal{O}_{X_M}$. Then the assignment $(\mathcal{X}, \mathcal{M}_X) \mapsto (X, \mathcal{M}_X)$ gives the desired functor, which is fully faithful by adjunction.

Definition 2.2.23 (cf. [Ogu18] Def. III.2.3.1). We say that a morphism $f : Y \rightarrow X$ of log adic spaces is an open immersion (resp. a closed immersion) if the underlying morphism of adic spaces is an open immersion (resp. a closed immersion) and if the morphism $f^\#: f^{-1}(\mathcal{M}_X) \rightarrow \mathcal{M}_Y$ is an isomorphism (resp. a surjection). We say that $f$ is an immersion if it is a composition of a closed immersion of log adic spaces followed by an open immersion of log adic spaces. We say that $f$ is strict if it is a strict morphism of log adic spaces.

Example 2.2.24. Let $(X, \mathcal{M}_X, \alpha_X)$ be a log adic space and $i : Z \rightarrow X$ an immersion of adic spaces. Let $(Z, \mathcal{M}_Z, \alpha_Z)$ be the log adic space associated with the pre-log structure $i^{-1}(\mathcal{M}_X) \rightarrow i^{-1}(\mathcal{O}_X) \rightarrow \mathcal{O}_Z$. Then the induced map $(Z, \mathcal{M}_Z, \alpha_Z) \rightarrow (X, \mathcal{M}_X, \alpha_X)$ of log adic spaces is a strict immersion, and it is an open immersion (resp. a closed immersion) exactly when the immersion $i$ of adic spaces is.

Remark 2.2.25. More generally, by the same argument as in [Ogu18] the paragraph after Def. III.2.3.1, a closed immersion is strict when it is exact.

Definition 2.2.26. We say that a morphism $f : Y \rightarrow X$ of locally noetherian log adic spaces is proper if the underlying morphism of adic spaces is proper (see
As usual, a proper log adic space over Spa(\(k, k^+\)) is a locally noetherian log adic space with a proper structure morphism to Spa(\(k, k^+\)).

2.3. Charts and fiber products. In this subsection, we introduce the notion of charts for log adic spaces. Compared with the corresponding notion for log schemes, a notable difference is that the definition of charts for a log adic space \(X\) involves not just \(\mathcal{O}_{X,\eta}\) but also \(\mathcal{O}_{X,\eta}^\times\). Based on this notion, we also introduce the category of coherent (resp. fine, resp. fs) log adic spaces and study the fiber products in it.

**Definition 2.3.1.** Let \((X, \mathcal{M}_X, \alpha)\) be a log adic space. Let \(P\) be a monoid and let \(P_X\) denote the associated constant sheaf of monoids on \(X\). A (global) chart of \(X\) modeled on \(P\) is a morphism of sheaves of monoids \(\theta : P_X \rightarrow \mathcal{M}_X\) such that \(\alpha(\theta(P_X)) \subset \mathcal{O}_{X,\eta}^\times\) and such that the log structure associated with the pre-log structure \(\alpha \circ \theta : P_X \rightarrow \mathcal{O}_{X,\eta}\) is isomorphic to \(\mathcal{M}_X\). We call the chart finitely generated (resp. fine, resp. fs) if \(P\) is finitely generated (resp. fine, resp. fs).

**Remark 2.3.2.** Giving a morphism \(\theta : P_X \rightarrow \mathcal{M}_X\) as in Definition 2.3.1 is equivalent to giving a homomorphism \(P \rightarrow \mathcal{M}_X(X)\) of monoids. If \(P\) is finitely generated, and if the underlying adic space \(X\) is over some affinoid adic space Spa(\(R, R^+\)), then giving \(P \rightarrow \mathcal{M}_X(X)\) is equivalent to giving a morphism \(f : (X, \mathcal{M}_X) \rightarrow (\text{Spa}(R(P), R^+(P)), P^{\text{pro}})\) of log adic spaces, whenever \(\text{Spa}(R(P), R^+(P))\) is étale sheafy. In this case, \(\theta\) is a chart if and only if the morphism \(f\) is strict.

**Remark 2.3.3.** In Remark 2.3.2, if the underlying adic space \(X\) is over some locally noetherian adic space \(Y\), then giving a morphism \(\theta : P_X \rightarrow \mathcal{M}_X\) is also equivalent to giving a morphism \(g : X \rightarrow Y(P)\) as in Example 2.2.19 in which case \(\theta\) is a chart if and only if the morphism \(g\) is strict. Moreover, if \(X\) is itself locally noetherian, then we can take \(Y = X\), and obtain a closed immersion \(h : X \rightarrow X(P)\), in which case \(\theta\) is a chart if and only if \(h\) is a strict closed immersion.

**Remark 2.3.4.** Let \(\theta : P_X \rightarrow \mathcal{M}_X\) be a chart of a log adic space \((X, \mathcal{M}_X, \alpha)\). By Lemma 2.1.4 and Remark 2.2.4, for each geometric point \(\pi\) of \(X\), we obtain a canonical isomorphism \(P/\alpha(\theta) - 1(\mathcal{O}_{X,\pi}) \xrightarrow{\sim} \mathcal{M}_{X,\pi}/\alpha^{-1}(\mathcal{O}_{X,\pi}) \cong \mathcal{M}_{X,\pi}\). In particular, the composition \(P_X \xrightarrow{\theta} \mathcal{M}_X \rightarrow \mathcal{M}_{X,\pi}\) is surjective.

**Definition 2.3.5.** A coherent (resp. fine, resp. fs) log adic space is a log adic space \(X\) that étale locally admits some charts modeled on finitely generated (resp. fine, resp. fs) monoids.

**Lemma 2.3.6.** Let \((X, \mathcal{M}, \alpha)\) be a log adic space, and \(\theta : S_X \rightarrow \mathcal{M}\) a chart modeled on some monoid \(S\). Suppose that there exists a finitely generated monoid \(S'\) such that \(\theta\) factors as \(S_X \xrightarrow{\theta'} S'_X \xrightarrow{\theta} \mathcal{M}\) and such that the composition \(\alpha \circ \theta' : S'_X \rightarrow \mathcal{O}_{X,\eta}\) factors through \(\mathcal{O}_{X,\eta}^\times\). Then, étale locally on \(X\), there exists a chart \(\theta'' : S''_X \rightarrow \mathcal{M}\) modeled on some finitely generated monoid \(S''\) such that \(\theta''\) factors through \(\theta'').

**Proof.** The proof is similar to [Ogu18] Prop. II.2.2.1, except that, compare with charts on log schemes, charts on log adic spaces are subject to the additional requirement that \(\alpha \circ \theta\) factors through \(\mathcal{O}_{X}^\times\).

Let \(\{s_i\}_{i \in I}\) be a finite set of generators of \(S'\). Since \(S_X \rightarrow \mathcal{M}\) is surjective (by Remark 2.3.4), étale locally on \(X\), there exist some \(s_i \in S\) and \(f_j \in \mathcal{O}_{X,\pi}^\times(X)\) such that \(\theta'(s_i) = \theta(s_i)f_j\), for all \(i \in I\). By [Hub96] (1) in the proof of Prop. 2.5.13, for each geometric point \(\pi\) of \(X\), we have \(\mathcal{O}_{X,\pi}^+ = \{f \in \mathcal{O}_{X,\pi} : |f(\pi)| \leq 1\}\) in \(\mathcal{O}_{X,\pi}\).
Up to further étale localization on $X$, we may assume that, for each $i \in I$, at least one of $f_i$ and $f_i^{-1}$ is in $\mathcal{O}_X^+(X)$. Consider the homomorphism $S' \oplus \mathbb{Z}_{\geq 0} \to \mathcal{M}(X)$ sending $(s'_i, 0) \mapsto \theta(s_i)$ and sending
\[
\begin{cases}
(0, e_i) \mapsto f_i, & \text{if } f_i \in \mathcal{O}_X^+(X); \\
(0, e_i) \mapsto f_i^{-1}, & \text{if } f_i \notin \mathcal{O}_X^+(X) \text{ but } f_i^{-1} \in \mathcal{O}_X^+(X).
\end{cases}
\]

Let $\beta$ denote the homomorphism $S \to S'$, and let $S''$ be the quotient of $S' \oplus \mathbb{Z}_{\geq 0}$ modulo the relations
\[
\begin{cases}
(s'_i, 0) \sim (\beta(s_i), e_i), & \text{if } f_i \in \mathcal{O}_X^+(X); \\
(s'_i, e_i) \sim (\beta(s_i), 0), & \text{if } f_i \notin \mathcal{O}_X^+(X) \text{ but } f_i^{-1} \in \mathcal{O}_X^+(X).
\end{cases}
\]

By construction, $S' \oplus \mathbb{Z}_{\geq 0}$ is generated by the images of $s'_i$ and $e_i$ resp. saturated factors through $\mathcal{M}(X)$ factors through a morphism $\theta'' : S'' \to \mathcal{M}(X)$, and the composition $\alpha \circ \theta''$ factors through $\mathcal{O}_X^+$, as desired.

It remains to check that the log structure associated with the pre-log structure $\alpha \circ \theta' : S'' \to \mathcal{M} \to \mathcal{O}_X$ coincides with $\mathcal{M}$; i.e., the natural morphism $aS_X \to aS''_X$ induced by $S \to S''$ is an isomorphism. It is injective because the composition $aS_X \to aS''_X \to \mathcal{M}$ is an isomorphism. It is also surjective, because the induced morphism $S_X/(\alpha \circ \theta)^{-1}(O_X^+) \to S''_X/(\alpha \circ \theta'')^{-1}(O_X^+)$ is surjective, since the target is generated by the images of $s'_i$ which lift to the images of $s_i$ in the source. \(\square\)

**Lemma 2.3.7.** Let $(X, \mathcal{M}_X, \alpha)$ be a locally noetherian coherent log adic space, $P$ a finitely generated monoid, and $P_X \to \mathcal{M}_X$ a morphism of sheaves of monoids. Suppose that $(X, \mathcal{M}_X, \alpha)$ is integral (resp. saturated), in which case $P_X \to \mathcal{M}_X$ factors through $P^\text{int}_X$ (resp. $P^\text{sat}_X$); and that the composition $P_X \to P^\text{int}_X \to \mathcal{M}_X$ (resp. $P_X \to P^\text{sat}_X \to \mathcal{M}_X$) is a chart. Then so is $P^\text{int}_X \to \mathcal{M}_X$ (resp. $P^\text{sat}_X \to \mathcal{M}_X$).

**Proof.** Suppose that $(X, \mathcal{M}_X, \alpha)$ is integral, and that the composition $P_X \to P^\text{int}_X \to \mathcal{M}_X$ is a chart. Then the induced morphism $aP_X \to aP^\text{int}_X$ is injective, because the composition $aP_X \to aP^\text{int}_X \to \mathcal{M}_X$ is an isomorphism. Since $P \to P^\text{int}$ is surjective, the composition $P^\text{int}_X \to \mathcal{M}_X \to \mathcal{O}_X$ factors through $\mathcal{O}_X^+$, and the induced map $aP_X \to aP^\text{int}_X$ is surjective and hence is an isomorphism.

Suppose that $(X, \mathcal{M}_X, \alpha)$ is saturated, and that the composition $P_X \to P^\text{sat}_X \to \mathcal{M}_X$ is a chart. Since $\mathcal{O}_X^+$ is integrally closed in $\mathcal{O}_X$, the composition $P^\text{sat}_X \to \mathcal{M}_X \to \mathcal{O}_X$ factors through $\mathcal{O}_X^+$. It remains to show that the induced morphism $aP_X \to aP^\text{sat}_X$ is an isomorphism. By Remark 2.3.4, it suffices to show that, at each geometric point $\overline{x}$ of $X$, if we denote by $\beta : P \to \mathcal{M}_{X, \overline{x}}$ and $\beta' : P^\text{sat} \to \mathcal{M}_{X, \overline{x}}$ the induced homomorphisms, then the canonical morphism
\[
P/(\alpha \circ \beta)^{-1}(\mathcal{O}_{X, \overline{x}}^+) \to P^\text{sat}/(\alpha \circ \beta')^{-1}(\mathcal{O}_{X, \overline{x}}^+)
\]
is an isomorphism. Let $\overline{\beta}$ denote the composition $P \to \mathcal{M}_{X, \overline{x}} \to \mathcal{M}_{X, \overline{x}}^\text{sat}$. Since $\ker(P^\text{sat}_{X, \overline{x}} \to \mathcal{M}_{X, \overline{x}}^\text{sat}) = \mathcal{M}_{X, \overline{x}}^\ast \alpha^{-1}(\mathcal{O}_{X, \overline{x}}^+)$, we obtain
\[
\ker(\overline{\beta}) = (\beta')^{-1}(\alpha^{-1}(\mathcal{O}_{X, \overline{x}}^+)).
\]
Since $P^\text{sat}/\ker(\overline{\beta}) \cong \mathcal{M}_{X, \overline{x}}^\text{sat} \cong P/(\alpha \circ \beta)^{-1}(\mathcal{O}_{X, \overline{x}}^+)^{\text{sat}}$, we obtain
\[
((\alpha \circ \beta)^{-1}(\mathcal{O}_{X, \overline{x}}^+))^{\text{sat}} = (\beta')^{-1}(\alpha^{-1}(\mathcal{O}_{X, \overline{x}}^+)).
\]
Since $(\alpha \circ \beta)^{-1}(\mathcal{O}_{X, \overline{x}}^+) \subset (\alpha \circ \beta')^{-1}(\mathcal{O}_{X, \overline{x}}^+) \subset (\beta')^{-1}(\alpha^{-1}(\mathcal{O}_{X, \overline{x}}^+))$, we obtain
\[
((\alpha \circ \beta')^{-1}(\mathcal{O}_{X, \overline{x}}^+))^{\text{sat}} = (\beta')^{-1}(\alpha^{-1}(\mathcal{O}_{X, \overline{x}}^+)).
\]
By Lemma 2.1.8 and the above, we see that the natural homomorphism
\[(2.3.9) \quad P^{\text{sat}}/(\alpha \circ \beta')^{-1}(\mathcal{O}_{X,\pi}^\times) \to P^{\text{sp}}/\ker(\mathcal{F}^{\text{sp}})\]
is injective, whose image is contained in \((P/(\alpha \circ \beta')^{-1}(\mathcal{O}_{X,\pi}^\times))^{\text{sat}}\). Moreover, the composition of \((2.3.8)\) and \((2.3.9)\) induces the canonical homomorphism
\[(2.3.10) \quad P/(\alpha \circ \beta)^{-1}(\mathcal{O}_{X,\pi}^\times) \to (P/(\alpha \circ \beta)^{-1}(\mathcal{O}_{X,\pi}^\times))^{\text{sat}}.\]

By Lemmas 2.1.8 and 2.2.6, \((2.3.10)\) is an isomorphism, and so is \((2.3.8)\), as desired.

**Proposition 2.3.11.** Let \((X, \mathcal{M}_X)\) be a locally noetherian coherent log adic space. Then \((X, \mathcal{M}_X)\) is fine (resp. fs) if and only if it is integral (resp. saturated).

**Proof.** If \((X, \mathcal{M}_X)\) is integral (resp. saturated), then it is fine (resp. fs) by Lemma 2.3.7. Conversely, if \((X, \mathcal{M}_X)\) is fine, then it is integral by Lemma 2.1.7. Suppose that \((X, \mathcal{M}_X)\) admits a fs chart \(\theta : X \to \mathcal{M}_X\). By Remark 2.3.4, we have an isomorphism \(P/(\alpha \circ \theta)^{-1}(\mathcal{O}_{X,\pi}^\times) \cong \mathcal{M}_{X,\pi}/\alpha^{-1}(\mathcal{O}_{X,\pi}^\times) \cong \mathcal{M}_{X,\pi}\) at each geometric point \(\pi\) of \(X\). By Lemma 2.1.8, \(\mathcal{M}_{X,\pi}\) is saturated. By [Ogu18, Prop. I.1.3.3], \(\mathcal{M}_{X,\pi}\) is also saturated. Thus, \((X, \mathcal{M}_X)\) is saturated, by Lemma 2.2.6.

**Lemma 2.3.12.** Let \((X, \mathcal{M}_X, \alpha)\) be a fs log adic space. For any geometric point \(\pi\) of \(X\), the monoid \(\mathcal{M}^\times_{X,\pi}\) is toric, and there is a splitting \(s\) of the canonical homomorphism \(\mathcal{M}^\times_{X,\pi} \to \mathcal{M}_{X,\pi}\) that factors through the preimage of \(\mathcal{O}_{X,\pi}^\times\) in \(\mathcal{M}_{X,\pi}\).

**Proof.** Let \(P := \mathcal{M}_{X,\pi}\), which is finitely generated because \(X\) is fs. By Proposition 2.3.11 and Lemma 2.2.6, \(\mathcal{M}_{X,\pi}\) is saturated, and so its sharp quotient \(P\) is toric. By Lemma 2.1.9, the surjective homomorphism \(f : \mathcal{M}_{X,\pi} \to P\) admits a section \(s_0\). We need to modify this into a section \(s : P \to \mathcal{M}_{X,\pi}\) such that \((\alpha \circ s)(P) \subset \mathcal{O}_{X,\pi}^\times\).

By [Hub98] (1) in the proof of Prop. 2.5.13, we have \(\mathcal{O}_{X,\pi}^\times = \{f \in \mathcal{O}_{X,\pi} : |f(\pi)| \leq 1\} \) and \(\{f \in \mathcal{O}_{X,\pi} : |f(\pi)| > 1\} \subset \mathcal{O}_{X,\pi}^\times\) in \(\mathcal{O}_{X,\pi}\). Let \(\{a_i\}_{i \in \mathbb{N}}\) be a finite set of generators of \(P\). Let \(\gamma_i := |\alpha(a_i)(\pi)|\), for all \(i\). If \(\gamma_i \leq 1\) for all \(i\), then we set \(f_0 := 1\) in \(\mathcal{O}_{X,\pi}\). Otherwise, there exists some \(i_0\) such that \(\gamma_{i_0} > 1\) and \(\gamma_{i_0} \geq \gamma_i\), for all \(i\). Then \(\alpha(a_{i_0})\) admits an inverse \(f_0\) in \(\mathcal{O}_{X,\pi}\), so that \(|f_0(\pi)| = \gamma_0^{-1}\). By [Ogu18] Cor. I.2.2.7, we can identify \(P\) with a submonoid of \(\mathbb{Z}^r\), for some \(r\), so that we can describe elements of \(P\) by \(r\)-tuples of integers. Then the homomorphism \(s : P \to \mathcal{M}_{X,\pi} : (n_1, \ldots, n_r) \mapsto f_0^{n_1+\cdots+n_r} s_0(n_1, \ldots, n_r)\) satisfies the desired property \((\alpha \circ s)(P) \subset \mathcal{O}_{X,\pi}^\times\).

**Proposition 2.3.13.** Let \((X, \mathcal{M}_X, \alpha)\) be an fs log adic space, and \(\pi\) any geometric point of \(X\). Then \(X\) admits, étale locally at \(\pi\), a chart modeled on \(\mathcal{M}^\times_{X,\pi}\).

**Proof.** By Lemma 2.3.12 and its proof, we have a splitting \(s : P := \mathcal{M}_{X,\pi} \to \mathcal{M}_{X,\pi}\) such that \((\alpha \circ s)(P) \subset \mathcal{O}_{X,\pi}^\times\). This \(s\) lifts to a morphism \(s_U : P_U \to \mathcal{M}_{X|U}\) for some étale neighborhood \(U\) of \(\pi\) in \(X\) such that \((\alpha \circ s_U)(P_U) \subset \mathcal{O}_{X,\pi}^\times(U)\) and such that the composition of \(P_U \xrightarrow{s_U} \mathcal{M}_{X|U} \to \mathcal{M}_{X|U}\) is an isomorphism. In particular, \(s_U\) is a chart of \(U\) modeled on \(P\), as desired.

**Example 2.3.14.** An fs log point is a log point (as in Example 2.2.8) that is an fs log adic space. In the setting of Example 2.2.3, by Remark 2.2.4 a log point \(s = (\text{Spa}(l, l^+), M)\) with \(l\) separably closed is an fs log point exactly when \(M/l^\times\) is
toric (i.e., sharp fs). In this case, by Lemmas 2.1.9 and 2.3.12, there always exists a homomorphism of monoids $M/l^\infty \to M$ splitting the canonical homomorphism $M \to M/l^\infty$ and defining a chart of $s$ modeled on $M$.

**Example 2.3.15.** A special case of Example 2.3.14 is a split fs log point i.e., a log point of the form $s = (X, \mathcal{M}_X) \cong (\text{Spa}(l, l^+), \mathcal{O}_{X_n}^\* \oplus \mathcal{P}_X)$ for some (necessarily) toric monoid $P$. This is equivalent to a log point $(\text{Spa}(L, L^+), M)$, where $L$ is the completion of a separable closure $\text{sep}$ of $l$, with a $\text{Gal}(\text{sep}/l)$-equivariant splitting of the homomorphism $M \to M/l^\infty$. We also remark that this is the same as a $\text{Gal}(\text{sep}/l)$-equivariant splitting of the homomorphism $M^\text{et} \to M^\text{et}/l^\infty$.

**Example 2.3.16.** Let $X$ be a normal rigid analytic variety over a nonarchimedean field $k$, and let $\iota : D \hookrightarrow X$ be a closed immersion of rigid analytic varieties. By viewing $X$ as a noetherian adic space, we equip $X$ with the log structure defined by setting $\mathcal{M}_X = \{ f \in \mathcal{O}_{X_n} : f$ is invertible on $X - D \}$, with $\alpha : \mathcal{M}_X \to \mathcal{O}_{X_n}$ the natural inclusion. This makes $X$ a locally noetherian fs log adic space. (The normality of $X$ is necessary for showing that the log structure $\mathcal{M}_X$ is indeed saturated.) The maximal open subspace of $X$ over which $\mathcal{M}_X$ is trivial is $X - D$. Note that, in Example 2.2.21, the log structure on $X \cong \mathbb{D}^n$ can be defined alternatively as above by the closed immersion $\iota : D := \{ T_1 \cdots T_n = 0 \} \hookrightarrow \mathbb{D}^n$.

The following special case is useful in many applications:

**Example 2.3.17.** Let $X$ be smooth rigid analytic variety over a nonarchimedean field $k$, and let $\iota : D \hookrightarrow X$ be a closed immersion such that, analytic locally, $X$ and $D$ are of the form $S \times \mathbb{D}^m$ and $S \times \{ T_1 \cdots T_m = 0 \}$, where $S$ is a smooth connected rigid analytic variety over $k$, and $\iota$ is the pullback of $\{ T_1 \cdots T_m = 0 \} \hookrightarrow \mathbb{D}^m$. In this case, we say that $D$ is a (reduced) normal crossings divisor of $X$. (This definition is justified by [Kie67, Thm. 1.18].) Then we equip $X$ with the fs log structure defined as in Example 2.3.16 which is compatible with the one of $\mathbb{D}^m$ as in Example 2.2.21 via pullback.

The following example will be useful when studying the behavior of local systems “along the boundary” (in the context of Example 2.3.17):

**Example 2.3.18.** Let $X$, $D$, and $k$ be as in Example 2.3.17. Suppose that $\{ D_j \}_{j \in J}$ is the set of irreducible components of $D$ (see [Con99]). For each $J \subset I$, as locally closed subspaces of $X$, consider $X_J := \bigcap_{j \in J} D_j$, $D_J := \bigcup_{J \subset J' \subset I} X_{J'}$, and $U_J := X_J - D_J$. By pulling back the log structure from $X$ to $X_J$ and $U_J$, respectively, we obtain log adic spaces $(X_J^\dagger, \mathcal{M}_{X_J}^\dagger)$ and $(U_J^\dagger, \mathcal{M}_{U_J}^\dagger)$ (with strict immersions to $X$). When $X_J$ is also smooth and so $D_J$ is a normal crossings divisor, we equip $X_J$ with the fs log structure defined by $D_J$ as in Example 2.3.16 whose restriction to $U_J$ is then the trivial log structure. If we also consider $D^\dagger := \bigcup_{j \in J} D_j$, and let $X^\dagger$ denote the same adic space $X$ but equipped with the fs log structure defined by $D^\dagger$ as in Example 2.3.16 then $\mathcal{M}_{X_J}$ and $\mathcal{M}_{U_J} = \mathcal{O}_{U_J, \text{et}}^\dagger$ are nothing but the log structures pulled back from $X^\dagger$. Moreover, since $D^\dagger \subset D$, there is a canonical morphism of log adic spaces $X \to X^\dagger$; and since $D_J = D^\dagger \cap X_J$, this morphism induces a canonical morphism of log adic spaces $X_J^\dagger \to X_J$, whose underlying morphism of adic spaces is an isomorphism. Since $X$ and $D$ is (analytic) locally of the form $S \times \mathbb{D}^m$ and $S \times \{ T_1 \cdots T_m = 0 \}$ for some smooth $S$ over $k$, it follows that $X_J$ is analytic locally of the form $S \times \mathbb{D}^{m - |J|}$, in which case the log structures $\mathcal{M}_{X_J}^\dagger$ and
Proposition 2.3.23. Let \( f : (X, \mathcal{M}_X, \alpha_X) \to (Y, \mathcal{M}_Y, \alpha_Y) \) be a morphism of log adic spaces. A chart of \( f \) consists of charts \( \theta_X : P_X \to \mathcal{M}_X \) and \( \theta_Y : Q_Y \to \mathcal{M}_Y \) and a homomorphism \( u : P \to Q \) of monoids such that the diagram

\[
\begin{array}{ccc}
P_X & \xrightarrow{u} & Q_Y \\
\downarrow{\theta_X} & & \downarrow{\theta_Y} \\
{f}^{-1}(M_X) & \xrightarrow{f^*} & M_Y
\end{array}
\]

commutes. We say that the chart is finitely generated (resp. fine, resp. fs) if both \( P \) and \( Q \) are finitely generated (resp. fine, resp. fs). When the context is clear, we shall simply say that \( u : P \to Q \) is the chart of \( f \).

Example 2.3.20. Let \( P := \mathbb{Z}_{\geq 0}^n \) and let \( Q \) be a toric submonoid of \( \mathbb{Z}_{\geq 0}^n \) containing \( P \), for some \( m \geq 1 \). Then the canonical homomorphism \( u : P \to Q \) induces a morphism \( f : Y := \text{Spa}(k(Q), k^+(Q)) \to X := \text{Spa}(k(P), k^+(P)) \cong \mathbb{A}^n_k \) of normal adic spaces, whose source and target are equipped with canonical log structures as in Examples 2.2.20 and 2.2.21 making \( f : Y \to X \) a morphism of fs log adic spaces. Moreover, these log structures on \( X \) and \( Y \) coincide with those on \( X \) and \( Y \) defined by \( D = \{T_1 \cdots T_n = 0\} \hookrightarrow X \) and its pullback to \( Y \), respectively, as in Example 2.3.16. A chart of \( f : Y \to X \) is given by the canonical charts \( P \to \mathcal{M}_X(X) \) and \( Q \to \mathcal{M}_Y(Y) \) and the above homomorphism \( u : P \to Q \).

Proposition 2.3.21. Let \( f : Y \to X \) be a morphism of coherent log adic spaces, and let \( P \to \mathcal{M}_X(X) \) be a chart modeled on a finitely generated monoid \( P \). Then, étale locally on \( Y \), there exist a chart \( Q \to \mathcal{M}_Y(Y) \) modeled on a finitely generated monoid \( Q \) and a homomorphism \( P \to Q \), which together provide a chart of \( f \).

Proof. We may assume that \( (X, \mathcal{M}_X) \) and \( (Y, \mathcal{M}_Y) \) are modeled on finitely generated monoids \( P \) and \( Q' \), respectively. Then the composition of \( P_Y \equiv f^{-1}(P_X) \to f^{-1}(\mathcal{M}_X) \to \mathcal{M}_Y \) induces a morphism \( P_Y \to (P \oplus Q')_Y \to \mathcal{M}_Y \). Note that \( P \oplus Q' \) is finitely generated, and that the composition \( (P \oplus Q')_Y \to \mathcal{M}_Y \to \mathcal{O}_Y \) factors through \( \mathcal{O}_Y^+ \). By applying Lemma 2.3.6 to \( S = Q' \) and \( S' = P \oplus Q' \), we see that, étale locally, \( (P \oplus Q')_Y \to \mathcal{M}_Y \) factors as \( (P \oplus Q')_Y \to Q_Y \to \mathcal{M}_Y \), where \( Q_Y \to \mathcal{M}_Y \) is a chart modeled on a finitely generated monoid \( Q \). Therefore, the composition \( P \to P \oplus Q' \to Q \) gives a chart of \( f \), as desired. \( \square \)

Proposition 2.3.22. Any morphism between fine (resp. fs) log adic spaces étale locally admits fine (resp. fs) charts.

Proof. By Proposition 2.3.13 up to étale localization, \( X \) admits a chart modeled on a fine (resp. fs) monoid \( P \). By Proposition 2.3.21 \( f \) admits, up to étale localization on \( Y \), a chart \( P \to Q \) with finitely generated \( Q \). By Lemma 2.3.7 the induced \( Q^\text{int} \to \mathcal{M}_Y \) (resp. \( Q^\text{ext} \to \mathcal{M}_Y \)) is also a chart of \( Y \), and hence the composition of \( P \to Q \to Q^\text{int} \) (resp. \( P \to Q \to Q^\text{ext} \)) is a fine (resp. fs) chart of \( f \). \( \square \)

Proposition 2.3.23. (1) The inclusion from the category of noetherian (resp. locally noetherian) fine log adic spaces to the category of noetherian (resp. locally noetherian) coherent log adic spaces admits a right adjoint \( X \mapsto \)}
By construction, both the functors to the category of noetherian (resp. locally noetherian) coherent log adic spaces.

Inclusion from the category of noetherian (resp. locally noetherian) fs log adic spaces is the right adjoint of the noetherian (resp. locally noetherian) coherent log adic spaces to the category of noetherian (resp. locally noetherian) adic space.

X_{int}, and the corresponding morphism of underlying adic spaces is a closed immersion.

(2) The inclusion from the category of noetherian (resp. locally noetherian) fs log adic spaces to the category of noetherian (resp. locally noetherian) fine log adic spaces admits a right adjoint X \mapsto X^{sat}, and the corresponding morphism of underlying adic spaces is finite and surjective.

Proof. In case (1) (resp. (2)), let \( \mathfrak{l} = \text{int} \) (resp. \( \mathfrak{s} \) sat) in the following.

Suppose that \( X = \text{Spa}(R, R^+) \) is noetherian affinoid and admits a global chart modeled on a finitely generated (resp. fine) monoid \( P \), so that we have a homomorphism \( P \to R \) of monoids, inducing a homomorphism \( \mathbb{Z}[P] \to R \) of rings. Let \( R^2 := R \otimes_{\mathbb{Z}[P]} \mathbb{Z}[P^2] \), and let \( R^2^+ \) denote the integral closure of \( R^2 \otimes_{\mathbb{Z}[P]} \mathbb{Z}[P^2] \) in \( R^2 \). Let \( X^2 := \text{Spa}(R^2, R^2^+) \), with the log structure induced by \( P^2 \to \mathcal{O}(X^2) = R \otimes_{\mathbb{Z}[P]} \mathbb{Z}[P^2] : a \mapsto 1 \otimes e^a \) (where \( e^a \) denotes the image of \( a \in P^2 \) in \( \mathbb{Z}[P^2] \), by our convention). Clearly, the natural projection \( X^2 \to X \) is a closed immersion (resp. finite and surjective morphism) of log adic spaces. We claim that, if \( (Y, \mathcal{M}_Y) \) is a fine (resp. fs) log adic space, then each morphism \( f : (Y, \mathcal{M}_Y) \to (X, \mathcal{M}_X) \) of log adic spaces factors through \( X^2 \), yielding \( \text{Hom}(Y, X) \cong \text{Hom}(Y, X^2) \). Indeed, by Proposition 2.3.11 the induced morphism \( P_Y \cong f^{-1}(P_X) \to f^{-1}(\mathcal{M}_X) \to \mathcal{M}_Y \) factors through \( P_{X^2} \), and hence \( Y \to X \) factors through \( Y \to X^2 \), as desired.

In general, there exists an étale covering of \( X \) by affinoids \( X_i = \text{Spa}(R_i, R_i^+) \) such that each \( X_i \) admits a global chart modeled on a finitely generated (resp. fine) monoid (see Definition 2.3.5). Consider \( \bar{X} = \coprod X_i \). By the affinoid case treated in the last paragraph, we obtain a finite morphism \( \bar{X}^2 \to \bar{X} \), which is equipped with a descent datum. By étale descent (see Proposition 2.3.22), \( \bar{X}^2 \to \bar{X} \) descends to a locally noetherian adic space \( \bar{X}^2 \to X \). Also, the étale sheaf of monoids descends (essentially by definition). Finally, by Proposition 2.3.22 and the local construction in the previous paragraph, the formation \( X \mapsto X^2 \) is functorial, as desired. \( \square \)

Remark 2.3.24. For a noetherian (resp. locally noetherian) coherent log adic space \( X \), we shall simply denote by \( X^{sat} \) the fs log adic space \( (X^{int})^{sat} \). By combining the two cases in Proposition 2.3.22, the functor \( X \mapsto X^{sat} \) from the category of noetherian (resp. locally noetherian) coherent log adic spaces to the category of noetherian (resp. locally noetherian) fs log adic spaces is the right adjoint of the inclusion from the category of noetherian (resp. locally noetherian) fs log adic spaces to the category of noetherian (resp. locally noetherian) coherent log adic spaces.

Remark 2.3.25. By construction, both the functors \( X \mapsto X^{int} \) and \( X \mapsto X^{sat} \) send strict and finite (resp. étale) morphisms to strict and finite (resp. étale) morphisms.

Remark 2.3.26. Again by construction, when \( X \) is a locally noetherian log adic space over a locally noetherian fs log adic space \( Y \) and admits a global chart modeled on a finitely generated (resp. fine) monoid \( P \), we have \( X^2 \cong X \times_{Y(P)} Y(P^2) \) as adic spaces, where \( \mathfrak{l} = \text{int} \) (resp. \( \mathfrak{s} \) sat), and where \( Y(P) \) and \( Y(P^2) \) are as in Example 2.2.19. (Note that the fiber product \( X \times_{Y(P)} Y(P^2) \) exists because the morphism \( Y(P^2) \to Y(P) \) is lif when \( P \) is finitely generated.)

Now, let us study fiber products in the category of locally noetherian coherent (resp. fine, resp. fs) log adic spaces:
Proposition 2.3.27. (1) Finite fiber products exist in the category of locally noetherian log adic spaces when the corresponding fiber products of the underlying adic spaces exist. Moreover, finite fiber products of locally noetherian coherent log adic spaces over locally noetherian coherent log adic spaces are coherent (when defined). The forgetful functor from the category of locally noetherian log adic spaces to the category of locally noetherian adic spaces respects finite fiber products (when defined).

(2) Finite fiber products exist in the category of locally noetherian fine (resp. fs) log adic spaces when the corresponding fiber products of the underlying adic spaces exist.

Proof. As for (1), let $Y \to X$ and $Z \to X$ be morphisms of locally noetherian log adic spaces such that the fiber product $W := Y \times_X Z$ of the underlying adic spaces is defined. Let $\text{pr}_Y$, $\text{pr}_Z$, and $\text{pr}_X$ denote the natural projections from $W$ to $Y$, $Z$, and $X$, respectively, and equip $W$ with the log structure associated with the pre-log structure $\text{pr}_Y^{-1}(\mathcal{M}_Y) \oplus \text{pr}_Z^{-1}(\mathcal{M}_Z) \to \mathcal{O}_{W_\alpha}$. Then the resulted log adic space clearly satisfies the desired universal property. Suppose moreover that $X$, $Y$, and $Z$ are all coherent. By Proposition 2.3.21, étale locally, $Y \to X$ and $Z \to X$ admit charts $P \to Q$ and $P \to R$, respectively, where $P$, $Q$, and $R$ are all finitely generated monoids, in which case $W$ is (by construction) modeled on the finitely generated monoid $S := Q \oplus_P R$, and hence is coherent.

As for (2), let $Y \to X$ and $Z \to X$ be morphisms of locally noetherian fine (resp. fs) log adic spaces such that the fiber product $Y \times_X Z$ of the underlying adic spaces is defined, in which case we equip it with the structure of a coherent log adic space as in (1). Then, by Proposition 2.3.23, $Y \times_X^\text{fine} Z := (Y \times_X Z)^{\text{int}}$ (resp. $Y \times_X^\text{fs} Z := (Y \times_X Z)^{\text{sat}}$) satisfies the desired universal property. □

Remark 2.3.28. Let $P \to Q$ and $P \to R$ be homomorphisms of finitely generated (resp. fine, resp. fs) monoids, and let $S^? := (Q \oplus_P R)^?$, where $? = \emptyset$ (resp. int, resp. sat). Let $Y$ be a locally noetherian fs log adic space. By Remark 2.3.2 and Proposition 2.3.27 (and the construction in its proof), $Y(S^?)$ is canonically isomorphic to the fiber product of $Y(Q)$ and $Y(R)$ over $Y(P)$ in the category of noetherian coherent (resp. fine, resp. fs) log adic spaces.

Remark 2.3.29. Let $P \to Q$ and $P \to R$ be fine (resp. fs) charts of morphisms $Y \to X$ and $Z \to X$, respectively, of locally noetherian fine (resp. fs) log adic spaces such that $Y \times_X Z$ is defined. Then $Y \times_X^\text{fine} Z$ (resp. $Y \times_X^\text{fs} Z$) is modeled on $(Q \oplus_P R)^{\text{int}}$ (resp. $(Q \oplus_P R)^{\text{sat}}$).

Remark 2.3.30. The forgetful functor from the category of locally noetherian fine (resp. fs) log adic spaces to the category of locally noetherian adic spaces does not respect fiber products (when defined), because the underlying adic spaces may change under the functor $X \mapsto X^{\text{int}}$ (resp. $X \mapsto X^{\text{sat}}$).

Convention 2.3.31. From now on, all fiber products of locally noetherian fs log adic spaces are taken in the category of fs ones unless otherwise specified. For simplicity, we shall omit the superscript “fs” from “×”.

We will need the following analogue of Nakayama’s Four Point Lemma [Nak97, Prop. 2.2.2]:

...
Proposition 2.3.32. Let $f : Y \to X$ and $g : Z \to X$ be two lft morphisms of locally noetherian fs log adic spaces, and assume that $f$ is exact. Then, given any two points $y \in Y$ and $z \in Z$ that are mapped to the same point $x \in X$, there exists some point $w \in W := Y \times_X Z$ that is mapped to $y \in Y$ and to $z \in Z$.

In order to prove Proposition 2.3.32, it suffices to treat the case where $X$, $Y$, and $Z$ are geometric points, and where $x$, $y$ and $z$ are the respective unique closed points. By [Hub96, Lem. 1.1.10], it suffices to prove the following:

Lemma 2.3.33. Let $f : Y \to X$ and $g : Z \to X$ be morphisms of fs log adic spaces such that the underlying adic spaces of $X$, $Y$, and $Z$ are Spa($l, l^+$) for the same complete separably closed nonarchimedean field $l$, and such that $f$ and $g$ are the identity morphism. Assume that $f$ is exact. Then $W = Y \times_X Z$ is nonempty.

Proof. Let $\pi$, $\pi'$, and $\pi''$ be the unique closed points of $X$, $Y$, and $Z$, respectively, and let $P = \overline{M}_{X, \pi}, Q = \overline{M}_{Y, \pi'}$, and $R = \overline{M}_{Z, \pi''}$. Let $u : P \to Q$ and $v : P \to R$ be the corresponding maps of monoids. By [Ogu18, Prop. I.4.2.1], $u$ is exact.

Let $\phi : P^{gp} \to Q^{gp} \oplus R^{gp}$ be a morphism defined by $\phi(v(a)) = (u(a), -v(a))$. Since $u$ is exact and $R$ is sharp, $\phi^{-1}(Q \oplus R)$ is trivial. By [Nak97, Lem. 2.2.6], the sharp monoid $S := Q \oplus P \to R$ is quasi-integral (i.e., if $a + b = a$, then $b = 0$), and the natural homomorphism $P \to Q \oplus_P R$ is injective.

Note that $f$ and $g$ admit charts modeled on $u : P \to Q$ and $v : P \to R$, respectively. This is because, by the proof of [Nak97, Lem. 2.2.3], there exist compatible homomorphisms $(\overline{M}_{X}(X))^{gp} \to l^\times, (\overline{M}_{Y}(Y))^{gp} \to l^\times$, and $(\overline{M}_{Z}(Z))^{gp} \to l^\times$ such that the compositions $l^\times \to (\overline{M}_{X}(X))^{gp} \to l^\times$, $l^\times \to (\overline{M}_{Y}(Y))^{gp} \to l^\times$, and $l^\times \to (\overline{M}_{Z}(Z))^{gp} \to l^\times$ are the identity homomorphisms. Therefore, the morphisms $f^*(\overline{M}_{X}) \to \overline{M}_{Y}$ and $g^*(\overline{M}_{X}) \to \overline{M}_{Z}$ are (noncanonically) isomorphic to $\text{Id} \oplus u : l^\times \oplus P \to l^\times \oplus Q$ and $\text{Id} \oplus v : l^\times \oplus P \to l^\times \oplus R$, respectively.

Consequently, $W \cong \text{Spa}(l, l^+) \times_{\text{Spa}(l(S), l^+(S))} \text{Spa}(l(S_{\text{sat}}), l^+(S_{\text{sat}}))$. The image of $\text{Spa}(l, l^+) \to \text{Spa}(l(S), l^+(S))$ consists of equivalence classes of valuations on $l(S)$ (bounded by 1 on $l^+$) whose support contains the ideal $I$ of $l(S)$ generated by $\{e_s : a \in S, a \neq 0\}$. On the other hand, the kernel of $l(S) \to l(S_{\text{sat}})$, which is generated by $\{e_a - e_b : a, b \in S, a = b \in S_{\text{sat}}\}$, is contained in $I$ because $S$ is quasi-integral. Thus, $W$ is nonempty, as desired.

3. Log smoothness and log differentials

3.1. Log smooth morphisms.

Definition 3.1.1. Let $f : Y \to X$ be a morphism between locally noetherian fs log adic spaces. We say that $f$ is log smooth (resp. log étale) if, étale locally on $Y$ and $X$, the morphism $f$ admits an fs chart $u : P \to Q$ such that

1. the kernel and the torsion part of the cokernel (resp. the kernel and cokernel) of $u^{gp} : P^{gp} \to Q^{gp}$ are finite groups of order invertible in $\mathcal{O}_X$; and
2. $f$ and $u$ induce a morphism $Y \to X \times_X Q$ of log adic spaces (cf. Remark 2.3.3) whose underlying morphism of adic spaces is étale.

Remark 3.1.2. In Definition 3.1.1, the fiber product in (2) exists and $f : Y \to X$ is lft, because $X(Q) \to X(P)$ and hence the first projection $X \times_X Q \to X(Q) \to X$ is lft when $Q$ is finitely generated. Hence, fiber products involving log smooth or log étale morphisms always exist.
**Proposition 3.1.3.** Base changes of log smooth (resp. log étale) morphisms (by arbitrary morphisms between locally noetherian fs log adic spaces, which are justified by Remark 3.1.2) are still log smooth (resp. log étale).

**Proof.** Suppose that \( Y \to X \) is a log smooth (resp. log étale) morphism of locally noetherian fs log adic spaces, with a chart \( P \to Q \) satisfying the conditions in Definition 3.1.1. Let \( Z \to X \) be any morphism of locally noetherian fs log adic spaces. By Proposition 2.3.22 we may assume that \( Z \to X \) admits an fs chart \( P \to R \). By Remark 2.3.26 \( Z \times_X Y \) is modeled on \( S := (R \oplus P)_{\text{sat}} \). By Remark 2.3.25 \( Z \times_X Y \to Z \times _Z R \) induces an étale morphism of underlying adic spaces. It remains to note that \( R_{\text{esp}} \to \left( (Q \oplus_P R)_{\text{sat}} \right)_{\text{esp}} \) satisfies the analogue of Definition 3.1.1(1), by the assumption on \( P_{\text{esp}} \to Q_{\text{esp}} \) and the fact that \( \left( (Q \oplus_P R)_{\text{sat}} \right)_{\text{esp}} \cong (Q \oplus P)_{\text{esp}} \cong Q_{\text{esp}} \oplus P_{\text{esp}} R_{\text{esp}} \). □

**Proposition 3.1.4.** Let \( f : Y \to X \) be a log smooth (resp. log étale) morphism of locally noetherian fs log adic spaces. Suppose that \( X \) is modeled on a global fs chart \( P \). Then, étale locally on \( Y \) and \( X \), there exists an injective fs chart \( u : P \to Q \) of \( f \) satisfying the conditions in Definition 3.1.1. Moreover, if \( P \) is torsionfree, we can choose \( Q \) to be torsionfree as well.

**Proof.** This is an analogue of the smooth and étale cases of [Kat89a, Lem. 3.1.6].

Suppose that, étale locally, \( f \) admits a chart \( u_1 : P_1 \to Q_1 \) satisfying the conditions in Definition 3.1.1. We may assume that \( X = \text{Spa}(R, R^+) \) is a noetherian log affinoid space. Let us begin with some preliminary reductions.

Firstly, we may assume that \( P_1 \to M_1 \) factors through \((P_1)_X \to M_X \). Indeed, by Lemma 2.3.6 étale locally, \( X \) admits an fs chart \( P_2 \) such that \((P_2)_X \to M_X \) factors through \((P_2)_1 \to M_1 \). Let \( Q_2 \) be \((P_2)_{\text{sat}} \). Then \( Q_{\text{esp}}^2 \cong (Q \oplus_P Q_{\text{sat}})_{\text{esp}} \). Let \( Q_{\text{esp}}^2 \) be two such \( P \)-charts of \( f \) that satisfy the conditions in Definition 3.1.1.

Secondly, we may assume \( u_1 : P_1 \to Q_1 \) is injective. Indeed, let \( K \) be the kernel of \( u_1^\text{esp} \), and let \( H \) be any finitely generated (commutative) group fitting in a cartesian diagram

\[
\begin{array}{ccc}
P_1^\text{esp} & \longrightarrow & H \\
\downarrow & & \downarrow \\
P_1^\text{esp}/K & \longrightarrow & Q_1^\text{esp}
\end{array}
\]

(which exists because \( K \) is finite and \( P_1^\text{esp} \) and \( Q_1^\text{esp} \) are finitely generated). For any geometric point \( \overline{y} \) of \( Y \), let \( Q_2 \) be the preimage of \( M_{Y, \overline{y}} \) under \( H \to Q_{\text{esp}}^1 \to M_{Y, \overline{y}}^\text{esp} \). Note that \( Q_2 \) is fs and \( P_1 \to Q_2 \) is injective. We claim that \( P_1 \to Q_2 \) is an fs chart of \( f \), étale locally at \( \overline{y} \) and \( f(\overline{y}) \), satisfying the conditions in Definition 3.1.1.

By Remark 2.2.4 \( Q_2 \) is the preimage of \( M_{Y, \overline{y}}^\text{esp} \) under \( H \to (M_{Y, \overline{y}})_{\text{esp}} \to (M_{Y, \overline{y}})^\text{esp} \). By Remark 2.3.4 \( Q_1 \to M_{Y, \overline{y}}^\text{esp} \) is surjective. Hence, \( Q_2 \to M_{Y, \overline{y}}^\text{esp} \) is also surjective, \( Q_2 \to M_{Y, \overline{y}} \) is the inverse of some element of \( Q_1 \), in which case \( X(Q_3) \to X(Q_1) \) is a rational localization. By [GR19, Lem. 12.1.18(iv)], \( Q_2 \) defines a chart étale locally at \( \overline{y} \). It remains to show that \( P_1 \to Q_2 \) satisfies the conditions in Definition 3.1.1. Since \( X(Q_3) \to X(Q_1) \) is a rational localization, and since \( X(Q_3) \to X(Q_2) \) is étale, because \( |K| \) is invertible in \( O_X \), it follows that \( Y \to X \times_{X(P_1)} X(Q_2) \) is also étale, as desired.
Thirdly, for any geometric point \( \overline{y} \) of \( Y \), we may assume that \( \mathcal{P}_1 \cong \mathcal{M}_{X,f(\overline{y})} \) and \( \mathcal{Q}_1 \cong \mathcal{M}_{Y,\overline{y}} \). Indeed, this can be achieved by replacing \( \mathcal{P}_1 \) and \( \mathcal{Q}_1 \), respectively, with their localizations with respect to the kernels of \( \mathcal{P}_1 \to \mathcal{M}_{X,f(\overline{y})} \) and \( \mathcal{Q}_1 \to \mathcal{M}_{Y,\overline{y}} \). The resulted \( u_1 : \mathcal{P}_1 \to \mathcal{Q}_1 \) is still an fs chart étale locally at \( \overline{y} \) and \( f(\overline{y}) \), by Remarks 2.1.11 and 2.1.12 and it still satisfies the conditions in Definition 3.1.1.

Now, following the same argument as in [Niz08, Lem. 2.8], up to further modifications of \( \mathcal{P}_1 \) and \( \mathcal{Q}_1 \), we can find \( H \) fitting into a cartesian diagram:

\[
\begin{array}{ccc}
\mathcal{P}_1 & \longrightarrow & H \\
\downarrow & & \downarrow \\
\mathcal{Q}_1 & \longrightarrow & Q_1^{\text{gp}}.
\end{array}
\]

Given such an \( H \), let \( Q \) be the preimage of \( \mathcal{M}_{Y,\overline{y}} \) under \( H \to \mathcal{M}_{Y,\overline{y}} \). Then \( u : \mathcal{P} \to \mathcal{Q} \) is an injective fs chart of \( H \) satisfying the conditions of Definition 3.1.1.

Finally, if \( P \) is torsionfree, let us show that we can take \( Q \) to be torsionfree as well. We learned the following argument from [Nak98, Prop. A.2]. Consider the torsion submonoid \( Q_{\text{tor}} \) of \( Q \), which is necessarily contained in \( Q^* \); and choose any splitting \( s \) of \( \pi : Q \to Q' := Q/Q_{\text{tor}} \). Let \( n \) be any integer invertible in \( \mathcal{O}_X \) which annihilates the torsion in \( \text{coker}(u^*) \). Since \( P \) is torsionfree, the composition \( u' : P \to Q \to Q' \) is injective, and \( Q_{\text{tor}} \) is also annihilated by \( n \). Let \( S \) be the étale \( R \)-algebra obtained from \( R(Q_{\text{tor}}) \) by formally joining the \( n \)-th roots of \( e^a \), for all \( a \in Q_{\text{tor}} \); and let \( S^+ \) be the integral closure of \( R^+[Q_{\text{tor}}] \) in \( S \). This defines a surjective étale morphism \( Z := \text{Spa}(S,S^+) \to X(Q_{\text{tor}}) = \text{Spa}(R(Q_{\text{tor}}),R^+[Q_{\text{tor}}]) \) over \( X \), with base change \( Z(Q') \to X(Q) \). Consider the composition \( v : P \to Q \to Q' \). Then \( u - v : P \to Q \) factors through \( P \to Q_{\text{tor}} \), which extends to some \( \phi : Q' \to S_{\text{tor}}^+ \); and \( a \mapsto \phi(a)a \), for \( a \in \mathcal{Q} \), induces an isomorphism between the two compositions \( g, h : \mathcal{Z}(Q') \to \mathcal{X}(Q) \to \mathcal{X}(P) \) induced by \( u, v \), respectively. Since \( u : \mathcal{P} \to \mathcal{Q} \) is a chart of \( f \), the induced morphism \( Y \to X \) is étale, whose pullback is an étale morphism \( Y \times_{X(Q_{\text{tor}})} Z \to X \times_{X(P),g} Z(Q') \). The target is isomorphic to \( X \times_{X(P),h} Z(Q') \), and hence is étale over \( X \times_{X(P)} Z(Q') \), by the above explanation. Consequently, the morphism \( Y \to X \times_{X(P)} Z(Q') \) induced by \( f \) and \( u' : P \to Q' \) is étale, and so \( u' \) is also an injective fs chart of \( f \), as desired. \[\square\]

**Proposition 3.1.6.** Compositions of log smooth (resp. log étale) morphisms are still log smooth (resp. log étale).

**Proof.** This follows from Definition 3.1.1 and Proposition 3.1.4. \[\square\]

**Proposition 3.1.7.** If \( f : Y \to X \) is log smooth (resp. log étale) and strict, then the underlying morphism of adic spaces is smooth (resp. étale).

**Proof.** By Proposition 3.1.4, we may assume that \( f : Y \to X \) admits an injective fs chart \( P \to Q \) as in Definition 3.1.1 and it suffices to show that \( \mathcal{X}(Q) \to \mathcal{X}(P) \) is smooth (resp. étale). At each geometric points \( \overline{y} \) of \( Y \), as in the proof of Proposition 3.1.4, we may assume that \( \mathcal{P} \cong \mathcal{M}_{X,f(\overline{y})} \) and \( \mathcal{Q} \cong \mathcal{M}_{Y,\overline{y}} \). Since \( f \) is strict, by Remark 2.2.5, \( \mathcal{M}_{X,f(\overline{y})} \cong \mathcal{M}_{Y,\overline{y}} \), and hence \( P^* = u^{-1}(Q^*) \). It remains to note that, by [Hub96, Cor. 1.6.10 and Prop. 1.7.1], \( X(Q^*) \to X(P^*) \) is smooth (resp. étale) when the kernel and the torsion part of the cokernel (resp. the kernel and cokernel) of \( P^* \to Q^* \) are finite groups of order invertible in \( \mathcal{O}_X \). \[\square\]
Definition 3.1.8. If $f$ satisfies the condition in Proposition 3.1.7, we say that $f$ is strictly smooth (resp. strictly étale), or simply smooth (resp. étale), when the context is clear.

Definition 3.1.9. Let $(k, k^+)$ be an affinoid field. A locally noetherian fs log adic space $X$ is called log smooth over $\text{Spa}(k, k^+)$ if there is a log smooth morphism $X \to \text{Spa}(k, k^+)$, where $\text{Spa}(k, k^+)$ is endowed with the trivial log structure. When $X$ is log smooth (resp. smooth) over $\text{Spa}(k, O_k)$, we simply say that $X$ is log smooth (resp. smooth) over $k$.

Local structures of log smooth log adic spaces can be described by toric charts, by the following proposition:

Proposition 3.1.10. Let $X$ be an fs log adic space log smooth over $\text{Spa}(k, k^+)$, where $(k, k^+)$ is an affinoid field. Then, étale locally on $X$, there exist a sharp fs monoid $P$ and a strictly étale morphism $X \to \text{Spa}(k(P), k^+(P))$ that is a composition of rational localizations and finite étale morphisms.

Proof. By definition, étale locally on $X$, there exists an fs monoid $Q$ whose torsion part has order prime to the characteristic of $k$ such that $X \to \text{Spa}(k(Q), k^+(Q))$ is étale. We may further assume that $X \to \text{Spa}(k(Q), k^+(Q))$ is a composition of rational localizations and finite étale morphisms. By Lemma 3.1.9 there is a decomposition $Q = \mathcal{O} \oplus Q^* \cong \mathcal{O} \oplus Q_{\text{tor-free}} \oplus Q_{\text{tor}} \cong \mathcal{O} \oplus \mathbb{Z}^r \oplus Q_{\text{tor}}$, for some $r$. Hence, $k(Q) \cong k(Q) \otimes_k k(W_1^+, \ldots, W_r^+) \otimes_k k(Q_{\text{tor}})$. Since $|Q_{\text{tor}}|$ is prime to $\text{char}(k)$, we see that $\text{Spa}(k(Q_{\text{tor}}), k^+(Q_{\text{tor}}))$ is strictly étale over $\text{Spa}(k, k^+)$. We may therefore assume that $Q_{\text{tor}}$ is trivial. Let $P := Q \oplus \mathbb{Z}_{\geq 0}$. Then the composition of $X \to \text{Spa}(k(Q) \otimes_k k(W_1^+, \ldots, W_r^+), k^+(Q) \otimes_k k(W_1^+, \ldots, W_r^+))$ gives the desired morphism. \qed

Corollary 3.1.11. Let $X$ and $(k, k^+)$ be as in Proposition 3.1.10. Suppose moreover that underlying adic space of $X$ is smooth over $\text{Spa}(k, k^+)$. Then, étale locally on $X$, there exists an strictly étale morphism $X \to \mathbb{D}^n$ (see Example 2.2.21) that is a composition of rational localizations and finite étale morphisms.

Proof. In the proof of Proposition 3.1.10 étale locally, we have a strictly étale morphism $X \to \text{Spa}(k(Q) \otimes_k k(W_1^+, \ldots, W_r^+), k^+(Q) \otimes_k k(W_1^+, \ldots, W_r^+))$ that is a composition of rational localizations and finite étale morphisms. Since $X$ is smooth over $\text{Spa}(k, k^+)$, by base change to $\text{Spa}(k, O_k)$, we see that $\text{Spa}(k(Q), O_k(Q))$ is smooth over $k$. This implies that $Q \cong \mathbb{Z}^r_{\geq 0}$ for some $t$ (see [EG00] Prop. 4.45), so that $k(Q) \cong k(T_1, \ldots, T_t)$. Therefore, we obtain an inclusion $X \hookrightarrow \text{Spa}(k(Q) \otimes_k k(W_1, \ldots, W_r), k^+(Q) \otimes_k k(W_1, \ldots, W_r)) \cong \mathbb{D}^{t+r}$ that is a composition of rational localizations and finite étale morphisms. In order to show that $X \to \mathbb{D}^{t+r}$ is strict, it suffices to note that the log structure associated with the pre-log structure induced by $\mathbb{Z}^{t+r}_{\geq 0} \to k(T_1, \ldots, T_t, W_1, \ldots, W_r) \to O_X$ coincides with the one associated with $Q \cong \mathbb{Z}^r_{\geq 0}$, because $W_1, \ldots, W_r$ are invertible on $X$. \qed

Definition 3.1.12. A strictly étale morphism $X \to \text{Spa}(k(P), k^+(P))$ as in Proposition 3.1.10 is called a toric chart. A strictly étale morphism $X \to \mathbb{D}^n$ as in Corollary 3.1.11 is called a smooth toric chart.
Example 3.1.13. Let $X$, $D$, and $k$ be as in Example 2.3.17. Note that, analytic (not just étale) locally, $X$ admits a smooth toric chart $X \to \mathbb{D}^n$, where $n = \dim(X)$. In order to see this, we may assume that there is a morphism (of adic spaces with trivial log structures) $S \to \mathbb{T}^{n-m} = \text{Spa}(k(T_1^+, \ldots, T_{n-m}^+), O_k(T_1^+, \ldots, T_{n-m}^+))$ that is a composition of finite étale morphisms and rational localizations. Then the composition of $X \cong S \times \mathbb{D}^n \to \mathbb{T}^{n-m} \times \mathbb{D}^m \hookrightarrow \tilde{\mathbb{D}}^{n-m} \times \mathbb{D}^m \cong \mathbb{D}^n$ is a desired smooth toric chart. In particular, $X$ is log smooth over $k$.

3.2. Log differentials. In this subsection, we develop a theory of log differentials from scratch. We first introduce log structures and log differentials for Huber rings.

Definition 3.2.1. (1) A pre-log Huber ring is a triple $(A, M, \alpha)$ consisting of a (not necessarily complete) Huber ring $A$, a monoid $M$, and a homomorphism $\alpha : M \to A$ of multiplicative monoids. We sometimes denote a pre-log Huber ring just by $(A, M)$, when the homomorphism $\alpha$ is clear from the context.

(2) A log Huber ring is a pre-log Huber ring $(A, M, \alpha)$ where $A$ is complete and where the induced homomorphism $\alpha^{-1}(A^\times) \to A^\times$ is an isomorphism.

(3) Given a pre-log Huber ring $(A, M, \alpha)$, let us still denote by $\alpha$ the composition of $M \xrightarrow{\alpha} A \xrightarrow{\text{can}} \hat{A}$, where $\hat{A}$ denotes the completion of $A$. Then we define the associated log Huber ring to be $(\hat{A}, aM, \hat{\alpha})$, where $aM$ is the pushout of $\hat{A}^\times \leftarrow \alpha^{-1}(A^\times) \to M$ in the category of monoids, which is equipped with the canonical homomorphism $\hat{\alpha} : aM \to \hat{A}$.

(4) A homomorphism $f : (A, M, \alpha) \to (B, N, \beta)$ of pre-log Huber rings consists of a continuous homomorphism $f : A \to B$ of Huber rings and a homomorphism of monoids $f^\sharp : M \to N$ such that $\beta \circ f^\sharp = f \circ \alpha$. We say that $f$ is strict if, in addition, $(B, N, \beta)$ is isomorphic (via $f^\sharp$) to the log Huber ring associated with the pre-log Huber ring $(B, M, \beta \circ f^\sharp)$. In this case, we sometimes write $N \cong f^*(M)$. In general, any homomorphism $f : (A, M) \to (B, N)$ of log Huber rings factors as $(A, M) \to (B, f^*(M)) \to (B, N)$.

Definition 3.2.2. Let $f : (A, M, \alpha) \to (B, N, \beta)$ be a homomorphism of pre-log Huber rings. Given a complete topological $B$-module $L$, a derivation from $(B, N, \beta)$ to $L$ over $(A, M, \alpha)$ (or an $(A, M, \alpha)$-derivation of $(B, N, \beta)$ to $L$) consists of a continuous $A$-linear derivation $d : B \to L$ and a map of monoids $\delta : N \to L$ such that $\delta(f^\sharp(m)) = 0$ and $\delta(\beta(n)) = \beta(n) \delta(n)$, for all $m \in M$ and $n \in N$. We denote by $\text{Der}_A^\log(B, L)$ the set of all $(A, M, \alpha)$-derivations from $(B, N, \beta)$ to $L$. It has a natural $B$-module structure induced by that of $L$. If $M = \alpha^{-1}(A^\times)$ and $N = \beta^{-1}(B^\times)$, we simply denote $\text{Der}_A^\log(B, L)$ by $\text{Der}_A(B, L)$, which is the usual $B$-module of continuous $A$-derivations from $B$ to $L$.

Remark 3.2.3. In Definition 3.2.2, $(d, \delta)$ naturally extends to a log derivation on $(B, aN, \beta)$ and $\text{Der}_A^\log(B, L)$ remains unchanged if we replace $(B, N, \beta)$ with $(B, aN, \beta)$. Also, $\delta$ naturally extends to a group homomorphism $\delta^\text{gp} : (aN)^\text{gp} \to L$.

Definition 3.2.4. A homomorphism $f : (A, M, \alpha) \to (B, N, \beta)$ of pre-log Huber rings is called topologically of finite type (or tft for short) if $A$ and $B$ are complete, $f : A \to B$ is topologically of finite type (as in [Hub94, Sec. 3]), and $N^\text{gp}/(f^*(M))^\text{gp} \beta^{-1}(B^\times)$ is a finitely generated abelian group.

Now, let $f : (A, M, \alpha) \to (B, N, \beta)$ be as in Definition 3.2.4. Let $(B \hat{\otimes}_A B)[N]$ denote the monoid algebra over $B \hat{\otimes}_A B$ associated with the monoid $N$, and for each
n ∈ N, let e^n denote its element corresponding to n (by our convention). Let I be its ideal generated by {e^{f^m(m)} - 1}_{m \in \mathbb{N}} and \{(\beta(n) \otimes 1) - (1 \otimes \beta(n)) e^n\}_{n \in \mathbb{N}}. Note that, if n ∈ β^{-1}(B^\times), then e^n = \beta(n) \otimes \beta(n)^{-1} in \((B \otimes_A B)[N])/I. Let \( J \) be the kernel of the homomorphism
\[
\Delta_{\log} : (B \otimes_A B)[N]/I \to B
\]
sending \( b_1 \otimes b_2 \) to \( b_1 b_2 \) and all \( e^n \) to 1. We set
\[
\Omega_{B/A}^{\log} := J/J^2,
\]
and define \( d_{B/A} : B \to \Omega_{B/A}^{\log} \) and \( \delta_{B/A} : N \to \Omega_{B/A}^{\log} \) by setting
\[
d_{B/A}(b) = (b \otimes 1) - (1 \otimes b)
\]
and
\[
\delta_{B/A}(n) = e^n - 1.
\]
A short computation shows that \( d_{B/A} \) is an \( A \)-linear derivation, and that \( \delta_{B/A} \) is a homomorphism of monoids satisfying the required properties in Definition 3.2.2.

As observed in Remark 3.2.3, \( \delta_{B/A} \) naturally extends to a group homomorphism \( \delta_{B/A}^{\ast} : N^{\ast} \to \Omega_{B/A}^{\log} \) such that \( \delta_{B/A}^{\ast}((f \ast)^{\ast}(M^{\ast})) = 0 \). Then \( \Omega_{B/A}^{\log} \) is generated as a \( B \)-module by \( \ker(B \otimes_A B \to B) \) and \( \{ \delta_{B/A}^{\ast}(n) \} \), where \( n \) runs over a set of representatives of generators of \( N^{\ast}/((f \ast)(M^{\ast}))^{\ast}\beta^{-1}(B^\times)) \). More precisely,
\[
\Omega_{B/A}^{\log} \cong (\Omega_{B/A} \oplus (B \otimes_{\mathbb{Z}} N^{\ast}))/R,
\]
where \( \Omega_{B/A} \) is the usual \( B \)-module of continuous differentials (see [Hub96, Def. 1.6.1 and (1.6.2)]), and \( R \) is the \( B \)-module generated by
\[
\{(\beta(n), -\beta(n) \otimes n) : n \in N\} \cup \{(0, 1 \otimes f^m(m)) : m \in M\}.
\]
In particular, \( \Omega_{B/A}^{\log} \) is a finite \( B \)-module. Therefore, \( \Omega_{B/A}^{\log} \) is complete with respect to its natural \( B \)-module topology, and \( d_{B/A} \) is continuous.

**Proposition 3.2.7.** Under the above assumption, \( (\Omega_{B/A}^{\log}, d_{B/A}, \delta_{B/A}) \) is a universal object among all \((A, M, \alpha)\)-derivations of \((B, N, \beta)\).

**Proof.** Let \((d, \delta)\) be a derivation from \((B, N, \beta)\) to some complete topological \( B \)-module \( L \) over \((A, M, \alpha)\). We turn the \( B \)-module \( B \oplus L \) into a complete topological \( B \)-algebra, which we denote by \( B \star L \), by setting \((b, x)(b', x') = (bb', bx' + b'x)\). Note that the \( A \)-linear derivation \( d \) gives rise to a continuous homomorphism of topological \( B \)-algebras \( B \otimes_A B \to B \star L \) sending \( x \otimes y \) to \((xy, xyd)\). We may further extend it to a homomorphism \( (B \otimes_A B)[N] \to B \star L \) by sending \( e^n \) to \((1, \delta(n))\), for each \( n \in N \). By the conditions in Definition 3.2.2, this homomorphism factors through \((B \otimes_A B)[N])/I \to B \star L \), which we denote by \( \varphi \). By construction, the composition of \( \varphi \) with the natural projection \( B \star L \to B \) just recovers the homomorphism \((3.2.5)\). Therefore, \( \varphi \) induces a continuous morphism of \( B \)-modules \( \overline{\varphi} : \Omega_{B/A}^{\log} = J/J^2 \to L \). Now, a careful chasing of definitions verifies that \( \overline{\varphi} \circ d_{B/A} = d \) and \( \overline{\varphi} \circ \delta_{B/A} = \delta \), as desired. \( \square \)

Given any complete topological \( B \)-module \( L \), there is a natural forgetful functor \( \text{Der}_A^{\log}(B, L) \to \text{Der}_A(B, L) \) defined by \((d, \delta) \to d\). The following lemma is obvious:
Lemma 3.2.8. If \( f : (A, M, \alpha) \to (B, N, \beta) \) is a strict homomorphism of log Huber rings, then the canonical morphism \( \text{Der}_A^\log(B, \mathcal{L}) \to \text{Der}_A(B, \mathcal{L}) \) is an isomorphism, for every complete topological \( B \)-module \( \mathcal{L} \). Consequently, the canonical morphism \( \Omega_{B/A} \to \Omega^\log_{B/A} \) is an isomorphism.

Consider the following commutative diagram

\[
\begin{array}{ccc}
(A, M, \alpha) & \xrightarrow{f} & (D, O, \mu) \\
\downarrow & & \downarrow \\
(B, N, \beta) & \xrightarrow{g} & (D', O', \mu')
\end{array}
\]

of solid arrows. Here \((D, O, \mu)\) is a log Huber ring, \(H\) is a closed ideal of \(D\) such that \(H^2 = 0\) (so that we may regard \(1 + H\) as a submonoid of \(O\) via \(\mu : O \to D\)), \(D' \cong D/H\), \(O' \cong O/(1 + H)\), and \(\mu' : O' \to D'\) is the induced log structure.

Definition 3.2.10. We say that a homomorphism \( f : (A, M, \alpha) \to (B, N, \beta) \) of log Huber rings is formally log smooth (resp. formally log unramified, resp. formally log étale) if, for any diagram as in (3.2.9), there exists a lifting (resp. at most one lifting, resp. a unique lifting) \( \hat{g} : (B, N, \beta) \to (D, O, \mu) \) of \(g\) as the dotted arrow in (3.2.9), making the whole diagram commute. If \(M = \alpha^{-1}(A^\times)\) and \(N = \beta^{-1}(B^\times)\), then we simply say that (the underlying ring homomorphism) \( f : A \to B \) is formally smooth (resp. formally unramified, resp. formally étale) (see [Hub96, Def. 1.6.5]).

Remark 3.2.11. If the homomorphism \( f : (A, M, \alpha) \to (B, N, \beta) \) is strict homomorphism of log Huber rings, then it is formally log smooth (resp. formally log unramified, resp. formally log étale) if and only if the underlying morphism \( f : A \to B \) is formally smooth (resp. formally unramified, resp. formally étale).

Remark 3.2.12. Let \(k\) be a nontrivial nonarchimedean field. By [Hub96, Prop. 1.7.11], a tft homomorphism \( f : A \to B \) of Tate \(k\)-algebras is formally smooth (resp. formally unramified, resp. formally étale) if and only if the induced morphism \(\text{Spa}(B, B^\circ) \to \text{Spa}(A, A^\circ)\) is smooth (resp. unramified, resp. étale) in the sense of classical rigid analytic geometry.

The following theorem establishes the first fundamental exact sequence for log differentials:

Theorem 3.2.13. (1) A composition \((A, M, \alpha) \xrightarrow{f} (B, N, \alpha) \xrightarrow{g} (C, L, \delta)\) of tft homomorphisms of log Huber rings leads to an exact sequence

\[
\Omega^\log_{B/A} \to \Omega^\log_{C/A} \to \Omega^\log_{C/B} \to 0
\]

of finite topological \(C\)-modules (cf. [Hub96, Prop. 1.6.3]), where the first map sends \(c \otimes d_{B/A}(b)\) and \(c \otimes \delta_{B/A}(n)\) to \(cd_{C/A}(g(b))\) and \(c\delta_{C/A}(g^t(n))\), respectively, and the second map sends \(d_{C/A}(c)\) and \(\delta_{C/A}(l)\) to \(d_{C/B}(c)\) and \(\delta_{C/B}(l)\), respectively.

(2) Moreover, if the homomorphism \( g : (B, N, \alpha) \to (C, L, \delta) \) is formally log smooth, then \(\Omega^\log_{B/A} \otimes_B C \to \Omega^\log_{C/A} \) is injective, and the short exact sequence

\[
0 \to C \otimes_B \Omega^\log_{B/A} \to \Omega^\log_{C/A} \to \Omega^\log_{C/B} \to 0
\]

is split in the category of topological \(C\)-modules.

(3) If \(g\) is formally log unramified, then \(\Omega^\log_{C/B} = 0\).
(4) If \( g \) is formally log étale, then \( \Omega_{C/A}^{\log} \cong C \otimes_B \Omega_{B/A}^{\log} \).

Proof. Since the homomorphisms of log Huber rings are all tft, \( \Omega_{B/A}^{\log} \otimes_B C, \Omega_{C/A}^{\log} \), and \( \Omega_{C/B} \) are finite \( C \)-modules. Thus, to prove the exactness in \( \Omega_{C/B} \), it suffices to show that, for any finite \( C \)-module \( O \), the induced sequence

\[
0 \to \text{Hom}_C(\Omega_{C/B}^{\log}, O) \to \text{Hom}_C(\Omega_{C/A}^{\log}, O) \to \text{Hom}_C(C \otimes_B \Omega_{B/A}^{\log}, O)
\]

is exact. By Proposition 3.2.7, this sequence is nothing but

\[
0 \to \text{Der}^\log_B(C, O) \to \text{Der}^\log_A(C, O) \to \text{Der}^\log_A(B, O),
\]

whose exactness is obvious.

In the rest of the proof, let \( O \) be a finite \( C \)-module, and let \( (d, \delta) : (B, N, \beta) \to O \) be an \( (A, M, \alpha) \)-derivation. Let \( C \ast O \) be the \( C \)-algebra defined as in the proof of Proposition 3.2.7 equipped with the log structure \( (\gamma \ast \text{Id}) : L \otimes O \to C \ast O : (a, b) \mapsto (\delta(a), \delta(a)b) \), and denote the resulted log Huber ring by \((C, L, \gamma) \ast O\).

We claim that there is a natural bijection between the set of \((A, M, \alpha)\)-derivations \((d, \delta) : (C, L, \gamma) \to O\) extending \((d, \delta)\) and the set of homomorphisms of log Huber rings \( h : (C, L, \gamma) \to (C, L, \gamma) \ast O\) making the diagram

\[
\begin{array}{ccc}
(B, N, \beta) & \longrightarrow & (C, L, \gamma) \ast O \\
\downarrow g & & \downarrow \gamma \\
(C, L, \gamma) & \longrightarrow & (C, L, \gamma)
\end{array}
\]

commute. Here the upper horizontal map is a homomorphism of log Huber rings sending \((b, n)\) to \(((g(b), d(b)), (g^2(n), \delta(n)))\), and the right vertical one is the natural projection. To justify the claim, for each map \( h' : (C, L) \to (C \ast O, L \oplus O) \) lifting the projection \((C \ast O, L \oplus O) \to (C, L)\), let us write \( h' = ((\text{Id}, \tilde{d}), (\text{Id}, \tilde{\delta}))\). Then a short computation shows that \( h' \) is a homomorphism of log Huber rings if and only if \( \tilde{d} \) is a derivation and \( \tilde{\delta} \) is a homomorphism of monoids such that \( \tilde{d}(\gamma(l)) = \gamma(l)\delta(l) \) for all \( l \in L \), and the claim follows.

Thus, if \((C, L, \gamma)\) is formally log unramified over \((B, N, \beta)\), then the natural map \( \text{Der}^\log_A(C, O) \to \text{Der}^\log_B(B, O) \) is injective for each finite \( C \)-module \( O \). In other words, \( \text{Hom}_C(\Omega_{C/A}^{\log}, O) \to \text{Hom}_C(C \otimes_B \Omega_{B/A}^{\log}, O) \) is injective, and so \( C \otimes_B \Omega_{B/A}^{\log} \to \Omega_{C/A}^{\log} \) is surjective, yielding (3). Similarly, if \((C, L, \gamma)\) is formally log smooth over \((B, N, \beta)\), then \( \text{Hom}_C(\Omega_{C/A}^{\log}, O) \to \text{Hom}_C(C \otimes_B \Omega_{B/A}^{\log}, O) \) is surjective, and we can justify (2) by taking \( O = C \otimes_B \Omega_{B/A}^{\log} \). Finally, (4) follows from (2) and (3). \( \square \)

Definition 3.2.14. Let \( u : P \to Q \) be a homomorphism of fine monoids, and let \( R \) be a Huber ring. Then we have the pre-log Huber ring \( P \to R(P) : a \mapsto e^a \) (resp. \( Q \to k(Q) : a \mapsto e^a \)), with the topology given in Lemma 2.2.11. In this case, we say that \( R(P) \) is a log Huber \( R \)-algebra. By abuse of notation, we shall still denote by \( R(P) \) (resp. \( R(Q) \)) the resulted log Huber \( R \)-algebras.

Proposition 3.2.15. Let \( u : P \to Q \) and \( R \) as in Definition 3.2.14. If the kernel and the torsion part of the cokernel (resp. the kernel and the cokernel) of \( u^\text{gp} : P^\text{gp} \to Q^\text{gp} \) are finite groups of orders invertible in \( R \), then \( R(Q) \) is formally log
smooth (resp. formally log étale) over \( R(P) \). In this case, the map \( \delta : Q^{\text{gp}} \rightarrow \Omega^1_{R(Q)/R(P)} \) induces an isomorphism

\[
\Omega^1_{R(Q)/R(P)} \cong R(Q) \otimes_{\mathbb{Z}} (Q^{\text{gp}}/u^{\text{gp}}(P^{\text{gp}})).
\]

Proof. Consider the commutative diagram

\[
\begin{array}{ccc}
R(P) & \longrightarrow & (D, O, \mu) \\
\downarrow & & \downarrow \\
R(Q) & \longrightarrow & (D', O', \mu')
\end{array}
\]

as in (3.2.9). This gives rise to a commutative diagram of monoids

\[
\begin{array}{ccc}
P & \longrightarrow & O \\
\downarrow & & \downarrow \\
Q & \longrightarrow & O',
\end{array}
\]

which in turn induces a commutative diagram of abelian groups

\[
\begin{array}{ccc}
P^{\text{gp}} & \longrightarrow & O^{\text{gp}} \\
\downarrow & & \downarrow \pi \\
Q^{\text{gp}} & \longrightarrow & (O')^{\text{gp}}
\end{array}
\]

Note that there is a natural bijection between the set of the homomorphisms of log Huber \( R \)-algebras \( R(Q) \rightarrow (D, O, \mu) \) extending (3.2.16) and the set of homomorphisms of monoids \( Q \rightarrow O \) extending (3.2.17). By using the cartesian diagram

\[
\begin{array}{ccc}
O & \longrightarrow & O^{\text{gp}} \\
\downarrow & & \downarrow \pi \\
O' & \longrightarrow & (O')^{\text{gp}}
\end{array}
\]

and the fact that \( P \) and \( Q \) are fine monoids, we see that there is also a bijection between the set of the desired homomorphisms \( R(Q) \rightarrow (D, O, \mu) \) and the set of group homomorphisms \( Q^{\text{gp}} \rightarrow O^{\text{gp}} \) extending (3.2.18).

Since \( H^2 = 0 \), we have \( \ker(O^{\text{gp}} \rightarrow (O')^{\text{gp}}) = 1 + H \cong H \). Let \( G = Q^{\text{gp}}/u^{\text{gp}}(P^{\text{gp}}) \). Since the kernel and the torsion part of the cokernel (resp. the kernel and the cokernel) of \( u^{\text{gp}} \) are finite groups of orders invertible in \( R \), the set of the desired homomorphisms \( Q^{\text{gp}} \rightarrow O^{\text{gp}} \) is a torsor under \( \text{Hom}(G, H) \cong \text{Hom}(G/G^{\text{tor}}, H) \), where \( G^{\text{tor}} \) denotes the torsion subgroup of \( G \). This proves the first statement of the proposition.

On the other hand, for any finite \( R(Q) \)-module \( L \), by the same argument as in the proof of Theorem 3.2.13, there is a bijection between \( \text{Der}^1_{R(P)}(R(Q), L) \) and the set of \( h : R(Q) \rightarrow R(Q) \ast L \) extending the following commutative diagram

\[
\begin{array}{ccc}
R(P) & \longrightarrow & R(Q) \ast L \\
\downarrow & & \downarrow \\
R(Q) & \longrightarrow & R(Q)
\end{array}
\]
Then \( \text{Der}_{R/P}^\log (R(Q), L) \cong \text{Hom}_{R(Q)}(\Omega^\log_{R(Q)/R(P)}, L) \cong \text{Hom}(G/G_{\text{tor}}, L) \), by the previous paragraph. The second statement of the proposition follows.

**Corollary 3.2.19.** Let \( u : P \to Q \) and \( R \) as in Definition 3.2.14 such that the kernel and the torsion part of the cokernel of \( u^\log : P^\log \to Q^\log \) are finite groups of orders invertible in \( R \). Let \( Q' \) be any fine monoid, and let \( S \) denote the log Huber ring associated with the pre-log Huber ring \( \tilde{Q} := Q \oplus Q' \to R(Q) \), so that the surjective homomorphism \( R(\tilde{Q}) \to S \) of log Huber rings is strict. Then the map \( \delta : \tilde{Q}^\log \to \Omega^\log_{S/R(P)} \) induces an isomorphism

\[
\Omega^\log_{S/R(P)} \cong R(Q) \otimes_{\mathbb{Z}} (\tilde{Q}^\log / u^\log(P^\log)).
\]

If, in addition, the torsion part of \((Q')^\log\) is a finite group whose order is invertible in \( R \), then we also have

\[
\Omega^\log_{S/R(P)} \cong R(Q) \otimes_{\mathbb{Z}} \Omega_{R(\tilde{Q})/R(P)}.
\]

**Proof.** By comparing the definitions of \( \Omega^\log_{S/R(P)} \) and \( \Omega^\log_{R(Q)/R(P)} \) as in (3.2.6), we obtain \( \Omega^\log_{S/R(P)} \cong \Omega^\log_{R(Q)/R(P)} \otimes (R(Q) \otimes_{\mathbb{Z}} (Q')^\log) \), because \( Q \to S \) maps \( Q' \) to zero. Since this isomorphism is compatible with the canonical maps \( Q^\log \to \Omega^\log_{R(Q)/R(P)} \), \( \tilde{Q}^\log \to \Omega^\log_{S/R(P)} \), and \( \tilde{Q}^\log \to \Omega^\log_{R(Q)/R(P)} \) (denoted by \( \delta \)), we can finish the proof by applying Proposition 3.2.15 to \( \Omega^\log_{R(Q)/R(P)} \) and \( \Omega^\log_{R(Q)/R(P)} \).

The next step is to define sheaves of log differentials for locally noetherian coherent log adic spaces, and show that their formation is compatible with fiber products in the category of locally noetherian coherent, fine, and fs log adic spaces.

**Definition 3.2.20.** Let \( f : (Y, \mathcal{M}_Y, \alpha_Y) \to (X, \mathcal{M}_X, \alpha_X) \) be a morphism of locally noetherian log adic spaces, and let \( \mathcal{F} \) be a sheaf of complete topological \( \mathcal{O}_{Y_{\text{et}}} \)-modules. By a derivation of \((Y, \mathcal{M}_Y, \alpha_Y)\) over \((X, \mathcal{M}_X, \alpha_X)\) valued in \( \mathcal{F} \), we mean a pair \((d, \delta)\), where \( d : \mathcal{O}_{Y_{\text{et}}} \to \mathcal{F} \) is a continuous \( \mathcal{O}_{X_{\text{et}}} \)-linear derivation and \( \delta : \mathcal{M}_Y \to \mathcal{F} \) is a morphism of sheaves of monoids such that \( \delta(f^{-1}(\mathcal{M}_X)) = 0 \) and \( d(\alpha_Y(m)) = \alpha_Y(m) \delta(m) \), for all sections \( m \) of \( \mathcal{M}_Y \).

**Construction 3.2.21.** Let \( f : (Y, \mathcal{M}_Y, \alpha_Y) \to (X, \mathcal{M}_X, \alpha_X) \) be a morphism of noetherian coherent log adic spaces, where \( X = \text{Spa}(A, A^+) \) and \( Y = \text{Spa}(B, B^+) \). Suppose that \( f \) induces a tft homomorphism of log Huber rings \((A, M, \alpha) \to (B, N, \beta)\), where \( M := \mathcal{M}_X(X) \) and \( N := \mathcal{M}_Y(Y) \), and where \( \alpha := \alpha_X(X) : M \to A \) and \( \beta := \alpha_Y(Y) : N \to B \). Let \( \Omega^\log_{Y/X} \) denote the coherent sheaf on \( Y_{\text{et}} \) associated with \( \Omega^\log_{B/A} \) (see (3.2.6)). For each \( \text{Spa}(C, C^+) \in Y_{\text{et}} \), by Theorem 3.2.13, the log differential \( \Omega^\log_{C/A} \) for \((A, M, \alpha) \to (C, N, (B \to C) \circ \beta)\) is naturally isomorphic to \( C \otimes_B \Omega^\log_{B/A} = \Omega^\log_{Y/X}(\text{Spa}(C, C^+)) \); and the maps \( d_{C/A} : C \to \Omega^\log_{C/A} \) and \( \delta_{C/A} : N \to \Omega^\log_{C/A} \) naturally extend to a continuous \( \mathcal{O}_{X_{\text{et}}} \)-linear derivation \( d_{Y/X} : \mathcal{O}_{Y_{\text{et}}} \to \Omega^\log_{Y/X} \) and a morphism of sheaves of monoids \( \delta_{Y/X} : N_Y \to \Omega^\log_{Y/X} \) satisfying \( \delta_{Y/X}(f^{-1}(\mathcal{M}_X)) = 0 \) and \( d_{Y/X}(\alpha_Y(n)) = \alpha_Y(n) \delta_{Y/X}(n) \), for all sections \( n \) of \( N_Y \) over objects of \( Y_{\text{et}} \). We may further extend \( \delta_{Y/X} \) to a morphism of sheaves of monoids \( \mathcal{M}_Y \to \Omega^\log_{Y/X} \) such that \( \delta_{Y/X}(f^{-1}(\mathcal{M}_X)) = 0 \) and \( d_{Y/X}(\alpha_Y(m)) = \alpha_Y(m) \delta_{Y/X}(m) \), for all sections \( m \) of \( \mathcal{M}_Y \) over objects of \( Y_{\text{et}} \).
Lemma 3.2.22. In Construction 3.2.21, the triple \((\Omega_{Y/X}^{\log}, d_{Y/X}, \delta_{Y/X})\) is a universal object among all derivations of \((Y, \mathcal{M}_Y, \alpha_Y)\) over \((X, \mathcal{M}_X, \alpha_X)\). Moreover, for any affinoid objects \(V \in Y_{\acute{e}t}\) and \(U \in X_{\acute{e}t}\) fitting into a commutative diagram

\[
\begin{array}{ccc}
V & \longrightarrow & Y \\
\downarrow & & \downarrow \\
U & \longrightarrow & X,
\end{array}
\]

we have a canonical isomorphism \((\Omega_{Y/X}^{\log}|_V, d_{Y/X}|_V, \delta_{Y/X}|_V) \cong (\Omega_{V/U}^{\log}, d_{V/U}, \delta_{V/U})\).

Proof. Let \((d, \delta)\) be a derivation of \((Y, \mathcal{M}_Y, \alpha_Y)\) over \((X, \mathcal{M}_X, \alpha_X)\) valued in some complete topological \(O_{Y_{\acute{e}t}}\)-module \(\mathcal{F}\). For each \(\text{Spa}(C, C^+) \in Y_{\acute{e}t}\), the derivation \((d, \delta)\) specializes to a derivation

\[
(C, \mathcal{M}_Y(\text{Spa}(C, C^+)), \alpha_Y(\text{Spa}(C, C^+))) \rightarrow \mathcal{F}(\text{Spa}(C, C^+))
\]

over \((A, \mathcal{M}_X(X), \alpha_X(X))\), which in turn restricts to a derivation

\[
(C, N, (B \rightarrow C) \circ \beta) \rightarrow \mathcal{F}(\text{Spa}(C, C^+))
\]

over \((A, M, \alpha)\). By the universal property of log differentials, it factors through a continuous \(C\)-linear morphism \(\Omega_{C/A}^{\log} \rightarrow \mathcal{F}(\text{Spa}(C, C^+))\). Moreover, we deduce from the universal property of log differentials a commutative diagram

\[
\begin{array}{ccc}
\Omega_{C_2/A}^{\log} & \longrightarrow & \mathcal{F}(\text{Spa}(C_2, C_2^+)) \\
\downarrow & & \downarrow \\
\Omega_{C_1/A}^{\log} & \longrightarrow & \mathcal{F}(\text{Spa}(C_1, C_1^+))
\end{array}
\]

for any \(\text{Spa}(C_1, C_1^+) \rightarrow \text{Spa}(C_2, C_2^+)\) in \(Y_{\acute{e}t}\). Consequently, for all \(\text{Spa}(C, C^+) \in Y_{\acute{e}t}\), the morphisms \(\Omega_{C_2/A}^{\log} \rightarrow \mathcal{F}(\text{Spa}(C, C^+))\) naturally extend to a continuous \(O_{X_{\acute{e}t}}\)-linear morphism \(\Omega_{Y/X}^{\log} \rightarrow \mathcal{F}\), whose compositions with \(d_{Y/X}\) and \(\delta_{Y/X}\) are equal to \(d\) and \(\delta\), respectively. This proves the first assertion of the lemma. The second assertion then follows from Theorem 3.2.13 \(\square\)

Construction 3.2.23. For a lft morphism \(f : Y \rightarrow X\) of noetherian coherent log adic spaces, by Proposition 2.3.21, there exist a finite index set \(I\) and \(\acute{e}tale\) coverings \(\{X_i \rightarrow X\}_{i \in I}\) and \(\{Y_i \rightarrow Y\}_{i \in I}\), respectively, by affinoid log adic spaces such that \(f\) induces a morphism \(Y_i \rightarrow X_i\) which fits into the setting of Construction 3.2.21 for each \(i \in I\). By Lemma 3.2.22, the pullbacks of the triples \((\Omega_{Y_i/X_i}^{\log}, d_{Y_i/X_i}, \delta_{Y_i/X_i})\) are canonically isomorphic to each other over the fiber products of \(Y_i\) over \(Y\). Thus, by Proposition A.10, we obtain a triple \((\Omega_{Y/X}^{\log}, d_{Y/X}, \delta_{Y/X})\) on \(Y_{\acute{e}t}\), where \((d_{Y/X}, \delta_{Y/X})\) gives a derivation of \(Y\) over \(X\) valued in \(\Omega_{Y/X}^{\log}\).

By Lemma 3.2.22, we immediately obtain the following:

Lemma 3.2.24. In Construction 3.2.23, \((\Omega_{Y/X}^{\log}, d_{Y/X}, \delta_{Y/X})\) is a universal object among all derivations of \(Y\) over \(X\). Consequently, \((\Omega_{Y/X}^{\log}, d_{Y/X}, \delta_{Y/X})\) is well defined, i.e., independent of the choice of \(\acute{e}tale\) coverings; and its definition extends to all lft morphisms \(f : Y \rightarrow X\) of locally noetherian coherent log adic spaces.
there is any risk of confusion, we shall denote this pushforward more precisely by $\Omega^\log_{Y/X}$ in Lemma 3.2.24. The sheaf of log differentials of $f$ and $(dY/X, \delta_{Y/X})$ is modeled on Proposition 3.2.26.

Log differentially the same argument as in the proof of [Ogu18, Prop. IV.1.1.9], we also have $L$ are affinoid, and that $\log$ adic spaces of $f$ of Definition 3.2.25. We call the $\Omega^\log_{Y/X}$ in Lemma 3.2.24 the sheaf of log differentials $\log$ adic spaces such that $\log$ adic spaces in which $Y \rightarrow X$ is lft. Then $f^*(\Omega^\log_{Y/X}) \cong \Omega^\log_{Y'/X'}$.

**Proposition 3.2.26.** Let

$$
\begin{array}{ccc}
Y' & \rightarrow & X' \\
\downarrow f & & \downarrow \\
Y & \rightarrow & X
\end{array}
$$

be a cartesian diagram in the category of locally noetherian coherent (resp. fine, resp. fs) log adic spaces in which $Y \rightarrow X$ is lft. Then $f^*(\Omega^\log_{Y/X}) \cong \Omega^\log_{Y'/X'}$.

**Proof.** By the construction of sheaves of log differentials, we may assume that $Y = \text{Spa}(B, B^+)$, $X = \text{Spa}(A, A^+)$, $X' = \text{Spa}(A', A^+)$, and $Y' = \text{Spa}(B', B^+)$ are affinoid, and that $Y \rightarrow X$ and $X' \rightarrow Y$ admit charts $P \rightarrow Q$ and $P \rightarrow P'$, respectively, given by finitely generated (resp. fine, resp. fs) monoids. Let $Q' := Q \oplus P P'$. By the proofs of Propositions 2.3.23 and 2.3.27, $B'$ is isomorphic to $B \otimes_{A A'} (\text{resp. } B \otimes_{A A'} \otimes_{\mathbb{Z}[Q']} \mathbb{Z}[[Q']^{\text{int}}], \text{resp. } B \otimes_{A A'} \otimes_{\mathbb{Z}[Q']} \mathbb{Z}[[Q']^{\text{sat}}])$ and $Y'$ is modeled on $Q'$ (resp. $(Q')^{\text{int}}$, resp. $(Q')^{\text{sat}}$). We need to show that $\Omega^\log_{B'/A'} \cong \Omega^\log_{B'/B} \otimes_{B'} B'$. Since $\text{Hom}_{B'}(\Omega^\log_{B'/A'}, L) \cong \text{Der}^\log_{A'}(B', L)$ and $\text{Hom}_{B'}(\Omega^\log_{B'/A'} \otimes_{B'} B', L) \cong \text{Hom}_{B'}(\Omega^\log_{B'/A'} \otimes_{B'} B', L)$, for each complete topological $B'$-module $L$, it suffices to show that $\text{Der}^\log_{A'}(B', L) \cong \text{Der}^\log_{A'}(B', L) \otimes_{B'} B'$, for each such $L$. In the case of coherent log adic spaces, given Remark 3.2.3, this follows from essentially the same argument as in the proof of [Ogu18, Prop. IV.1.1.3] (in the log scheme case). Since $(Q')^{\text{gp}} \cong ((Q')^{\text{int}})^{\text{gp}} \cong ((Q')^{\text{sat}})^{\text{gp}}$, by essentially the same argument as in the proof of [Ogu18, Prop. IV.1.1.9], we also have $\text{Der}^\log_{A'}(B', L) \cong \text{Der}^\log_{A'}(B \otimes_{A A'} A', \text{sat}) \otimes_{B A'} B' \cong \text{Der}^\log_{A'}(B, L) \otimes_{B} B'$, yielding the desired isomorphism in the remaining cases of fine and fs log adic spaces. 

**Theorem 3.2.27.**

1. A composition of lft morphisms $Y \rightarrow X \rightarrow S$ of locally noetherian coherent log adic spaces naturally induces an exact sequence $f^*(\Omega^\log_{Y/S}) \rightarrow \Omega^\log_{Y/X} \rightarrow \Omega^\log_{Y/S} \rightarrow 0$ of coherent $\Omega_Y$-modules.

2. If $f$ is a log smooth morphism of locally noetherian fs log adic spaces, then $f^*(\Omega^\log_{Y/S}) \rightarrow \Omega^\log_{Y/X}$ is injective, and $\Omega^\log_{Y/X}$ is a vector bundle whose rank is equal to the relative dimension of $f$.

3. If $f$ is a log étale morphism of locally noetherian fs log adic spaces, then $f^*(\Omega^\log_{Y/S}) \cong \Omega^\log_{Y/S}$ and $\Omega^\log_{Y/X} = 0$.

**Proof.** By the construction of sheaves of log differentials, the assertion [1] follows from Theorem 3.2.13[1]. The remaining assertions [2] and [3] follow from Proposition 3.2.15 Proposition 3.2.26 and Theorem 3.2.13[2] and [4].

**Corollary 3.2.28.** Suppose that $X \rightarrow Y \rightarrow S$ are morphisms of locally noetherian fs log adic spaces such that $g$ is log smooth; such that the underlying morphism of adic spaces of $f$ is an isomorphism; and such that the canonical homomorphism...
\[ M_{X, \pi} \to M_{\tilde{X}, \pi} \] of fs monoids splits as a direct summand at each geometric point \( \pi \) of \( X \). Then \( \tilde{\Omega}_{X/S, \pi} \cong \Omega_{X/S, \pi} \oplus (O_{X, \pi} \otimes (M_{\tilde{X}, \pi}/M_{X, \pi})) \) at each \( \pi \). Moreover, if there is a strict closed immersion \( \iota: \tilde{X} \to Y \) to a log adic space \( Y \) log smooth over \( S \), then we also have \( \tilde{\Omega}_{Y/S} \cong \iota^*(\Omega_{Y/S}) \).

**Proof.** This follows from Theorem 3.2.27 and Corollary 3.2.19. \( \square \)

**Definition 3.2.29.** Let \( X \) be a log adic space log smooth over a nonarchimedean field \( k \), as in Definition 3.1.9. Then we set \( \Omega_{X, \log}^a := \wedge^a \Omega_X^\log \), for each integer \( a \geq 0 \). More generally, for any \( X \) over \( X \) such that \( \tilde{X} \to X \to S = \Spa(k, k^+) \) is as in Corollary 3.2.28 and such that \( \tilde{X} \) admits a strict closed immersion to a log adic space \( Y \) log smooth over \( k \), we also set \( \Omega_{\tilde{X}, \log}^a := \wedge^a \tilde{\Omega}_{\tilde{X}}^\log \), which is canonically isomorphic to the pullback of \( \Omega_{\tilde{Y}, \log}^a \), for each integer \( a \geq 0 \).

**Example 3.2.30.** In Example 2.3.18 the morphisms \( X \leftarrow X^J \to X \) satisfy the requirements of the morphisms \( Y \leftarrow X \to X \) in the second half of Corollary 3.2.28 and hence we have a canonical isomorphism \( \Omega_{X^J, \log} = (X_J \to X)^*(\Omega_X^\log) \) and locally (depending on the choices of coordinates) some isomorphisms \( \Omega_{X^J, \log} \cong \Omega_{X^J, \log}^a \otimes O^J \) of vector bundles on \( X^J_{/an} \cong X_{J, an} \).

### 4. Kummer étale topology

**4.1. The Kummer étale site.**

**Definition 4.1.1.** A homomorphism \( u : P \to Q \) of saturated monoids is called Kummer if it is injective and if the following conditions hold:

(1) For any \( a \in Q \), there exists some integer \( n \geq 1 \) such that \( na \in u(P) \).

(2) The quotient \( Q^{\gp}/u^{\gp}(P^{\gp}) \) is a finite group.

**Definition 4.1.2.** A morphism (resp. finite morphism) \( f : Y \to X \) of locally noetherian fs log adic spaces is called Kummer (resp. finite Kummer) if it admits, étale locally on \( X \) and \( Y \) (resp. étale locally on \( X \)), an fs chart \( u : P \to Q \) such that the homomorphism \( u \) is Kummer. Such a chart \( u \) is called a Kummer chart of \( f \). Moreover, \( f \) is called Kummer étale (resp. finite Kummer étale) if the Kummer chart \( u \) above can be chosen such that \( |Q^{\gp}/u^{\gp}(P^{\gp})| \) is invertible in \( O_Y \), and such that \( f \) and \( u \) induce a morphism \( Y \to X \times X(P) \times Q \) of log adic spaces (cf. Remark 2.3.3) whose underlying morphism of adic spaces is étale (resp. finite étale). A Kummer morphism is called a Kummer cover if it is surjective.

**Remark 4.1.3.** Definition 4.1.2 can be extended beyond the case of locally noetherian fs log adic spaces, with suitable \( P \) and \( Q \), such that all adic spaces involved are étale sheafy. However, we will not pursue this generality in this paper.

**Remark 4.1.4.** Any Kummer homomorphism \( u : P \to Q \) as in Definition 4.1.1 is exact. Accordingly, any Kummer morphism \( f : Y \to X \) as in Definition 4.1.2 is exact. In particular, Proposition 2.3.32 is applicable to Kummer morphisms.

**Definition 4.1.5.** (1) For any saturated torsionfree monoid \( P \) and any positive integer \( n \), let \( \frac{1}{n}P \) be the saturated torsionfree monoid such that the inclusion \( P \to \frac{1}{n}P \) is isomorphic to the \( n \)-th multiple map \([n] : P \to P\).
Let \( X \) be a locally noetherian log adic space modeled on a torsionfree fs monoid \( P \), and \( n \) any positive integer. Then we have the log adic space \( X^\pm := X \times_{X(P)} X(\frac{1}{n}P) \), with a natural chart modeled on \( \frac{1}{n}P \).

The structure morphism \( X^\pm \to X \) is a finite Kummer cover with a Kummer chart given by the natural inclusion \( P \hookrightarrow \frac{1}{n}P \), which is finite Kummer étale when \( n \) is invertible in \( O_X \). Such morphisms will play an important role in Sections 4.3 and 4.4. More generally, we have the following:

**Proposition 4.1.6.** Let \( X \) be a locally noetherian log adic space with a chart modeled on an fs monoid \( P \). Let \( u : P \to Q \) be a Kummer homomorphism of fs monoids such that \( G := Q^{\text{gp}} / u^{\text{gp}}(P^{\text{gp}}) \) is a finite group. Consider the fiber product

\[
Y := X \times_{X(P)} X(Q),
\]

which is equipped with a canonical action of the group object

\[
G^D_X := X(G)
\]

over \( X \), which is an analogue of a diagonalizable group scheme that is Cartier dual to the constant group scheme \( G \). Then the following hold:

1. The natural projection \( f : Y \to X \) is a finite Kummer cover, which is finite (see [Hub96, (1.4.4)]) and surjective.
2. When \( X \) and hence \( Y \) are affinoid, we have a canonical exact sequence

\[
0 \to O_X(X) \to O_Y(Y) \to O_{Y \times_X Y}(Y \times_X Y).
\]
3. The morphism \( G^D_X \times_X Y \to Y \times_X Y \) induced by the action and the second projection is an isomorphism.
4. When \( |G| \) is invertible in \( O_X \), the group \( G^D_X \) is étale over \( X \) (which is simply the constant group \( \text{Hom}(G, O_X(X)^\times) \)) when \( O_X(X) \) contains all the \( |G| \)-th roots of unity); and the cover \( f : Y \to X \) is a Galois finite Kummer étale cover with Galois group \( G^D_X \), which is open.

For the proof of Proposition 4.1.6 we need the following general construction, which will also be useful later in Section 4.4.

**Lemma 4.1.7.** Let \( Y = \text{Spa}(S, S^+) \to X = \text{Spa}(R, R^+) \) be a finite morphism of noetherian adic spaces, and let \( \Gamma \) be a finite group which acts on \( Y \) by morphisms over \( X \). Then \((T, T^+) := (S^T, (S^+)^T) \) is a Huber pair, and \( Z := \text{Spa}(T, T^+) \) is a noetherian adic space finite over \( X \). Moreover, the canonical morphism \( Y \to X \) factors through a finite, open, and surjective morphism \( Y \to Z \), which induces a homeomorphism \( Y/\Gamma \sim Z \) of underlying topological spaces and identifies \( Z \) as the categorical quotient \( Y/\Gamma \) in the category of adic spaces.

**Proof.** For analytic adic spaces, and for any finite group \( \Gamma \) such that \( |\Gamma| \) is invertible in \( S \), this essentially follows from [Han16, Thm. 1.2] without the noetherian hypothesis. Nevertheless, we have the noetherian hypothesis, but not the analytic or invertibility assumptions here. Moreover, we have a base space \( X \) over which \( Y \) is finite. Hence, we can resort to the following more direct arguments.

Since \( R \) is noetherian, and since \( S \) is a finite \( R \)-module, \( T = S^T \) is also a finite \( R \)-module, and \( T^+ = (S^+)^T \) is the integral closure of \( R^+ \) in \( T \). Therefore, \((T, T^+) \) has a canonical structure of a Huber pair such that \( Z := \text{Spa}(T, T^+) \) is a noetherian adic space finite over \( X = \text{Spa}(R, R^+) \). Moreover, \( Y \to Z \) is also finite. By [Hub96, (1.4.2) and (1.4.4)] and [Hub94, Sec. 2], if \( \{s_1, \ldots, s_r\} \) is any set of generators of
S as an $T$-module, then the topology of $S$ is generated by $\sum_{i=1}^{r} U_i s_i$, where $U_i$ runs over a basis of the topology of $T$, for all $i$. Now, suppose that $w : T \to \Gamma_w$ is any continuous valuation, and that $v : S \to \Gamma_v$ is any valuation extending $w$. Note that $v$ and $w$ factor through the domains $\overline{S} := S[w^{-1}(0)]$ and $\overline{T} := T[w^{-1}(0)]$, respectively, and Frac($\overline{S}$) is a finite extension of Frac($\overline{T}$). Therefore, we may assume that $\Gamma_v$ and $\Gamma_w$ are generated by $v(S)$ and $w(T)$, respectively, and that $\Gamma_w$ is a finite index subgroup of $\Gamma_v$. For each $\gamma \in \Gamma_v$, the subgroup $\{ s \in S : v(s) < \gamma \}$ of $S$ is open because it contains $\sum_{i=1}^{r} U_i s_i$, where $U_i := \{ t \in T : w(t) < \gamma - v(s_i) \}$ is open by the continuity of $w$. Consequently, Cont($S$) $\to$ Cont($T$) is surjective. This replaces the main argument in Step 1 of the proof of [Han16, Thm. 3.1] where the Tate assumption is used. After this step, the remaining arguments in the proof of [Han16, Thm. 3.1] works verbatim and shows that Spa($S, S^+$) $\to$ Spa($T, T^+$) induces a homeomorphism Spa($S, S^+$) $\to$ Spa($T, T^+$).

Since $Y$ and $Z$ are finite over $X$, and since $T = S^\Gamma$, by [Han16, (1.4.4)] and Proposition [A.9], the canonical morphism $O_Z \to (Y \to Z)_* (O_Y)$ factors through an isomorphism $O_Z \cong (Y \to Z)_* (O_Y)^\Gamma$. (This provides a replacement of [Han16, Thm. 3.2].) Thus, the canonical morphism $Y \to Z$ factors through an isomorphism $Y/\Gamma \cong Z$ of adic spaces, as in [Han16, Thm. 1.2], as desired.

Now we are ready for the following:

**Proof of Proposition 4.1.6.** Let us identify $P$ as a submonoid of $Q$ via $u : P \to Q$. Since the assertions are local in nature on $X$, we may assume that $X = \text{Spa}(R, R^+)$ and hence $Y$ is affinoid. Then the homomorphism $O_X(X) \to O_Y(Y)$ is injective, because it is the base change of $\mathbb{Z}[P] \to \mathbb{Z}[Q]$ from $\mathbb{Z}[P]$ to $R$, and $\mathbb{Z}[P]$ is a direct summand of $\mathbb{Z}[Q]$ as $\mathbb{Z}[P]$-modules, as explained in the proof of [INT13, Lem. 2.1]. Also, the homomorphism is finite because $Q$ is finitely generated and $u$ is Kummer, and hence [1] follows. Since the canonical sequence $O_X(X) \to O_Y(Y) \to O_{X \times_X Y}(Y \times_X Y)$ is exact by [Niz08, Lem. 3.28], [2] also follows.

By [INT13, Lem. 3.3], $(Q \oplus P) \cong Q \oplus G$. Hence, we have $X(G) \times_X X(Q) \cong X(Q) \times_X \mathbb{P}(Q)$ (see Remark [2.3.28]), and the action of $G^\Gamma = X(G)$ on $Y$ induces $X(G) \times_X Y \to Y \times_X Y$. This verifies [3].

As for [4], since it can be verified étale locally on $X$, we may assume that $R$ contains all $[G]$-th roots of unity. In this case, $X(G) \cong \Gamma_X$, where $\Gamma := \text{Hom}(G, O_X(X)^\times)$, and the action of $G^\Gamma$ is induced by the canonical actions of $\Gamma$ on $X(Q)$ and $X(Q^{\text{gp}})$, by sending $q$ to $\gamma(q)q$, for each $q \in Q^{\text{gp}}$ and $\gamma \in \Gamma$. Note that $(R(Q))^\Gamma = (R(Q^{\text{gp}}))^\Gamma \cap R(Q) = R(Q^{\text{gp}}) \cap R(Q) = R(P)$, where the last one follows from the assumptions that $u : P \to Q$ is Kummer and that $P$ is saturated; and the formation of $\Gamma$-invariants commutes the base change from $R(P)$ to $R$, because $[\Gamma]$ is invertible in $R$. Thus, if $Y = \text{Spa}(S, S^+)$, then the morphism $Y \to X = \text{Spa}(R, R^+) \cong \text{Spa}(S^\Gamma, (S^+)^\Gamma)$ is open and induces an isomorphism $Y/\Gamma \cong X$, by Lemma [4.1.7]. Moreover, for any subgroup $\Gamma' \subseteq \text{Hom}(G, R^\times)$, which is of the form $\text{Hom}(G', R^\times)$ for some quotient $G'$ of $G$, we have $(R(Q))^\Gamma = R(Q')$ and $(R(Q'^{\text{gp}}))^\Gamma = R(Q'^{\text{gp}})$, where $Q'$ is the preimage of $\ker(G \to \Gamma)$ under the canonical homomorphism $Q \to G = Q^{\text{gp}}/u^{\#}(P^{\text{gp}})$, so that $Y/\Gamma' \cong X \times_X (P^\times Q') \to X$ is a finite Kummer étale cover. Consequently, $f : Y \to X$ is a Galois finite Kummer étale cover with Galois group $\Gamma$, as desired.
Definition 4.1.8. Kummer (resp. Kummer étale) covers \( f : Y \to X \) as in Proposition 4.1.6 are called standard Kummer (resp. standard Kummer étale) covers.

Corollary 4.1.9. Kummer étale morphisms are open.

Proof. By definition, Kummer étale morphisms are locally compositions of standard Kummer étale covers and strictly étale morphisms, both of which are open. □

In the remainder of this subsection, let us study some general properties of Kummer étale morphisms. Our goal is to introduce the Kummer étale site.

Lemma 4.1.10. Let \( f : Y \to X \) be a Kummer étale morphism of locally noetherian fs log adic spaces. Suppose that \( X \) is modeled on an fs monoid \( P \). Then \( f \) admits, étale locally on \( X \) and \( Y \), a Kummer chart \( P \to Q \) as in Definition 4.1.2. Moreover, if \( P \) is torsionfree (resp. sharp), then we can choose \( Q \) to be torsionfree (resp. sharp).

Proof. Étale locally on \( Y \) and \( X \), let \( P_1 \to Q_1 \) be a Kummer chart of \( f \) as in Definition 4.1.2. As in Proposition 3.1.4, by localization, we may assume that \( \overline{P}_1 \cong \overline{M}_{X,\overline{x}} \) and \( \overline{Q}_1 \cong \overline{M}_{Y,\overline{y}} \), for geometric points \( \overline{y} \) of \( Y \) and \( \overline{x} = f(\overline{y}) \) of \( X \). By the same argument as in [Niz08, Lem. 2.8], we can find a group \( H \) as in the diagram (3.1.5), and we let \( Q \) denote the preimage of \( \overline{M}_{Y,\overline{y}} \) under \( H \to \overline{M}_{Y,\overline{y}} \). Then \( u : P \to Q \) is also a Kummer chart of \( f \). By the last assertion of Proposition 3.1.4 if \( P \) is torsionfree, then we may assume that \( Q \) is torsionfree as well. Finally, suppose that \( P \) is sharp and that \( Q \) is chosen to be torsionfree. For any \( q \in Q^* \), there is some \( n \geq 1 \) such that \( nq \) and \( -nq \) are both in \( u(P) \) and hence in \( u(P^*) = \{0\} \). Since \( Q \) is torsionfree, this forces \( q = 0 \). Thus, \( Q \) is also sharp, as desired. □

Lemma 4.1.11. Let \( f : Y \to X \) be a Kummer étale morphism of locally noetherian fs log adic spaces. For any geometric points \( \overline{y} \) of \( Y \) and \( \overline{x} = f(\overline{y}) \) of \( X \), the induced homomorphism of sharp fs monoids \( \overline{M}_{X,\overline{x}} \to \overline{M}_{Y,\overline{y}} \) is exact, and is Kummer with cokernel annihilated by some positive integer invertible in \( \mathcal{O}_Y \).

Proof. By Proposition 2.3.13, \( X \) admits, étale locally at \( \overline{x} \), a chart modeled on \( P = \overline{M}_{X,\overline{x}} \). By Lemma 4.1.10, \( f \) admits, étale locally at \( \overline{x} \) and any \( \overline{y} \) such that \( \overline{x} = f(\overline{y}) \), a Kummer chart \( u : P \to Q \) as in Definition 4.1.2 with a sharp \( Q \). Since \( u \) is exact and \( \overline{M}_{Y,\overline{y}} \) is a quotient of \( Q \), by [Ogu18, Prop. 1.4.2.1], \( v : \overline{M}_{X,\overline{x}} \to \overline{M}_{Y,\overline{y}} \) is also exact. This forces \( v \) to be injective because \( \overline{M}_{X,\overline{x}} \) is sharp, and it satisfies the conditions in Definition 4.1.11 as \( u \) does. In particular, \( v \) is Kummer, as desired. □

Definition 4.1.12. In Lemma 4.1.11, the ramification index of \( f \) at \( \overline{y} \) is defined to be the smallest positive integer \( n \) that annihilates the cokernel of \( \overline{M}_{X,f(\overline{y})} \to \overline{M}_{Y,\overline{y}} \). The ramification index of a Kummer étale morphism \( f \) is the least common multiple of the ramification indices among the geometric points \( \overline{y} \) of \( Y \). The ramification index of a Kummer étale morphism is 1 if and only if \( f \) is strictly étale.

Lemma 4.1.13. A morphism \( f : Y \to X \) of locally noetherian fs log adic spaces is Kummer étale if and only if it is log étale and Kummer, and if and only if it is log étale and exact. It is finite Kummer étale if and only if it is log étale and finite Kummer.

Proof. If \( f \) is Kummer étale (resp. finite Kummer étale), then it is log étale and Kummer (resp. finite Kummer) by definition, and it is exact by Lemma 4.1.11 and Remark 2.2.5. Conversely, assume that \( f \) is log étale and exact. By the same
argument as in the proof of Proposition 3.1.4, \( f \) admits \( \text{étale} \) locally at geometric points \( \mathfrak{p} \) of \( Y \) and \( \mathfrak{p} = f(\mathfrak{m}) \) of \( X \), an injective \( \text{fs} \) chart \( u : P \to Q \) such that \( P \cong \mathcal{M}_{X, \mathfrak{p}} \) and \( Q \cong \mathcal{M}_{Y, \mathfrak{p}} \), and such that \( n := |\text{coker}(u_{\text{gp}})| \) is invertible in \( \mathcal{O}_Y \).

Since \( f \) is exact, \( \pi : P \to Q \) is exact. By \cite{Ogu18} Prop. I.4.2.1, \( u \) is also exact. Thus, for each \( q \in Q \), we have \( nq \in Q \cap u_{\text{gp}}(P_{\text{gp}}) = u(P) \) in \( Q_{\text{gp}} \). This shows that \( u : P \to Q \) is a Kummer chart, as desired. Alternatively, if \( f \) is log \( \text{étale} \) and finite Kummer, then any finite Kummer chart \( u : P \to Q \) is necessarily a finite Kummer \( \text{étale} \) chart, by comparing the induced morphism between geometric stalks with the one given by any chart as in Definition 3.1.1 and so \( f \) is finite Kummer \( \text{étale} \).

**Proposition 4.1.14.** Kummer \( \text{étale} \) (resp. finite Kummer \( \text{étale} \)) morphisms as in Definition 4.1.2 are stable under compositions and base changes under arbitrary morphisms between locally noetherian \( \text{fs} \) log adic spaces (which are justified by Remark 3.1.2 and Proposition 3.1.3).

**Proof.** The stability under compositions follows from Proposition 3.1.6 and Lemma 4.1.13. As for the stability under base changes, it suffices to note that, if \( P \to Q \) is a Kummer homomorphism (of \( \text{fs} \) monoids), and if \( P \to R \) is any homomorphism between \( \text{fs} \) monoids, then the induced homomorphism \( R \to (R \oplus P)_{\text{sat}} \) is also Kummer, because it is injective as the composition \( R \to (R \oplus P)_{\text{sat}} \to R_{\text{gp}} \oplus P_{\text{gp}} \) \( Q_{\text{gp}} \) is, and because it satisfies the conditions in Definition 4.1.1 as \( P \to Q \) does.

**Proposition 4.1.15.** Let \( f : Y \to X \) and \( g : Z \to X \) be two Kummer \( \text{étale} \) morphisms of locally noetherian \( \text{fs} \) log adic spaces. Then any morphism \( h : Y \to Z \) such that \( f = g \circ h \) is also Kummer \( \text{étale} \).

**Proof.** By Lemma 4.1.13 it suffices to show that \( h \) is log \( \text{étale} \) and exact. By Lemma 4.1.11, \( \text{étale} \) locally at each geometric point \( \mathfrak{p} \) of \( Y \), with \( \mathfrak{p} = f(\mathfrak{m}) \) and \( \pi = h(\mathfrak{m}) \), both \( f \) and \( g \) admit Kummer charts \( u : P \to Q \) and \( v : P \to R \) such that \( P \cong \mathcal{M}_{X, \mathfrak{m}} \), such that \( Q \) and \( R \) are sharp, and such that \( \text{coker}(u_{\text{gp}}) \) and \( \text{coker}(v_{\text{gp}}) \) are annihilated by some integer \( n \) invertible in \( \mathcal{O}_Y \). Then \( \pi \) and \( \mathfrak{m} \) are identified with the Kummer homomorphisms \( \mathcal{M}_{X, f(\mathfrak{m}) \to Y, \pi} \) and \( \mathcal{M}_{X, \mathfrak{m} \to Y, \pi} \) induced by \( f \) and \( g \), respectively. Consequently, the homomorphism \( \mathcal{M}_{Z, \mathfrak{p} \to Y, \pi} \) induced by \( h \) is also Kummer, and hence exact. Hence, \( h \) is exact, by Remark 4.1.4. Since \( P \cong \mathcal{M}_{X, \mathfrak{m}} \), the kernels of \( Q_{\text{gp}} \to \mathcal{M}_{Y, \pi} \) and \( R_{\text{gp}} \to \mathcal{M}_{Z, \pi} \) are also annihilated by \( n \). As in the proof of Proposition 2.3.32, \( h \) admits a chart \( w : R \to Q' \) such that \( (Q \oplus R)_Y \to \mathcal{M}_Y \) factors through \( Q'_Y \to \mathcal{M}_Y \). Since the argument there is based on the explicit quotient construction in the proof of Lemma 2.3.6, we may assume that the kernel \( (Q')_{\text{gp}} \to \mathcal{M}_{Y, \pi} \) is also annihilated by \( n \). Since \( n \) annihilates \( \text{coker}(w_{\text{gp}}) \) because \( \pi \) can be identified with the above \( \mathcal{M}_{Z, \mathfrak{p} \to Y, \pi} \), the kernel and cokernel of \( w_{\text{gp}} \) are both annihilated by \( n \). This shows that \( h \) is log \( \text{étale} \), as desired.

By Proposition 2.3.32 and Remark 4.1.4 and by Propositions 4.1.14 and 4.1.15, we are now ready for the following:

**Definition 4.1.16.** Let \( X \) be a locally noetherian \( \text{fs} \) log adic space. The Kummer \( \text{étale site} \) \( X_{\text{kéti}} \) has as underlying category the full subcategory of the category of locally noetherian \( \text{fs} \) log adic spaces consisting of objects that are Kummer \( \text{étale} \) over \( X \), and has coverings given by the topological coverings.

**Remark 4.1.17.** Let \( X \) be as in Definition 4.1.16.
(1) For each \( U \in X_{\text{ét}} \), we can view \( U \) as a log adic space by restricting the log structure \( \alpha : \mathcal{M}_X \to \mathcal{O}_{X_{\text{ét}}} \) to \( U_{\text{ét}} \). This gives rise to a strictly étale morphism \( U \to X \) of log adic spaces, which is Kummer étale by definition. Therefore, we obtain a natural projection of sites \( \mathcal{E}_X \to X_{\text{ét}} \), which is an isomorphism when the log structure on \( X \) is trivial.

(2) For any morphism \( f : Y \to X \) of locally noetherian fs log adic spaces, we have a natural morphism of sites \( f_{k\text{ét}} : Y_{k\text{ét}} \to X_{k\text{ét}} \), because base changes of Kummer étale morphisms are still Kummer étale, by Proposition 4.1.14.

**Remark 4.1.18.** By definition, the Kummer étale topology on \( X \) is generated by surjective (strictly) étale morphisms and standard Kummer étale covers.

### 4.2. Abhyankar’s lemma

An important class of finite Kummer étale covers arise in the following way:

**Proposition 4.2.1** (rigid Abhyankar’s lemma). Let \( X \) be a smooth rigid analytic variety over a nonarchimedean field \( k \) of characteristic zero, and let \( D \) be a normal crossings divisor of \( X \). We equip \( X \) with the fs log structure induced by \( D \) as in Example 2.3.17. Suppose that \( h : V \to U := X - D \) is a finite étale surjective morphism of rigid analytic varieties over \( k \). Then it extends to a finite surjective and Kummer étale morphism of log adic spaces \( f : Y \to X \), where \( Y \) is a normal rigid analytic variety with its log structures defined by the preimage of \( D \).

**Proof.** By [Han19, Thm. 1.6] (which was based on [Lüt93, Thm. 3.1 and its proof]), \( h : V \to U \) extends to a finite ramified cover \( f : Y \to X \), for some normal rigid analytic variety \( Y \) (viewed as a noetherian adic space). Just as \( X \) is equipped with the log structure defined by \( D \), we equip \( Y \) with the log structure defined by the preimage of \( D \). The question is whether the map \( f \) is Kummer étale (with respect to the log structures on \( X \) and \( Y \)), and this question is local in nature in the analytic topology of \( X \). As in Example 2.3.17, we may assume that there is an affinoid smooth rigid analytic variety \( S \) over \( k \) such that \( X = S \times \mathcal{D}^r \cong S(\mathbb{Z}_{>0}) \) (see Example 2.2.19) for some \( r \in \mathbb{Z}_{>0} \), with \( D = S \times \{ T_1 \cdots T_e = 0 \} \). Thus, we can conclude the proof by combining the following Lemmas 4.2.2 and 4.2.3. □

For simplicity, let us introduce some notation for following two lemmas. We write \( P := \mathbb{Z}_{>0} \) and identify \( \mathcal{D}^r \) with \( \text{Spa}(k(P), k^+(P)) \) as in Example 2.2.21. For each \( m \in \mathbb{Z}_{\geq 1} \), we also write \( \frac{1}{m} P = \frac{1}{m} \mathbb{Z}_{\geq 0} \). For each power \( \rho \) of \( p \), we denote by \( \mathbb{D}_\rho \) the (one-dimensional) disc of radius \( \rho \), so that \( \mathbb{D} = \mathbb{D}_1 \) when \( \rho = 1 \). We also denote by \( \mathbb{D}_\rho^\times \) the punctured disc of radius \( \rho \), and by \( \mathbb{D}^\times \) the punctured unit disc.

For any rigid analytic variety with a canonical morphism to \( \mathbb{D}^r \), we denote with a subscript “\( \rho \)” (resp. “\( \times \)”) its pullback under \( \mathbb{D}_\rho^r \to \mathbb{D}^r \) (resp. \( (\mathbb{D}^\times)^r \to \mathbb{D}^r \)).

**Lemma 4.2.2.** Suppose that \( X = S \times \mathcal{D}^r \cong S(P) \), \( D \), and \( U = X - D \cong S(P)^\times \) are as in the proof of Proposition 4.2.1. Assume there is some \( \rho \leq 1 \) such that, for each connected component \( Y' \) of \( Y_\rho \), there exist \( d_1, \ldots, d_e \in \mathbb{Z}_{\geq 1} \) such that induced cover \( Y' \to X' := X_\rho \) is refined (i.e., admits a further cover) by some finite ramified cover \( Z := S(P')_{\rho'} \to X' \), where \( P' = \oplus_{1 \leq i \leq r} (\frac{1}{m} \mathbb{Z}_{\geq 0}) \). Then, up to replacing \( k \) with a finite extension, we have \( Y' \cong S(Q)_\rho \) for some sharp fs monoid \( Q \) such that \( P \subset Q \subset P' \). Consequently, the morphism \( Y_\rho \to X' = X_\rho \) is finite Kummer étale. Moreover, if \( mQ \subset P \) for some \( m \in \mathbb{Z}_{\geq 1} \), and if \( X^{\frac{1}{m}} := S(\frac{1}{m} P) \), then
the finite (a priori ramified) cover $Y \times_X X_{P}^{\frac{1}{m}} \to X_{P}^{\frac{1}{m}}$ splits completely (i.e., the source is a disjoint union of sections) and is hence strictly étale.

Proof. Let $Y'$ (resp. $W$) be the preimage of $U' := U_{\rho} \cong S(P)_{\rho}^{\times}$ in $Y'$ (resp. $Z$). Up to replacing $k$ with a finite extension containing all $d_{j}$-th roots of unity for all $j$, the finite étale cover $W \to U'$ is Galois with Galois group $G' := \text{Hom}((P')^{\text{gp}}/P^{\text{gp}}, k^{\times})$ (see Proposition 4.2.2), and $Y'$ is (by the usual arguments, as in [Gro71, V]) the quotient of $W$ by some subgroup $G$ of $G'$ (as in Lemma 4.1.7), which is isomorphic to $S(Q)_{\rho}^{\times}$ for some monoid $Q$ such that $P \subset Q \subset P'$ and $Q = Q^{\text{gp}} \cap P'$. These conditions imply that $Q$ is toric, and hence $S(Q)$ is normal because $\text{Spa}(k(Q), k^{+}(Q))$ is (see Example 2.2.20 and the references given there). Since $Y'$ and $S(Q)_{\rho}$ are both normal and are both finite ramified covers of $X'$ extending the same finite étale cover $V'$ of $U'$, they are canonically isomorphic by [Han19, Thm. 1.6], as desired. Finally, for the last assertion of the lemma, it suffices to note that, for any $Q$ as above satisfying $mQ \subset P$, up to replacing $k$ with a finite extension containing all $m$-th roots of unity, the connected components of the finite étale cover refining $Y' \to S$ are all of the form $S(\frac{1}{m}P)^{\text{sat}}$, where $(Q \oplus P(\frac{1}{m}P))^{\text{sat}}$ is the product of $\frac{1}{m}P$ with a finite group annihilated by $m$. 

Lemma 4.2.3. The (cover-refinement) assumption in Lemma 4.2.2 holds up to replacing $k$ with a finite extension and $S$ with a strictly finite étale cover and we may assume that the positive integers $d_{1}, \ldots, d_{r}$, there (for various $Y'$) are no greater than the degree $d$ of $f : Y \to X$. Moreover, we can take $\rho = p^{-b(d,p)}$, where $b(d,p)$ is defined as in [Lüt93, Thm. 2.2], which depends on $d$ and $p$ but not on $r$; and we can take $m = d!$ in the last assertion of Lemma 4.2.2.

Proof. We shall proceed by induction on $r$. When $r = 0$, the assumption in Lemma 4.2.2 means, for each connected component $Y'$ of $Y$, the strictly étale cover $Y' \to X = S$ splits completely. This can always be achieved up to replacing $S$ with a Galois strictly finite étale cover refining $Y' \to S$ for all $Y'$.

In the remainder of this proof, suppose that $r \geq 1$, and that the lemma has been proved for all strictly smaller $r$. Let $\rho = p^{-b(d,p)}$ be as above. Fix some $a \in k$ such that $|a| = \rho$. We shall denote normalizations of fiber products by $\tilde{\times}$ (instead of $\times$).

Consider the submonoid $P_{1} := \mathbb{Z}_{>0}^{a} \oplus \{0\}$ of $P = \mathbb{Z}_{>0}^{a}$. Let $X_{1} := S \times \mathbb{D}_{>1}^{a} \cong S(P_{1})$, which we identify with the subspace $S \times \mathbb{D}_{>1}^{a} \times \{a\}$ of $X = S \times \mathbb{D}_{>1} \cong S(P)$. Let $Y_{1} := Y \times_{X} X_{1}$. Note that the degree of $Y_{1} \to X_{1}$ is also $d$. By induction, up to replacing $k$ with a finite extension and $S$ with a strictly finite étale cover, for each connected component $Y'_{1}$ of $(Y_{1})_{\rho}$, there exist $1 \leq d_{1}, \ldots, d_{r-1} \leq d$ such that the induced finite ramified cover $Y'_{1} \to (X_{1})_{\rho}$ is refined by $S(P'_{1})_{\rho} \to (X_{1})_{\rho}$, where $P'_{1} := \oplus_{1 \leq i \leq r-1}(\frac{1}{d_{i}}\mathbb{Z}_{>0})$. Let $X_{1}^{\frac{1}{d_{i}}} := S^{\frac{1}{d_{i}}}(P_{1})$, with $\frac{1}{d_{i}}P_{1} = \frac{1}{d_{i}}\mathbb{Z}_{>0}$.

Let $\tilde{X} := X_{1}^{\frac{1}{d_{i}}} \times_{X} X \cong X_{1}^{\frac{1}{d_{i}}} \times \mathbb{D}$ and $\tilde{Y} := X_{1}^{\frac{1}{d_{i}}} \times_{X} Y$, and let $\tilde{f} : \tilde{Y} \to \tilde{X}$ denote the induced finite ramified cover, which is also of degree $d$. Then the (strictly finite étale) pullback of $\tilde{f}$ to $(X_{1}^{\frac{1}{d_{i}}})^{\times} \times \{a\}$ can be identified with the pullback of $Y_{1} \to X_{1}$ to $(X_{1}^{\frac{1}{d_{i}}})^{\times}$, which splits completely, by the inductive hypothesis and the last assertion in Lemma 4.2.2. Hence, since $\rho = p^{-b(d,p)}$, for each connected component $Y'$ of $Y_{1}^{\times}$, by applying [Lüt93, Lem. 3.2] to the morphism $(Y')^{\times} \to \tilde{X}_{\rho}^{\times}$ induced by $\tilde{f}$, we obtain a rigid analytic function $\tilde{T}$ on $(Y')^{\times}$ such that $\tilde{T}^{d_{i}} = T_{r}$, where $T_{r}$ is the coordinate on the $r$-th factor of $\mathbb{D}^{r}$. By [Bar76, Sec. 3] (see also [Han19, Thm.
2.6), \( \tilde{T} \) extends to a rigid analytic function on the normal \( \tilde{Y} \), which still satisfies 
\( \tilde{T}^{d_r} = T_r \). Therefore, we can view \( \tilde{T} \) as \( T_r^{\tilde{P}} \), and \( \tilde{Y} \rightarrow \tilde{X} \cong (X_1^n \times D)_{/\rho} \) factors through \( S(\tilde{P})_{/\rho} \rightarrow (X_1^n \times D)_{/\rho} \), where \( \tilde{P} := (\frac{1}{n} P_1) \oplus (\frac{1}{d_r} \mathbb{Z}_{\geq 0}) \). Since these are finite ramified covers of the same degree \( d_r \) from connected and normal rigid analytic varieties, we obtain an induced isomorphism 
\( \tilde{Y} \cong S(\tilde{P})_{/\rho} \).

Since each connected component \( Y' \) of \( Y_{/\rho} \) is covered by some connected component \( \tilde{Y}' \) of \( \tilde{Y}_{/\rho} \), by Lemma 4.2.2 up to replacing \( k \) with a finite extension, \( Y' \cong S(Q)_{/\rho} \) for some monoid \( Q \) satisfying \( P \subset Q \subset \tilde{P} = (\frac{1}{n} P_1) \oplus (\frac{1}{d_r} \mathbb{Z}_{\geq 0}) \), for some \( 1 \leq d_r \leq d \) and \( \tilde{P} \) determined by \( \tilde{Y}' \) as above. By the construction of \( \tilde{Y}' \), the monoid \( (\frac{1}{n} P_1) \oplus (\frac{1}{d_r} \mathbb{Z}_{\geq 0}) \) sat is the product of \( \tilde{P} \) with a finite group, which forces \( \{0\}^{r-1} \oplus (\frac{1}{d_r} \mathbb{Z}_{\geq 0}) \subset Q \). By the construction of \( Y' \) and the induction hypothesis, the projection from \( \tilde{P} \) to \( \frac{1}{n} P_1 \) maps \( Q \) into \( \oplus_{1 \leq i \leq r-1} (\frac{1}{d_r} \mathbb{Z}_{\geq 0}) \) for some \( 1 \leq d_1, \ldots, d_r-1 \leq d \). Thus, \( P \subset Q \subset P' := \oplus_{1 \leq i \leq r-1} (\frac{1}{d_r} \mathbb{Z}_{\geq 0}) \), as desired.

**Remark 4.2.4.** Proposition 4.2.1 can be regarded as the Abhyankar’s lemma (cf. [Gro71, XIII, 5.2]) in the rigid analytic setting, because of the last assertion in Lemma 4.2.2.

More generally, we have the following basic but useful facts:

**Lemma 4.2.5.** Let \( X \) be a noetherian fs log adic space modeled on a sharp fs monoid \( P \), and let \( f : Y \rightarrow X \) be a Kummer étale (resp. finite Kummer étale) morphism. Then \( Y \times_X X^{\hat{\hat{\pi}}} \rightarrow X^{\hat{\hat{\pi}}} \) is étale (resp. finite étale) for some positive integer \( n \). If \( X \) is defined over \( \text{Spa}(k, k^+) \), where \( (k, k^+) \) is an affinoid field, then we can take \( n \) to be invertible in \( k \).

**Proof.** Since \( X \) is noetherian, by taking the least common multiple of the positive integers obtained on finitely many members in an étale covering, it suffices to work étale locally on \( X \). By Lemma 4.1.10 up to étale localization on \( X \), there exists an étale covering \( \{Y_i \rightarrow X\}_{i \in I} \) indexed by a finite set \( I \) such that each Kummer étale morphism \( Y_i \rightarrow X \) admits a Kummer chart \( P \rightarrow Q_i \) with a sharp \( Q_i \). Then there exists some positive integer \( n \), which we may assume to be invertible in \( k \) when \( X \) is defined over \( \text{Spa}(k, k^+) \), such that \( P \rightarrow \frac{1}{n} P \) factors as \( P \rightarrow Q_i \rightarrow \frac{1}{n} P \) for some injective \( u_i \), for all \( i \in I \). Hence, the induced morphism \( Y_i \times_X X^{\hat{\hat{\pi}}} \rightarrow X^{\hat{\hat{\pi}}} \) is étale, for each \( i \in I \), because it admits a Kummer chart \( \frac{1}{n} P \rightarrow (Q_i \oplus_{/P} (\frac{1}{n} P)) \) sat \( \cong ((Q_i \oplus_{/P} Q_i) \oplus_{/Q_i} (\frac{1}{n} P)) \) sat \( \cong G_i \oplus (\frac{1}{n} P) \), where \( G_i := (Q_i)_{/\text{gp}}/u_i^{\text{gp}}(P^{\text{gp}}) \) has order invertible in \( O_{Y_i} \) by assumption. By étale descent, \( Y \times_X X^{\hat{\hat{\pi}}} \) is also étale.

**Lemma 4.2.6.** Let \( X \) be a noetherian fs log adic space modeled on a sharp fs monoid \( P \). Let \( \{U_i \rightarrow X\}_{i \in I} \) be a Kummer étale covering indexed by a finite set \( I \). Then there exists a Kummer étale covering \( \{V_j \rightarrow X\}_{j \in J} \) indexed by a finite set \( J \) refining \( \{U_i \rightarrow X\}_{i \in I} \) such that each \( V_j \rightarrow X \) admits a chart \( P \rightarrow \frac{1}{n_j} P \) for some integer \( n_j \) invertible in \( O_{V_j} \), and such that \( \{V_j \times_X X^{\hat{\hat{\pi}}} \rightarrow X^{\hat{\hat{\pi}}}\}_{j \in J} \) is an étale covering of \( X^{\hat{\hat{\pi}}} \) for some integer \( n \) divided by all \( n_j \). If \( X \) is defined over \( \text{Spa}(k, k^+) \), where \( (k, k^+) \) is an affinoid field, then we can take all \( n_j \) and \( n \) to be the same and invertible in \( k \).

**Proof.** Since \( X \) is noetherian, by Lemma 4.1.10 we may replace \( \{U_i \rightarrow X\}_{i \in I} \) with a finite refinement \( \{U_j \rightarrow X\}_{j \in J} \) such that each \( U_j \rightarrow X \) admits a Kummer chart
$P \to Q_j$ with a sharp $Q_j$. Then there exists some positive integer $n_j$ invertible in $O_{U_j}$ such that $P \to \frac{1}{n_j}P$ factors as $P \to Q_j \to \frac{1}{n_j}P$ for some injective $u_j$, for each $j \in J$. Therefore, each $V_j := U_j \times_{U_j(Q_j)} U_j(\frac{1}{n_j}P) \to X$ is Kummer étale with a Kummer chart $P \to \frac{1}{n_j}P$, and the induced morphism $V_j \to X^{\frac{1}{n_j}} = X \times_{X(P)} X(\frac{1}{n_j}P)$ is étale (as in Definition 4.1.2). In this case, $V_j \times_X X^{\frac{1}{n_j}} \to X^{\frac{1}{n_j}}$ is also étale (cf. the proof of Lemma 4.2.5). Since $\prod_{j \in J} U_j \to X$ is surjective by assumption, $\{V_j \times_X X^{\frac{1}{n_j}}\}_{j \in J} \to X^{\frac{1}{n_j}}$ is an étale covering of $X^{\frac{1}{n_j}}$, as desired. □

4.3. Coherent sheaves. In this subsection, we show that, when $X$ is a locally noetherian fs log adic space, the presheaf $\mathcal{O}_{X_{\text{két}}}$ (resp. $\mathcal{O}^{+}_{X_{\text{két}}}$) on $X_{\text{két}}$ defined by $U \mapsto \mathcal{O}_{U}(U)$ (resp. $U \mapsto \mathcal{O}^{+}_{U}(U)$) is indeed a sheaf, generalizing a well-known unpublished result of Kato’s [Kat91] for log schemes. We also study some problems related to the Kummer étale descent of coherent sheaves.

**Theorem 4.3.1.** Let $X$ be a locally noetherian fs log adic space.

- (1) The presheaves $\mathcal{O}_{X_{\text{két}}}$ and $\mathcal{O}^{+}_{X_{\text{két}}}$ are sheaves.
- (2) If $X$ is affinoid, then $H^i(X_{\text{két}}, \mathcal{O}_{X_{\text{két}}}) = 0$, for all $i > 0$.

A key input is the following:

**Lemma 4.3.2.** Let $X$ be an affinoid noetherian fs log adic spaces, endowed with a chart modeled on a sharp fs monoid $P$. Let $Y \to X$ be a standard Kummer cover (see Definition 4.1.8). Then the Čech complex $C^*(Y/X) : 0 \to \mathcal{O}(X) \to \mathcal{O}(Y) \to \mathcal{O}(Y \times_X Y) \to \cdots$ (where we omit the subscripts of the structure sheaf $\mathcal{O}$ for simplicity) is exact.

**Proof.** This is essentially [Niz08 Lem. 3.28], based on the idea in [Kat91 Lem. 3.4.1]. Suppose that $Y \to X = \text{Spa}(R, R^+)$ is associated with a Kummer homomorphism $u : P \to Q$ as in Proposition 4.1.6. Then $C^*(Y/X)$ is already known to be exact at the first three terms, and $\mathcal{O}(Y \times_X Y \times_X \cdots \times_X Y) \cong \mathcal{O}(Y) \otimes_R R[G] \otimes_R R[G] \cdots \otimes_R R[G]$, where $G = Q^{\text{sp}}/u^{\text{sp}}(P^{\text{sp}})$, in which case we can write the differentials $C^*(Y/X)$ explicitly and construct a contracting homotopy for $C^*(Y/X)$, by the same argument as in the proof of [Niz08 Lem. 3.28]. □

We emphasize that Lemma 4.3.2 also works for standard Kummer covers that are not necessarily Kummer étale.

**Proof of Theorem 4.3.1.** (1) It suffices to prove the statement for $\mathcal{O}_{X_{\text{két}}}$. Since the sheafiness for the étale topology is known for all locally noetherian adic spaces, by Lemma 4.2.6 the statement is reduced to Lemma 4.3.2.

(2) By Propositions 2.3.13 and A.10 we may reduce to the case where $X$ is affinoid with a global sharp fs chart $P$. By Lemma 4.2.6 any Kummer étale covering $\{U_i \to X\}_{i \in I}$ of $X$ admits some refinement $\{V_j \to X\}_{j \in J}$ as finite Kummer étale covering such that $\{V_j \times_X X^{\frac{1}{n_j}} \to X^{\frac{1}{n_j}}\}_{j \in J}$ is an étale covering, for some $m$, and such that each $V_j \times_X X^{\frac{1}{n_j}} \to V_j$ is a composition of étale morphisms and standard étale covers. Thus, by Lemma 4.3.2 the Čech complex $\mathcal{O}(X) \to \mathcal{O}(X^{\frac{1}{n_j}}) \to \mathcal{O}(X^{\frac{1}{n_j}} \times_X X^{\frac{1}{n_j}}) \to \cdots$ is exact. Then by Proposition A.10 the Čech complex $\mathcal{O}(X) \to \oplus_{j} \mathcal{O}(V_j) \to \oplus_{j,j'} \mathcal{O}(V_j \times_X V_{j'}) \to \cdots$ is also exact, as desired. □
Corollary 4.3.3. Let $X$ be a locally noetherian fs log adic space. Let $\epsilon_{an} : X_{k\acute{e}t} \to X_{an}$ and $\epsilon_{\acute{e}t} : X_{k\acute{e}t} \to X_{\acute{e}t}$ be the natural projections of sites. Then we have canonical isomorphisms $\mathcal{O}_{X_{an}} \cong R_{an,s}(\mathcal{O}_{X_{k\acute{e}t}})$ and $\mathcal{O}_{X_{\acute{e}t}} \cong R_{\acute{e}t,s}(\mathcal{O}_{X_{k\acute{e}t}})$. As a result, the pullback functor from the category of vector bundles on $X_{an}$ (resp. $X_{\acute{e}t}$) to the category of $\mathcal{O}_{X_{k\acute{e}t}}$-modules is fully faithful (cf. Proposition [A.10]).

Proposition 4.3.4. Let $X$ be a locally noetherian fs log adic space. Then the presheaf $\mathcal{M}_{X_{k\acute{e}t}}$ assigning $U \mapsto \mathcal{M}_{U}(U)$ is a sheaf on $X_{k\acute{e}t}$.

Proof. The proof is similar to [Kat91] Lem. 3.5.1. Since $\mathcal{M}_{X}$ is already a sheaf on the étale topology, by replacing $X$ with its strict localization at a geometric point $\mathfrak{p}$, it suffices to show the exactness of $0 \to \mathcal{M}_{X}(X) \to \mathcal{M}_{Y}(Y) = \mathcal{M}_{Y \times_{X} Y}(Y \times_{X} Y)$, where $X = \text{Spa}(R, R^{+})$ admits a chart modeled on a sharp fs monoid $P \cong \mathcal{M}_{X_{k\acute{e}t}}$, for some strictly local ring $R$ (see Proposition [2.3.13]), and where $Y \to X$ is a standard Kummer étale cover with a Kummer chart $u : P \to Q$ with a sharp $Q$ such that the order of $G := \text{coker}(u^{gp})$ is invertible in $R$ (see Corollary [4.11.10]).

Let $R^{+} := \mathcal{O}^{	imes}_{Y}(Y)$ and $R^{0} := \mathcal{O}^{	imes}_{Y \times_{X} Y}(Y \times_{X} Y)$. By definition, $R^{0} \cong R \otimes_{R_{P}} R(Q)$, where $R(P) \to R$ and $R(P) \to R(Q)$ are given by the charts. By Proposition [4.4.6], $R^{0} \cong R \otimes_{R_{P}} R((Q \otimes_{P} Q)^{\text{gp}}) \cong R \otimes_{R_{P}} R(Q \otimes G)$. Let $I$ (resp. $I'$, resp. $I''$) be the ideal of $R$ (resp. $R'$, resp. $R''$) generated by the image of $P \setminus \{0\}$ (resp. $Q \setminus \{0\}$, resp. $Q \setminus \{0\}$). Let $V$ (resp. $V'$, resp. $V''$) be the subgroup of elements in $R^{0}$ (resp. $(R')^{0}$, resp. $(R'')^{0}$) congruent to $1$ modulo $I$ (resp. $I'$, resp. $I''$). By Lemma [4.3.2] 0 $\to$ $R$ $\to$ $R' \cong R''$ and hence 0 $\to$ $R^{0}$ $\to$ $(R')^{0}$ $\cong (R'')^{0}$ are exact. This implies the exactness of 0 $\to$ $V$ $\to$ $V'$ $\to$ $V''$, since $R^{0}/V \cong R/I \cong R'/I' \cong (R')^{0}/V'$. By some diagram chasing, it suffices to show that

$$0 \to \mathcal{M}_{X}(X)/V \to \mathcal{M}_{Y}(Y)/V' \cong \mathcal{M}_{Y \times_{X} Y}(Y \times_{X} Y)/V''$$

is exact. By construction, this can be identified with

$$0 \to ((R/I)^{0} \oplus P \to (R/I)^{0} \cong ((R/I[V])^{0} \oplus Q,$$

where the double arrows are $(x, q) \to (x, q)$ and $(x, q) \to (x \bar{q}, q)$, with $\bar{q}$ denoting the image of $q$ in $G = Q^{gp}/u^{gp}(P^{gp})$, which is exact because $u : P \to Q$ is exact. □

As a byproduct, let us show that representable presheaves are sheaves on $X_{k\acute{e}t}$. The log scheme version can be found in [Ill02] Thm. 2.6], which can be further traced back to [Kat91] Thm. 3.1.

Proposition 4.3.5. Let $Y \to X$ be a morphism of locally noetherian fs log adic spaces. Then the presheaf $\hom(\cdot, Y)$ on $X_{k\acute{e}t}$ is a sheaf.

Proof. We follow the idea of [Kat91] Thm. 3.1. It suffices to show that the presheaf $\hom(\cdot, Y)$ on $X_{k\acute{e}t}$ is a sheaf, because $\hom_{X}(\cdot, Y)$ is just the sub-presheaf of sections of $\hom_{X}(\cdot, Y)$ with compatible morphisms to $X$. We may assume that $Y = \text{Spa}(R, R^{+})$ is affinoid with a chart modeled on a sharp fs monoid $P$.

Consider the presheaves $\mathcal{F} : T \mapsto \hom((R, R^{+}), (\mathcal{O}_{T}(T), \mathcal{O}_{T}^{+}(T)))$, $\mathcal{G} : T \mapsto \hom(P, \mathcal{M}_{T}(T))$, and $\mathcal{H} : T \mapsto \hom(P, \mathcal{O}_{T}(T))$ on $X_{k\acute{e}t}$, where the first Hom is in the category of Huber pairs, and where the latter two are in the category of monoids. We claim that $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ are all sheaves on $X_{k\acute{e}t}$. As for $\mathcal{F}$, it suffices to show that $\mathcal{F}' : T \mapsto \hom_{cont}(R, \mathcal{O}_{T}(T))$ is a sheaf, where the homomorphisms are continuous ring homomorphisms; or that $\mathcal{F}'' : T \mapsto \hom(R, \mathcal{O}_{T}(T))$ is a sheaf, where we consider all ring homomorphisms. Choose any presentation $R \cong \mathbb{Z}[T_{i}]/(f_{j})_{j \in J}$,
so that \( F''(T) \equiv \ker(O_T(T)^I \to O_T(T)) \). Then \( F'' \) is a sheaf on \( X_{\text{ét}} \) as \( T \mapsto O_T(T) \) (see Theorem 4.3.1). As for \( G \) and \( H \), consider any presentation \( \mathbb{Z}_r^s \equiv \mathbb{Z}_{\geq 0}^s \to P \to 0 \) of the finitely generated monoid \( P \), which exists by [Ogu18] Lem. I.2.1.9. Then \( G(T) \) (resp. \( H(T) \)) is the equalizer of \( M_T(T)^s \equiv M_T(T)^f \) (resp. \( O_T(T)^s \equiv O_T(T)^f \)). Hence, both presheaves are sheaves on \( X_{\text{ét}} \) as \( T \mapsto M_T(T) \) and \( T \mapsto O_T(T) \) are (see Theorem 4.3.1 and Proposition 4.3.3).

Having established the claim, since \( \text{Hom}(\cdot, Y) \) is the fiber product of the morphisms \( F \to H \) and \( G \to H \) induced by \( P \to R \) and \( M_T(T) \to O_T(T) \), respectively, it is also a sheaf, as desired. \( \square \)

In the remainder of this subsection, we study coherent sheaves on the Kummer étale site.

**Definition 4.3.6.** Let \( X \) be a locally noetherian fs log adic space.

(1) An \( O_{X_{\text{ét}}} \)-module \( F \) is called an analytic coherent sheaf if it is isomorphic to the inverse image of a coherent sheaf on the analytic site of \( X \).

(2) An \( O_{X_{\text{ét}}} \)-module \( F \) is called a coherent sheaf if there exists a Kummer étale covering \( \{ U_i \to X \} \), such that each \( F|_{U_i} \) is an analytic coherent sheaf.

The following results are analogues of [Kat91] Prop. 6.5, the proof of which is completed in [Niz08] Prop. 3.27.

**Theorem 4.3.7.** Suppose that \( X \) is an affinoid noetherian fs log adic space. Then \( H^i(X_{\text{ét}}, F) = 0 \), for all \( i > 0 \), in the following two situations:

(1) \( F \) is an analytic coherent \( O_{X_{\text{ét}}} \)-module.

(2) \( F \) is a coherent \( O_{X_{\text{ét}}} \)-module, and \( X \) is over an affinoid field \((k, k^+)\).

**Proof.**

(1) As in the proof of Theorem 4.3.1 by Lemma 4.2.6 and Proposition A.10 it suffices to show the exactness of the Čech complex \( C^*_\mathbb{P}(Y/X) : 0 \to F(X) \to F(Y) \to F(Y \times_X Y) \to \cdots \), where \( X \) is affinoid with a sharp fs chart \( P \), and where \( Y \to X \) is a standard Kummer cover. By Proposition 4.1.6 the morphisms \( Y \to X \), \( Y \times_X Y \to X \), \( Y \times_X Y \times_X Y \to X \), etc are finite, and hence \( C^*_\mathbb{P}(Y/X) \cong C^*(Y/X) \otimes_{O_X} \mathbb{P}(X) \mathbb{P}(F)(X) \), where \( C^*(Y/X) \) is as in Lemma 4.3.2. Since the contracting homotopy used in the proof of Lemma 4.3.2 (based on the proof of [Niz08] Lem. 3.28) is \( O_X(X) \)-linear, \( C^*_\mathbb{P}(Y/X) \) is also exact, as desired.

(2) First assume that \( X \) is modeled on a sharp fs monoid \( P \). By definition, there exists a Kummer étale covering \( \{ U_i \to X \} \), such that each \( F|_{U_i} \) is analytic coherent. By Lemma 4.2.6 and Proposition A.10 we may assume that \( F|_{U_i} \) is analytic coherent, where \( U = X^+ \) for some \( n \) invertible in \( k \). Let \( G := (\mathbb{Z}^n)^{op}/\mathbb{Z}^n \). Since \( H^j((U \times_X \cdots \times_X U)_{\text{ét}}, F) = 0 \), for all \( j > 0 \), by [1], it suffices to show that \( H^i(C^*_\mathbb{P}(U/X)) = 0 \), for all \( i > 0 \). As in the proof of Lemma 4.3.2 by Proposition 4.1.6 \( O_{X_{\text{ét}}}(U \times_X \cdots \times_X U) \equiv O_{X_{\text{ét}}}(U) \otimes_k k[G] \otimes_k \cdots \otimes_k k[G] \), and we can identify the complex \( F(U) \to F(U \times_X U) \to F(U \times_X U \times_X U) \to \cdots \) with the complex computing the group cohomology \( H^i(G, F(U)) \). Since \( |G| \) is invertible in \( k \) as \( n \) is, and since \( F(U) \) is a \( k \)-vector space, we have \( H^i(G, F(U)) = 0 \), for all \( i > 0 \).

Let \( \varepsilon_{\text{ét}} : X_{\text{ét}} \to X_{\text{ét}} \) denote the natural projection of sites. Then the argument above shows that \( R^i\varepsilon_{\text{ét}}_* \equiv 0 \), for all \( j > 0 \), and that \( \varepsilon_{\text{ét}}_* \equiv (F) \) is a coherent sheaf on \( X_{\text{ét}} \). Since these statements are étale local in nature, they extend to all \( X \) considered in the statement of the theorem, by
Kummer étale descent of objects (coherent sheaves, log adic spaces, etc) are usually not effective, mainly because fiber products of Kummer étale covers do not correspond to fiber products of structure rings. Here is a standard counterexample.

**Example 4.3.8.** Let \( k \) be a nonarchimedean field. Consider the unit disk \( \mathbb{D} = \text{Spa}(k(T), O_k(T)) \) equipped with the log structure modeled on the chart \( \mathbb{Z}_{\geq 0} \to k<T> : 1 \to T \). By Proposition 4.1.6 we have a standard Kummer étale Galois cover \( f_n : \mathbb{D} \to \mathbb{D} \) corresponding to the chart \( \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0} : 1 \to n \), where \( n \) is invertible in \( k \), with Galois group \( \mu_n \). Then the ideal sheaf \( \mathcal{I} \) of the origin, a \( \mu_n \)-invariant invertible sheaf on \( \mathbb{D} \), does not descend via \( f_n \).

Kummer étale descent of morphisms are more satisfactory.

**Proposition 4.3.9.** Let \( X \) be a locally noetherian fs log adic space, and let \( f : Y \to X \) be a Kummer étale cover. Let \( \text{pr}_1, \text{pr}_2 : Y \times_X Y \to Y \) denote the two projections. Suppose that \( \mathcal{E} \) and \( \mathcal{F} \) are analytic coherent \( \mathcal{O}_{X_\kappa} \)-modules; and that \( g' : f'^*(\mathcal{E}) \to f'^*(\mathcal{F}) \) is a morphism on \( Y \) such that \( \text{pr}_1^*(g') = \text{pr}_2^*(g') \) on \( Y \times_X Y \). Then there exists a unique morphism \( g : \mathcal{E} \to \mathcal{F} \) such that \( f^*(g) = g' \).

**Proof.** By Lemma 4.2.6 and Proposition A.10 we may assume that \( X \) is affinoid and that \( Y \to X \) is a standard Kummer cover. We need to show that

\[
0 \to \text{Hom}_A(M,N) \to \text{Hom}_B(B \otimes_A M, B \otimes_A N) \to \text{Hom}_C(C \otimes_A M, C \otimes_A N)
\]

is exact, where \( A := O_X(X), B := O_X(X_{\kappa}), C := O_X(Y \times_X Y), M := \mathcal{E}(X), \) and \( N := \mathcal{F}(X) \), and where the third arrow is the difference between two pullbacks as usual. Equivalently, we need to show that

\[
0 \to \text{Hom}_A(M,N) \to \text{Hom}_A(M, B \otimes_A N) \to \text{Hom}_A(M, C \otimes_A N)
\]

is exact. By the left exactness of \( \text{Hom}_A(M, \cdot) \), we are reduced to showing that the sequence \( 0 \to N \to B \otimes_A N \to C \otimes_A N \) is exact. But this is just the first three terms in the complex \( C^*(Y/X) \) in the proof of Theorem 4.3.7[1].

To wrap up the subsection, let us introduce a convenient basis for the Kummer étale topology.

**Lemma 4.3.10.** Let \( X \) be a locally noetherian fs log adic space. Let \( \mathcal{B} \) be the full subcategory of \( X_{\kappa} \) consisting of affinoid adic spaces \( V \) with fs global charts. Then \( \mathcal{B} \) is a basis for \( X_{\kappa} \), and we have an isomorphism of topoi \( X_{\kappa} \sim \mathcal{B}^{\sim} \).

**Proof.** By [AGV73 III, 4.1], it suffices to show that every object in \( X_{\kappa} \) has a covering by objects in \( \mathcal{B} \). But this is clear.

**Lemma 4.3.11.** Let \( X \) be a locally noetherian fs log adic space, and let \( \mathcal{F} \) be as in Lemma 4.3.10 Suppose that \( \mathcal{F} \) is a rule that functorially assigns to each \( V \in \mathcal{B} \) a finite \( O_{X_\kappa}(V) \)-module \( \mathcal{F}(V) \) such that \( \mathcal{F}(V) \otimes_{O_V(V)} O_{V'}(V') \sim \mathcal{F}(V') \) for all \( V' \to V \) that are either étale morphisms or standard Kummer étale covers. Then \( \mathcal{F} \) defines an analytic coherent sheaf on \( X_{\kappa} \).

**Proof.** This follows from Propositions 4.3.9 and A.10 and Lemma 4.3.10. □
4.4. Descent of Kummer étale covers.

**Definition 4.4.1.** Let $X$ be a locally noetherian fs log adic space $X$. Let $X_{\text{f\'et}}$ denote the full subcategory of $X_{\text{f\'et}}$ consisting of log adic spaces that are finite Kummer étale over $X$. Let $\mathcal{F}_{\text{\acute{e}t}}$ denote the fibered category over the category of locally noetherian fs log adic spaces such that $\mathcal{F}_{\text{\acute{e}t}}(X) = X_{\text{f\'et}}$.

The goal of this subsection is to show that Kummer étale covers satisfy effective descent in $\mathcal{F}_{\text{\acute{e}t}}$. We first study $X_{\text{f\'et}}$ when $X$ is as in Examples 2.2.8 and 2.2.9

**Definition 4.4.2.**

(1) A log geometric point is a log point $\zeta = (\text{Spa}(l, l^+), \mathcal{M}, \alpha)$ (as in Examples 2.2.8 and 2.2.9) such that:

(a) $l$ is a complete separably closed nonarchimedean field; and

(b) if $M := \Gamma(\text{Spa}(l, l^+), \mathcal{M})$, then $M/l^\times$ is uniquely $n$-divisible (see Definition 2.2.14) for all positive integers $n$ invertible in $l$.

(2) Let $X$ be a locally noetherian fs log adic space. A log geometric point of $X$ is a morphism of log adic spaces $\eta : \zeta \to X$ from a log geometric point $\zeta$.

(3) Let $X$ be a locally noetherian fs log adic space. A Kummer étale neighborhood of a log geometric point $\eta : \zeta \to X$ is a lifting of $\eta$ to a composition $\zeta \to \tilde{U} \xrightarrow{\phi} X$ in which $\phi$ is Kummer étale.

**Construction 4.4.3.** For each geometric point $\xi : \text{Spa}(l, l^+) \to X$, let us construct some log geometric point $\tilde{\xi}$ above it (i.e., the morphism $\tilde{\xi} : X \to X$ of underlying adic spaces factors through $\xi : X \to X$) as follows. By Proposition 2.3.13 up to étale localization on $X$, we may assume that $X$ admits a chart modeled on a sharp fs monoid $P$, so that we have a strict closed immersion $X \to X(P)$ as in Remark 2.3.3. We equip $\text{Spa}(l, l^+)$ with the log structure $P^{\log}$ associated with the pre-log structure given by the composition of $P \to \mathcal{O}_X(X) \to l$, so that $(\text{Spa}(l, l^+), P^{\log})$ is an fs log point with a chart given by $P \to l$. We shall still denote this fs log point by $\xi$. For each positive integer $m$, let $P \mapsto \frac{1}{m}P$ be as in Definition 4.1.5. Consider $\xi^{\frac{1}{m}} := (\text{Spa}(l, l^+), X(\xi^{\frac{1}{m}})_{\text{red}}$, equipped with the natural log structure modeled on $\frac{1}{m}P$. Note that $\xi^{\frac{1}{m}}$ is different from the $\xi^{\frac{1}{m}}$ as in Definition 4.1.5, because we are taking the reduced subspace, so that the underlying adic space of $\xi^{\frac{1}{m}}$ is still isomorphic to $\text{Spa}(l, l^+)$. Then $\tilde{\xi} := \lim_{\frac{1}{m}} \xi^{\frac{1}{m}}$

where the inverse limit runs through all positive integers $m$ invertible in $l$, is a log geometric point above $\xi$. The underlying adic space of $\tilde{\xi}$ is isomorphic to $\text{Spa}(l, l^+)$, endowed with the natural log structure modeled on

$P_{\geq 0} := \lim_{\frac{1}{m}} \frac{1}{m}P,$

with the direct limit over all positive integers $m$ invertible in $l$.

**Lemma 4.4.4.** Let $\zeta : X$ be a log geometric point of a locally noetherian fs log adic space. Then the functor $\mathcal{S}h(X_{\text{f\'et}}) \to \mathcal{S}ets : \mathcal{F} \mapsto \mathcal{F}_{\zeta} := \lim_{\phi} \mathcal{F}(U)$ from the category of sheaves on $X_{\text{f\'et}}$ to the category of sets, where the direct limit is over Kummer étale neighborhoods $U$ of $\zeta$, is a fiber functor. The fiber functors defined by log geometric points form a conservative system.
Proof. By Proposition 2.3.32 and Remark 4.1.4 the category of Kummer étale neighborhood of ζ is filtered, and hence the first statement follows. Since every point of X admits some geometric point and hence some log geometric point above it (see Construction 4.4.3), and since every object U in X_{ké}t is covered by liftings of log geometric points of X, the second statement also follows. □

Definition 4.4.5. For each pro-finite group G, let G-FSets denote the category of finite sets (with discrete topology) with continuous actions of G.

Definition 4.4.6. Let l be a separably closed field. For each positive integer m, let \( \mu_l(m) \) denote the group of m-th roots of unity in l. Let \( \mu_\infty(l) := \lim_m \mu_l(m) \) and \( \tilde{\mathbb{Z}}'(1)(l) := \lim_m \mu_l(m) \), where the limits run through all positive integers m invertible in l. When \( \text{char}(l) = 0 \), we shall write \( \tilde{\mathbb{Z}}(1)(l) \) instead of \( \tilde{\mathbb{Z}}'(1)(l) \). When the context is clear, we shall simply write \( \mu_l, \mu_\infty, \) and \( \tilde{\mathbb{Z}}'(1), \) without \( l \).

Proposition 4.4.7. Let \( \xi = (\text{Spa}(l, l^+) , M) \) be an fs log point with l complete and separably closed. Let \( \xi \) be a log geometric point constructed as in Construction 4.4.3. Let \( M := M_\xi \) and so \( \tilde{M} \cong M / l^x \). Then the functor \( F_\xi : Y \mapsto \text{Hom}_\xi(\tilde{\xi}, Y) \) induces an equivalence of categories

\[ \xi_{ké}{\text{t}} \cong \text{Hom}(\tilde{\mathbb{M}}^{gp}, \tilde{\mathbb{Z}}'(1)(l)) \text{-FSets}. \]

Proof. For simplicity, we shall omit the symbols \( l \) as in Definition 4.4.6. Let \( P := \tilde{M} \), a sharp and fs monoid. By Lemma 2.1.9, we have a splitting \( M \cong l^x \oplus P \) such that \( l^x \oplus P \cong M \) defines a chart for \( \xi \). For each \( m \) invertible in \( l \), the cover \( \tilde{\xi}(\tilde{\xi}) \rightarrow \xi \) is given by \( M \cong l^x \oplus P \rightarrow l^x \oplus P \cong M \). Note that any finite Kummer étale cover of \( \xi \) is a finite disjoint union of fs log adic spaces of the form

\[ \xi_Q := \xi \times_{\xi(P)} \xi(Q), \]

where \( P \rightarrow Q \) is a Kummer homomorphism of sharp fs monoids whose cokernel is annihilated by an integer invertible in \( l \). We have

\[ F_\xi(\xi_Q) = \text{Hom}_\xi(\tilde{\xi}, \xi_Q) \cong \text{Hom}_{l^x \oplus P}(l^x \oplus Q, l^x \oplus P_{\geq 0}) \cong \text{Hom}(Q^{gp} / P^{gp}, \mu_\infty). \]

The last group has a natural transitive action of

\[ \text{Aut}_\xi(\tilde{\xi}) \cong \text{Hom}_P(P_{\geq 0}, l^x) \cong \text{Hom}(P_{\geq 0}^{gp} / P^{gp}, \mu_\infty) \]

\[ \cong \lim_m \text{Hom}(P^{gp} \otimes \mathbb{Z}(\mathbb{Z}/\mathbb{Z}), \mu_m) \cong \text{Hom}(P^{gp}, \tilde{\mathbb{Z}}'(1)). \]

Hence, \( F_\xi \) is indeed a functor from \( \xi_{ké}{\text{t}} \) to \( \text{Hom}(\tilde{\mathbb{M}}^{gp}, \tilde{\mathbb{Z}}'(1)) \text{-FSets}. \)

Let us verify that \( F_\xi \) is fully faithful. By working with connected components, it suffices to show that, for any \( Q_1 \) and \( Q_2 \), the natural map

\[ \text{Hom}_\xi(\xi_{Q_1}, \xi_{Q_2}) \rightarrow \text{Hom}_P(\text{Hom}(Q_1^{gp} / P^{gp}, \mu_\infty), \text{Hom}(Q_2^{gp} / P^{gp}, \mu_\infty)) \]

is bijective. Note that \( \text{Hom}_\xi(\xi_{Q_1}, \xi_{Q_2}) \cong \text{Hom}_P(Q_2, l^x \oplus Q_1) \). By the unique divisibility of \( P_{\geq 0} \), the sharp monoids \( Q_1 \) and \( Q_2 \) can be viewed as submonoids of \( P_{\geq 0} \). If \( Q_2 \not\subset Q_1 \), then both sides of (4.4.8) are zero. Otherwise, \( Q_2 \subset Q_1 \), and (4.4.8) sends the natural inclusion \( Q_2 \hookrightarrow Q_1 \) in \( \text{Hom}_P(Q_2, l^x \oplus Q_1) \) to the map induced by restriction from \( Q_1^{gp} / P^{gp} \) to \( Q_2^{gp} / P^{gp} \). Consequently, (4.4.8) is bijective, because both sides of (4.4.8) are principally homogeneous under compatible actions of \( \text{Aut}_\xi(\xi_{Q_2}) \cong \text{Hom}(Q_2^{gp} / P^{gp}, \mu_\infty). \)
Finally, let us verify that $F_\xi$ is essentially surjective. Since any discrete finite set $S$ with a continuous action of $\text{Hom}(P^{\text{gp}}, \hat{\mathbb{Z}}(1)) \cong \text{Hom}((P_{\mathbb{Q}_{\geq 0}})^{\text{gp}}/P^{\text{gp}}, \mu_\infty)$ is a disjoint union of orbits, we may assume the action on $S$ is transitive. Then $S$ is a quotient space of $\text{Hom}((P_{\mathbb{Q}_{\geq 0}})^{\text{gp}}/P^{\text{gp}}, \mu_\infty)$, which corresponds by Pontryagin duality to a finite subgroup $G \subset (P_{\mathbb{Q}_{\geq 0}})^{\text{gp}}/P^{\text{gp}}$. Let $Q$ denote the preimage of $G$ under $P_{\mathbb{Q}_{\geq 0}} \to (P_{\mathbb{Q}_{\geq 0}})^{\text{gp}}/P^{\text{gp}}$. Then $Q^{\text{gp}}/P^{\text{gp}} \cong G$ and $F_\xi(Q) = S$, as desired. 

**Proposition 4.4.9.** Let $(X, \mathcal{M}_X)$ be a locally noetherian fs log adic space. Let $\xi = \text{Spa}(l, l^+)$ be a geometric point of $X$, and let $X(\xi)$ be the strict localization of $X$ at $\xi$, with its log structure pulled back from $X$. Without loss of generality, let us assume that $l \cong \pi(x)$, the completion of a separable closure of the residue field $k(x)$ of $\mathcal{O}_{X,x}$. Let $M := \mathcal{M}_{X,\xi}$ and so $\overline{M} \cong M/l^\infty$. Let us view $\xi$ and $X(\xi)$ as log adic spaces by equipping them with the log structures pulled back from $X$. Let $\xi$ be the log geometric point constructed in Construction 4.4.3. Then the functor $H_\xi : Y \mapsto \text{Hom}_X(\xi, Y)$ induces an equivalence of categories

$$X(\xi)_{\text{f\acute{e}t}} \cong \text{Hom}(\overline{M}^{\text{gp}}, \hat{\mathbb{Z}}(1)(l))-\text{FSets},$$

Moreover, we have $H_\xi = F_\xi \circ \iota^{-1}$, where $F_\xi$ is as in Proposition 4.4.7 and $\iota^{-1} : X(\xi)_{\text{f\acute{e}t}} \to \xi_{\text{f\acute{e}t}}$ is the natural restriction functor induced by $\iota : \xi \to X(\xi)$.

Note that, if $x$ is an analytic point of $X$, then Proposition 4.4.9 follows immediately from Proposition 4.4.7 because $\xi \cong X(\xi)$ in this case. Nevertheless, the proof below works for non-analytic points as well.

**Proof.** It suffices to show that $\iota^{-1}$ is an equivalence of categories. Write $P = \overline{M} = M^{\text{gp}}$ and $X(\xi) = \text{Spa}(R, R^+)$. By Lemma 2.1.9, we can choose a splitting $1 \overset{\text{split}}{\to} R^+ \oplus P \cong M$ defines a chart for $X(\xi)$. Note that objects in $X(\xi)_{\text{f\acute{e}t}}$ (resp. $\xi_{\text{f\acute{e}t}}$) are finite disjoint unions of fs log adic spaces of the form

$$X(\xi)_Q := X(\xi) \times_{X(\xi)(P)} X(\xi)/Q$$

(resp. $\xi_Q$), where $P \to Q$ is a Kummer homomorphism of sharp monoids. Then $\iota^{-1}$ sends $X(\xi)_Q$ to $\xi_Q$, and hence is essentially surjective. To see that $\iota^{-1}$ is fully faithful, it suffices to show that the canonical map

$$\text{Hom}_{X(\xi)}(X(\xi)_Q, X(\xi)_Q) \to \text{Hom}_\xi(\xi_1, \xi_2)$$

is bijective. By definition, $X(\xi)_Q \cong \text{Spa}(R_Q, R_Q^+)$, for $i = 1, 2$, where $R_Q = R \otimes_{R(P)} R(Q) \cong R \otimes_{R(P)} R[Q]$ are strictly local rings with residue field $l$. Therefore,

$$\text{Hom}_{X(\xi)}(X(\xi)_Q, X(\xi)_Q) \cong \text{Hom}_P(Q_2, R_Q^+) \cong \text{Hom}_P(\xi_1, \xi_2),$$

and hence the map 4.4.11 is bijective, as desired. 

Now, we are ready to prove the main result of this subsection; i.e., Kummer étale covers satisfy effective descent in the fibered category $\text{F}\text{\acute{e}t}$.

**Theorem 4.4.12.** Let $X$ be a locally noetherian fs log adic space, and let $f : Y \to X$ be a Kummer étale cover. Let $\text{pr}_1, \text{pr}_2 : Y \times_X Y \to Y$ denote the two projections. Suppose that $Y \in \text{Y}_{\text{f\acute{e}t}}$ and that there exists an isomorphism $\text{pr}_1^{-1}(Y) \cong \text{pr}_2^{-1}(Y)$ satisfying the usual cocycle condition. Then there exists a unique $\tilde{X} \in X_{\text{f\acute{e}t}}$ (up to isomorphism) such that $\tilde{Y} \cong \tilde{X} \times_X Y$. 

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4.4.9 In particular, \( \bar{\mathbb{K}} \) is bijective, and hence the corresponding canonical morphism \( \bar{\mathbb{K}} \rightarrow \tilde{X} \times_X Y \) is an isomorphism. Consequently, \( \bar{\mathbb{K}} \) is compatible with a sharp fs chart \( P \), and where \( Y \rightarrow X \) is a standard Kummer étale cover induced by a Kummer homomorphism of sharp monoids \( u : P \rightarrow \hat{Q} \), where \( G := \bar{Q}^{\mathrm{gp}}/\bar{u}^\mathrm{gp}(\hat{P}) \) is a finite group of order invertible in \( \mathcal{O}_X \). By Proposition \( \ref{4.1.6} \) up to étale localization on \( X \), we may assume that the morphism \( Y \rightarrow X \) is a Galois cover with Galois group \( \Gamma := \text{Hom}(G, \mathcal{O}_X(X)^\times) \); that \( |G| \) is invertible in \( \mathcal{O}_X \), and \( \mathcal{O}_X(X)^\times \) contains all the \(|G|\)-th roots of unity; and that \( Y \times_X Y \cong \Gamma_X \times_Y Y \). Then the descent datum is equivalent to an action of \( \Gamma \) on \( \tilde{Y} \) over \( X \) lifting the action of \( \Gamma \) on \( Y \) over \( X \). Let us write \( X = (\text{Spa}(\mathcal{R}, \mathcal{R}^+), \mathcal{M}_X) \) and \( \tilde{Y} = (\text{Spa}(\mathcal{S}, \mathcal{S}^+), \mathcal{M}_{\tilde{Y}}) \). By Lemma \( \ref{4.1.11} \) \( (\tilde{R}, \tilde{R}^+) := (\tilde{S}^\Gamma, (\tilde{S}^\Gamma)^P) \) is a Huber pair, and \( \tilde{X} := \text{Spa}(\tilde{R}, \tilde{R}^+) \) is a noetherian adic space finite over \( X \). Moreover, the morphism \( \tilde{Y} \rightarrow \tilde{X} \) is finite, open, surjective, and invariant under the \( \Gamma \)-action on \( \tilde{Y} \). The étale sheaf of monoids \( \mathcal{M}_{\tilde{X}} \) defined by \( \mathcal{M}_{\tilde{X}}(U) := (\mathcal{M}_{\tilde{Y}}(\tilde{X} \times_X U))^\Gamma \), for each \( U \in \tilde{X} \), is fine and saturated, and defines a log structure on \( \tilde{X} \). We claim that the log adic space \( \tilde{X} \) thus obtained gives the desired descent.

Let us first verify that the canonical morphism \( \tilde{Y} \rightarrow \tilde{X} \times_X Y \) induced by the structure morphisms \( \tilde{Y} \rightarrow \tilde{X} \) and \( \tilde{Y} \rightarrow Y \) is an isomorphism. Since the morphism is between spaces that are finite over \( X \), and since the formation of \( \Gamma \)-invariants is compatible with (strict) base change (as \( |\Gamma| \) is invertible in \( \mathcal{O}_X \)), we may assume that \( X = \tilde{X}(\xi) \) is strict local, and so is \( Y \cong \tilde{Y}(\xi) \) for simplicity. By Proposition \( \ref{4.2.9} \) \( \tilde{Y} \rightarrow X \) corresponds to the set \( \tilde{\Gamma} := \text{Hom}(\tilde{Q}^{\mathrm{gp}}/\tilde{u}^\mathrm{gp}(\tilde{P}), \mu_\infty) \) with \( \Gamma \)-action; \( Y \rightarrow X \) corresponds to \( \Gamma \) itself (with its canonical \( \Gamma \)-action); and \( Y \rightarrow Y \) corresponds to a surjective \( \Gamma \)-equivariant map \( \tilde{\Gamma} \rightarrow \Gamma \). Since \( \tilde{Y} \rightarrow X \) is Kummer, we have \( (\mathcal{M}_{\tilde{X}})_{\tilde{Y}}^{\mathrm{gp}} = (\mathcal{M}_{\tilde{Y}})_{\tilde{Y}}^{\Gamma} \), and hence \( \tilde{X} \cong \tilde{X}(\xi)_{\tilde{P}} \) for the fs monoid \( \tilde{P} = \tilde{Q} \cap \tilde{P}^\mathrm{gp} \) such that \( \text{Hom}(\tilde{P}^{\mathrm{gp}}/\tilde{u}^\mathrm{gp}(\tilde{P}), \mu_\infty) \cong \tilde{\Gamma} / \Gamma \), by explicitly computing \( \tilde{R} = \tilde{S}^\Gamma \cong (\tilde{R} \otimes_{\tilde{R}[\tilde{P}]} R[\tilde{Q}])^\Gamma \) using the identifications in the proof of Proposition \( \ref{4.2.9} \). In particular, \( \tilde{X} \rightarrow X \) is finite Kummer étale, and \( \tilde{Y} \rightarrow \tilde{X} \) corresponds to the quotient \( \tilde{\Gamma} \rightarrow \tilde{\Gamma} / \Gamma \) under \( H_{\tilde{\xi}} \). Since \( \tilde{\Gamma} \) is an abelian group, the canonical map \( \tilde{\Gamma} \rightarrow (\tilde{\Gamma} / \Gamma) \times \Gamma \) is bijective, and hence the corresponding canonical morphism \( \tilde{Y} \rightarrow \tilde{X} \times_X Y \) is indeed an isomorphism. Consequently, \( \tilde{Y} \cong \tilde{X} \times_X Y \cong \tilde{X} \times_X \tilde{\mathcal{O}}(\tilde{P}) \tilde{X}(\tilde{Q}) \rightarrow \tilde{X} \) is finite Kummer étale. We need to show that the morphism \( \tilde{X} \rightarrow X \) is also finite Kummer étale (without the assumption that \( X \) is strict local). By construction, and by essentially the same arguments as in the proof of Proposition \( \ref{4.1.15} \) \( \tilde{X} \rightarrow X \) is finite Kummer. Hence, by Lemma \( \ref{4.1.13} \) it suffices to show that \( X \rightarrow X \) is log étale. By Lemma \( \ref{4.1.11} \) and its proof, up to étale localization on \( X \), we may assume that \( \tilde{Y} \rightarrow \tilde{X} \rightarrow X \) admit compatible global charts \( P \rightarrow \tilde{P} \rightarrow \tilde{Q} \), and that the induced morphism \( \tilde{X} \rightarrow X' := X \times_X (\tilde{P}) \tilde{X}(\tilde{P}) \) is strict. Let us write \( (\mathcal{O}_{\tilde{X}}(\tilde{X}'), \mathcal{M}_{\tilde{X}}(\tilde{X}')) = (A, M) \), \( (\mathcal{O}_X(\tilde{X}), \mathcal{M}_X(\tilde{X})) = (B, N) \), and \( (\mathcal{O}_{\tilde{Y}}(\tilde{Y}), \mathcal{M}_{\tilde{Y}}(\tilde{Y})) = (C, L) \), for simplicity. By...
Lemma 4.1.13 and Propositions 3.2.15 and 4.1.15. \( \tilde{Y} \to \tilde{X}' \) is log étale and, accordingly, \((A, M) \to (C, L)\) is formally log étale (as in Definition 3.2.10). Since \( \tilde{X} \to \tilde{X}' \) is strict, \((A, M) \to (B, N)\) is also finite Kummer \( \Gamma \)-invariants is compatible with arbitrary base changes when \( |\Gamma| \) is invertible, and because of Remark 2.1.6. Thus, we have obtained an extended commutative diagram for \((A, M) \to (C, L)\) and the base change \((\tilde{D}, \tilde{O}) \to (\tilde{D}', \tilde{O}')\) of \((D, O) \to (D', O')\). Since \((A, M) \to (C, L)\) is formally log étale, \((C, L) \to (\tilde{D}, \tilde{O})\), whose pre-composition with \((B, N) \to (C, L)\) is \( \Gamma \)-invariant and hence factors through \((D, O)\). This implies that the strict homomorphism \((A, M) \to (B, N)\) is also formally log étale, and hence \( A \to B \) is formally étale (see Remark 3.2.11). Consequently, \( \tilde{X} \to \tilde{X}' \) is étale, because the proof of [Hub96 Prop. 1.7.1] uses only the properties of being lift and formally étale; and \( \tilde{X} \to X \) is log étale as \( \tilde{X}' \to X \) is, as desired.

Corollary 4.4.13. Let \( X \) be a locally noetherian fs log adic space, and let \( f : Y \to X \) be a finite Kummer étale cover. Let \( \Gamma \) be a finite group which acts on \( Y \) by morphisms over \( X \). Then the canonical morphisms \( Y \to Z := X/\Gamma \to X \) induced by \( f \) (by Lemma 4.1.7) are both finite Kummer étale covers.

Proof. By Lemma 4.1.7 both morphisms \( Y \to Z \) and \( Z \to X \) are finite, and \( Y \to Z \) is finite Kummer. By Proposition 4.1.15 it suffices to show that \( Z \to X \) is finite Kummer étale. Then the first projection \( \tilde{f} : \tilde{Y} := Y \times_X Y \to Y \) is a pullback of \( Y \to X \), which inherits an action of \( \Gamma \). By [Hub96 Lem. 1.7.6], under the noetherian hypothesis, the formation of quotients by \( \Gamma \) as in Lemma 4.1.7 is compatible with base changes under étale morphisms of affinoid adic spaces. By Proposition 4.1.6 and Remark 4.1.18 up to étale localization on \( X \), we may assume that \( Y \to X \) is a composition of a finite étale \( f_1 \) morphism and a standard Kummer étale cover \( f_2 \) as in Proposition 4.1.6 in which case \( \tilde{Y} \) is a disjoint union of sections of \( \tilde{f} \). Then \( \Gamma \) acts on \( \tilde{Y} \) by permuting such sections, and we have a quotient \( \tilde{Z} := \tilde{Y}/\Gamma \to \tilde{Y} \), which is clearly finite Kummer étale. Moreover, the pullbacks of \( \tilde{Z} \to Y \) along the two projections are isomorphic to each other by interchanging the factors, and hence \( \tilde{Z} \to Y \) descends to a finite Kummer étale cover of \( X \), by Theorem 4.4.12. We claim that this cover is canonically isomorphic to \( Z \to X \). Since the set of sections of \( f : \tilde{Y} \to \tilde{X} \) is a disjoint union of subsets formed by the sections of pullback of the finite étale morphism \( f_1 \), we can reduce the claim to the extremal cases where either \( f = f_1 \) is finite étale or \( f = f_2 \) is standard Kummer étale. In the former case, the claim follows from the usual theory for finite étale covers of schemes, as in [Gro71 V]. In the latter case, the claim follows from Proposition 4.1.6. □

Definition 4.4.14. Let \( X \) be a locally noetherian fs log adic space, and let \( \Lambda \) be a commutative ring.
(1) A sheaf \( \mathcal{F} \) on \( X_{k\acute{e}t} \) is called a constant sheaf of sets (resp. constant sheaf of \( \Lambda \)-modules) if it is the sheafification of a constant presheaf given by some set \( S \) (resp. some \( \Lambda \)-module \( M \)).

(2) A sheaf \( \mathcal{F} \) on \( X_{k\acute{e}t} \) is called locally constant if there exists a Kummer étale covering \( \{ U_i \} \) \( i \in I \rightarrow X \) such that all \( \mathcal{F}|_{U_i} \) are constant sheaves. We denote by \( \text{Loc}(X_{k\acute{e}t}) \) the category of locally constant sheaves of finite sets on \( X_{k\acute{e}t} \).

**Theorem 4.4.15.** Let \( X \) be a locally noetherian fs log adic space. The functor
\[
\phi : X_{k\acute{e}t} \rightarrow \text{Loc}(X_{k\acute{e}t}) : Y \mapsto \text{Hom}_X(\cdot, Y)
\]
is an equivalence of categories. Moreover:

1. Fiber products exist in \( X_{k\acute{e}t} \) and \( \text{Loc}(X_{k\acute{e}t}) \), and \( \phi \) preserves fiber products.
2. Categorical quotients by finite groups exist in \( X_{k\acute{e}t} \) and \( \text{Loc}(X_{k\acute{e}t}) \), and \( \phi \) preserves such quotients.

**Proof.** By Proposition 4.3.5, representable presheaves on \( X_{k\acute{e}t} \) are sheaves. By Proposition 4.1.6 and Remark 4.1.18 any \( Y \in X_{k\acute{e}t} \) is Kummer étale locally (on \( X \)) a disjoint union of finitely many copies of \( X \). Hence, \( \text{Hom}_X(\cdot, Y) \) is indeed a locally constant sheaf of finite sets, and the functor \( \phi \) is defined. The functor \( \phi \) is fully faithful for formal reasons. Since any locally constant sheaf of finite set is Kummer étale locally represented by objects in \( X_{k\acute{e}t} \), these objects glue to a global object \( Y \) by Theorem 4.4.12 and the full faithfulness of \( \phi \). This shows that \( \phi \) is also essentially surjective, as desired. As for the statements (1) and (2), by Kummer étale localization, we just need to note that the statements become trivial after replacing the source and target of the functor \( \phi \) with the categories of finite disjoint unions of copies of \( X \) and of constant sheaves of finite sets, respectively. \( \square \)

Next, let us define the Kummer étale fundamental groups.

**Lemma 4.4.16.** Let \( X \) be a connected locally noetherian fs log adic space, and \( \eta : \zeta \rightarrow X \) a log geometric point. Let \( \text{FSets} \) denote the category of finite sets. Consider the fiber functor
\[
(4.4.17) \quad F : X_{k\acute{e}t} \rightarrow \text{FSets} : Y \mapsto Y_\zeta := \text{Hom}_X(\zeta, Y).
\]
Then \( X_{k\acute{e}t} \) together with the fiber functor \( F \) is a Galois category.

**Proof.** We already know that the final object, fiber products (see Proposition 4.1.14), categorical quotients by finite groups (see Corollary 4.4.13), and finite coproducts exist in \( X_{k\acute{e}t} \) (and \( \text{FSets} \)). It remains to verify the following conditions:

1. \( F \) preserves fiber products, finite coproducts, and quotients by finite groups.
2. \( F \) reflexes isomorphisms (i.e., \( F(f) \) being an isomorphism implies \( f \) also being an isomorphism).

(We refer to [Gro71, V, 4] for the basics on Galois categories.)

As for condition (1), since \( F \) is defined Kummer étale locally at the log geometric point \( \zeta \), it suffices to verify the condition after restricting \( F \) to the category of finite disjoint unions of \( X \), in which case the condition clearly holds.

As for condition (2), note that \( F \) factors through the equivalence of categories \( \phi \) in Theorem 4.4.15 and induces the stalk functor \( \text{Loc}(X_{k\acute{e}t}) \rightarrow \text{FSets} : F \mapsto F_\zeta := \lim \mathcal{F}(U) \), where the direct limit is over Kummer étale neighborhoods \( U \) of \( \zeta \). Since \( X \) is connected, the stalk functors at any two log geometric points are isomorphic. Thus, whether \( f \) is an isomorphism can be checked at just one stalk. \( \square \)
Corollary 4.4.18. Let $X$ and $\zeta \to X$ be as in Lemma 4.4.16. Then the fiber functor $F$ in (4.4.17) is pro-representable. Let $\pi^\kappa_1(X, \zeta)$ be the automorphism group of $F$. Then $F$ induces an equivalence of categories
\[(4.4.19)\quad X^{\kappa_1} \xrightarrow{\sim} \pi^\kappa_1(X, \zeta)\text{-Fsets},\]
which is the composition of the equivalence of categories $\phi$ in Theorem 4.4.15 with the equivalence of categories
\[(4.4.20)\quad \text{Loc}(X^{\kappa_1}) \xrightarrow{\sim} \pi^\kappa_1(X, \zeta)\text{-Fsets}\]
induced by the stalk functor $\mathcal{F} \to \mathcal{F}_\zeta$.

Remark 4.4.21. In Corollary 4.4.18, since stalk functors at any two log geometric points $\zeta$ and $\zeta'$ are isomorphic, the fundamental groups $\pi^\kappa_1(X, \zeta)$ and $\pi^\kappa_1(X, \zeta')$ are isomorphic. We shall omit $\zeta$ from the notation when the context is clear.

Corollary 4.4.22. Let $(X, \mathcal{M}_X)$, $\xi = \text{Spa}(l, l^+)$, $X(\xi)$, and $M := \mathcal{M}_{X, \xi}$ be as in Proposition 4.4.9. In particular, the underlying adic spaces of $\xi$ (resp. $X(\xi)$) is a geometric point (resp. a strictly local adic space). Then we have
\[\pi^\kappa_1(X(\xi)) \cong \pi^\kappa_1(\xi) \cong \text{Hom}(\mathcal{M}^{\text{gp}}_l, \mathbb{Z}(1)(l)).\]
Since $\mathcal{M}$ is sharp and fs, we have $\mathcal{M}^{\text{gp}}_l \cong \mathbb{Z}^r$ for some $r$, and we obtain a non-canonical isomorphism $\pi^\kappa_1(\xi) \cong (\mathbb{Z}(1)(l))^r$.

Remark 4.4.23. For any connected locally noetherian fs log adic space $X$ and any log geometric point $\xi$ of $X$, the natural inclusion from the category of finite étale covers to that of finite Kummer étale covers is fully faithful, and hence induces a canonical surjective homomorphism $\pi^\kappa_1(X, \xi) \to \pi^\kappa_1(X, \xi)$ (see [Gro71, V, 6.9]).

Example 4.4.24. Let $(k, k^+)$ be an affinoid field, and let $s = (\text{Spa}(k, k^+), \mathcal{M})$ be an fs log point as in Example 2.3.14. Let $\tilde{s} = (\text{Spa}(k, k^+), M)$ be a geometric point over $s$, where $K$ is the completion of a separable closure $k^{\text{sep}}$ of $k$, and let $\tilde{s}$ be a log geometric point over $\tilde{s}$. Then we have a canonical short exact sequence
\[1 \to \pi^\kappa_1(\tilde{s}, \tilde{s}) \to \pi^\kappa_1(s, \tilde{s}) \to \pi^\kappa_1(s, s) \to 1,
where $\pi^\kappa_1(\tilde{s}, \tilde{s}) \cong \text{Hom}(\mathcal{M}^{\text{gp}}_l, \mathbb{Z}(1)(k^{\text{sep}}))$ (as in Corollary 4.4.22) and $\pi^\kappa_1(s, \tilde{s}) \cong \text{Gal}(k^{\text{sep}}/k)$. If $s$ is a split fs log point as in Example 2.3.15, then any choice of a Gal($k^{\text{sep}}/k$)-equivariant splitting of $M \to \mathcal{M}$ lifts this exact sequence, inducing an isomorphism
\[\pi^\kappa_1(s, \tilde{s}) \cong \text{Hom}(\mathcal{M}^{\text{gp}}_l, \mathbb{Z}(1)(k^{\text{sep}})) \times \text{Gal}(k^{\text{sep}}/k).
Example 4.4.25. Let $(k, k^+)$ be an affinoid field. Consider the point $0 = (\text{Spa}(k, k^+)$ of $\text{Spa}(k(Z_{\geq 0}), k^{+}(Z_{\geq 0})) \cong \text{Spa}(k(T), k^{+}(T))$ defined by $T = 0$. Let us denote by $0^\partial$ the log adic space with underlying adic space $0$ and with log structure pulled back from $\text{Spa}(k(Z_{\geq 0}), k^{+}(Z_{\geq 0}))$, which is the split fs log point $(X, \mathcal{M}_X) = (\text{Spa}(k, k^+), \mathcal{O}^\times_{X, 0} \oplus (Z_{\geq 0} X)$ as in Example 2.3.15. Let $0^\partial$ and $0^\partial$ be defined by $0^\partial$ as in Example 4.4.24 (with $s = 0^\partial$ there). Then
\[\pi^\kappa_1(0^\partial, 0^\partial) \cong \mathbb{Z}(1)(k^{\text{sep}}) \times \text{Gal}(k^{\text{sep}}/k).
For each $n$ invertible in $k$, and each $r \geq 1$, we have a $\mathbb{Z}/n$-local system $\mathcal{J}_n^r$ on $0^\partial_0$ defined by the representation of $\pi^\kappa_1(0^\partial, 0^\partial)$ on $(\mathbb{Z}/n)^r$ such that a topological generator of $\mathbb{Z}(1)(k^{\text{sep}})$ acts as the standard upper triangular principal unipotent.
matrix $J_r$ and $\text{Gal}(k_{\text{sep}}/k)$ acts on $\ker(J_r - 1)$ trivially. (The resulting local system is independent of the choice of the generator of $\hat{\mathbb{Z}}'(1)(k_{\text{sep}})$ up to isomorphism.) Moreover, for each $m \geq 1$ with $m$ invertible in $k$, we also have the $\mathbb{Z}/n$-local system $\mathbb{Z}_{m,n}^\partial$ defined by the representation of $\pi_1^\kappa((0^\partial, \tilde{0}^\partial))$ induced from the trivial representation of $m\hat{\mathbb{Z}}'(1)(k_{\text{sep}}) \rtimes \text{Gal}(k_{\text{sep}}/k)$ on $\mathbb{Z}/n$. (These local systems will be useful for defining unipotent and quasi-unipotent nearby cycles in Section 6.4.)

**Example 4.4.26.** Let $s$, $\pi$, and $\tilde{s}$ be as in Example 4.4.24 and let $f : X \to s$ be any strict lift morphism of log adic spaces. Let $\xi$ be a geometric point of $X$ above $\pi$, and let $\tilde{\xi}$ be a log geometric point above $\xi$ and $\tilde{s}$. Let $X(\xi)$ denote the strict localization of $X$ at $\xi$. Then, by Proposition 4.4.9 and Corollary 4.4.22, we have $\pi_1^\kappa(X(\xi), \tilde{\xi}) \cong \pi_1^\kappa(\tilde{s}, \tilde{s}) \cong \text{Hom}(M^\text{gp}, \hat{\mathbb{Z}}'(1)(k_{\text{sep}}))$.

**Lemma 4.4.27.** Let $(X, \mathcal{M}_X)$, $\xi = \text{Spa}(l, l^+)$, $X(\xi)$, and $M := \mathcal{M}_{X, \xi}$ be as in Proposition 4.4.9. Let $\varepsilon : X_{\kappa} \to X_{\text{et}}$ be the natural projection of sites, as before. Then, for each sheaf $\mathcal{F}$ of finite abelian groups over $X_{\kappa}$, we have

$$
(R^i \varepsilon_{\text{et},*}(\mathcal{F}))_{\xi} \cong H^i(\pi_1^\kappa(\tilde{s}, \tilde{s}), \mathcal{F}_{\tilde{\xi}}).
$$

**Proof.** By definition, $(R^i \varepsilon_{\text{et},*}(\mathcal{F}))_{\xi} \cong \lim \mathcal{H}^i(U_{\kappa}, \mathcal{F})$, where the direct limit is over the filtered category of étale neighborhoods $i_U : \xi \to U$ of $x$ in $X$. Up to étale localization, we may assume that $X$ admits a chart modeled on $P := \mathcal{M}$. Consider the morphism $i^{-1} : \lim U_{\kappa} \to x_{\kappa}$, where the direct limit of sites $\lim \mathcal{U}_{\kappa}$ is as in [AGV73 VI, 8.2.3], induced by the morphisms $i_U : U_{\kappa} \to x_{\kappa}$. Since each Kummer étale covering of $\xi$ can be further covered by some standard Kummer étale covers induced by $n$-th multiple maps $[n] : P \to P$, for some integers $n \geq 1$ invertible in $l$, and since coverings of the latter kind are in the essential image of $i^{-1}$, by Proposition 4.4.7, we obtain an equivalence $\xi_{\kappa} \cong \lim \mathcal{U}_{\kappa}$ of the associated topoi. Then $H^i(\mathcal{U}_{\kappa}, \mathcal{F}) \cong H^i(\xi_{\kappa}, \xi^{-1}(\mathcal{F})) \cong H^i(\pi_1^\kappa(\tilde{s}, \tilde{s}), \mathcal{F}_{\tilde{\xi}})$, as desired.

Let $(X, \mathcal{M}_X)$ be a locally noetherian fs log adic space, and let $\mathcal{M}_{X_{\kappa}}$ be as in Proposition 4.4.4. For each positive integer $n$ invertible in $\mathcal{O}_X$, there is an exact sequence $1 \to \mu_n \to \mathcal{M}_{X_{\kappa}}^\text{gp} \xrightarrow{[n]} \mathcal{M}_{X_{\kappa}}^\text{gp} \to 1$ of sheaves on $X_{\kappa}$, whose pushforward along $\varepsilon : X_{\kappa} \to X_{\text{et}}$ induces a morphism

$$
\mathcal{M}_{X_{\kappa}}^\text{gp}/n\mathcal{M}_{X_{\kappa}}^\text{gp} \to R^1 \varepsilon_{\text{et},*}(\mu_n).
$$

At every geometric point of $X$, it is clear that this morphism is nothing but the inverse of the isomorphism in Lemma 4.4.27. Therefore, we obtain the following:

**Lemma 4.4.29.** The above morphism (4.4.28) is an isomorphism and, for each $i$, the canonical morphism $\wedge^i(R^1 \varepsilon_{\text{et},*}(\mu_n)) \to R^i \varepsilon_{\text{et},*}(\mu_n)$ is an isomorphism.

4.5. **Localization and base change functors.** In this subsection, we study the behavior of sheaves on Kummer étale sites under certain direct image and inverse image functors. (The readers are referred to [AGV73 IV] for general notions concerning sites, topoi, and the functors and morphisms among them.)

For any morphism $f : Y \to X$ of locally noetherian fs log adic spaces, since pulling back by $f$ respects fiber products, we have a morphism of topoi $(f^{-1}, f_*) : Y_{\kappa} \to X_{\kappa}$. Concretely, we have the direct image (or pushforward) functor

$$
f_* : \text{Sh}(Y_{\kappa}) \to \text{Sh}(X_{\kappa}) : \mathcal{F} \to (U \mapsto f_*(\mathcal{F})(U) := \mathcal{F}(U \times_X Y)).
$$
and the inverse image (or pullback) functor
\[ f^{-1} : \text{Sh}(X) \to \text{Sh}(W) \]
sending \( G \in \text{Sh}(X) \) to the sheafification of \( V \mapsto \lim_{\longrightarrow} G(U) \), where \( U \) runs over the objects in \( Y \) such that \( V \to X \) factors through \( f^{-1}(U) \to Y \). It is formal that \( f_* \) is the right adjoint of \( f^{-1} \). Moreover, \( f^{-1} \) is exact, and \( f_* \) is left exact.

For any Kummer étale morphism \( f : Y \to X \), the functor \( f_* \) is also called the base change functor, while the functor \( f^{-1} \) is simply \( f^{-1}(F)(U) := F(U) \), because any object \( U \) of \( Y \) gives an object in \( X \) by composition with \( f \). Moreover, we have the localization functor
\[ f_! : \text{Sh}(Y) \to \text{Sh}(X) \]
sending \( F \in \text{Sh}(Y) \) to the sheafification of the presheaf
\[ f_!(F)(U) := \prod_{U \to X} F(U, \alpha). \]

We shall also denote by \( f_! : \text{Sh}_{ab}(Y) \to \text{Sh}_{ab}(X) \) the induced functor between the categories of abelian sheaves, in which case the above coproduct becomes a direct sum. It is also formal that \( f_! \) is left adjoint to \( f^{-1} \), and that \( f_! \) is right exact.

**Lemma 4.5.1.** Let \( f : V \to W \) be a finite Kummer étale morphism in \( X \). If \( f \) has a section \( g : W \to V \), then there exists a finite Kummer étale morphism \( W' \to W \) and an isomorphism \( h : V \to W' \) such that the composition \( h \circ g \) is the natural inclusion \( W \to W' \).

**Proof.** We may assume that \( W \) is connected. Let \( G := \pi^k(W) \) (see Remark 4.4.21). Via (4.4.19), the finite Kummer étale cover \( V \to W \) (resp. \( W \to W' \)) corresponds to a finite set \( S \) (resp. a single tone \( S_0 \)) with a continuous \( G \)-action (resp. the trivial action), and the section \( g : W \to V \) corresponds to a \( G \)-equivariant map \( g_* : S_0 \to S \). This gives rise to an \( G \)-equivariant decomposition \( S = g_*(S_0) \times S' \), and hence to the desired decomposition \( h : V \to W' \times S' \), by Corollary 4.4.18.

**Proposition 4.5.2.** Let \( f : Y \to X \) be a finite Kummer étale morphism of locally noetherian fs log adic spaces. Then we have a natural isomorphism \( f_! \to f_* : \text{Sh}_{ab}(Y) \to \text{Sh}_{ab}(X) \). Consequently, both functors are exact.

**Proof.** Let \( F \) be an abelian sheaf on \( Y \). For any \( U \in X \), each \( h \in \text{Hom}_X(U, \alpha) \) induces a section \( U \to U \times_X Y \) of the natural projection \( U \times_X Y \to U \). By Lemma 4.5.1, we obtain a decomposition \( U \times_X Y \cong U \times U' \) identifying \( U \to U \times_X Y \) with \( U \to U \times U' \), which gives rise to a canonical isomorphism \( F(U) \to F(U \times_X Y) \) because \( F \) is a sheaf. By combining such maps, we obtain a map of presheaves \( (f_!(F))(U) = \oplus_{h \in \text{Hom}_X(U, \alpha)} F(U, \alpha) \to (f_*(F))(U) = F(U \times_X Y) \), which induces a canonical morphism \( f_! \to f_* \) by sheafification.

By the above construction, it remains to show that, étale locally on \( U \), there exists a finite Kummer étale cover \( V \to U \) such that \( \text{Hom}_X(V, Y) \) is a finite set and such that the sections \( V \to X \) given by \( h \in \text{Hom}_X(V, Y) \) induces \( \prod h \in \text{Hom}_X(V, Y) V \to V \times_X Y \). Note that this is true in the special case where \( Y \to X \) is strictly finite étale, because \( Y \) is étale locally a finite disjoint union of copies of \( X \). In general, up to étale localization on \( X \), we may assume that \( X \) is affine and modeled on a sharp fs monoid \( P \), and that \( Y \to X \) factors as a composition \( Y \to X \to X_Q := X \times_Y P, X(Q) \to X \), where the first morphism is strictly finite étale,
and where the second morphism is the standard Kummer étale cover induced by a Kummer homomorphism \( u : P \to Q \) of sharp fs monoids. Let \( G = Q^{\text{gp}}/u^{\text{gp}}(P^{\text{gp}}) \).

By Proposition 4.1.6, \( Y \times X_{Q} \cong Y \times X_{Q} (X_{Q} \times X) \equiv Y \times X \times (G) \to X_{Q} \) is strictly étale. Hence, as explained above, there exists a finite Kummer étale cover \( V \to U \times X_{Q} \) such that \( \coprod_{h \in \text{Hom}_{Q}(V, Y \times X_{Q})} V \cong V \times X_{Q} (Y \times X) \). Since \( \coprod_{h \in \text{Hom}_{Q}(V, Y)} V \cong \coprod_{h \in \text{Hom}_{Q}(V, Y \times X_{Q})} V \cong V \times X_{Q} (Y \times X) \equiv V \times X Y \), the composition of \( V \to U \times X_{Q} \to X \) gives the desired finite Kummer étale cover.

### Lemma 4.5.3

Let \( X \) be a locally noetherian fs log adic space. Let \( \iota : Z \to X \) be a strict closed immersion of log adic spaces, and \( j : W \to X \) an open immersion of log adic spaces, as in Definition 2.2.23 such that \( W = X - Z \). For \( ? = \text{an}, \text{ét}, \text{két} \), let \( (\iota_{\gamma}^{-1}, \iota_{\gamma}^{-*}) \) and \( (j_{\gamma}^{-1}, j_{\gamma}^{-*}) \) denote the associated morphisms of topoi, and let \( j_{??,i} \) denote the left adjoint of \( j_{??}^{-1} \) (which is defined as explained above).

1. For each abelian sheaf \( F \) on \( X_{??} \), we have the excision short exact sequence
   \[ 0 \to j_{??,i}^{-1}(F) \to F \to \iota_{??,i}\iota_{??}^{-1}(F) \to 0 \] in \( \text{Sh}_{\text{Ab}}(X_{??}) \).
2. For each abelian sheaf \( G \) on \( Z_{??} \), the adjunction morphism \( \iota_{??}^{-1} \iota_{??,*}(G) \to G \)
   is an isomorphism in \( \text{Sh}_{\text{Ab}}(Z_{??}) \), and hence \( \iota_{??,*} \) is exact and fully faithful.
3. For each abelian sheaf \( H \) on \( W_{??} \), the adjunction morphism \( H \to j_{??,i}^{-1} j_{??,!*}(H) \)
   is an isomorphism in \( \text{Sh}_{\text{Ab}}(W_{??}) \), and hence \( j_{??,!} \) is exact and fully faithful.

**Proof.** These follow easily from the definitions of the objects involved, by evaluating the sheaves at points (resp. geometric points, resp. log geometric points) when \( ? = \text{an} \) (resp. ét, resp. két). (See [Hub96, Prop. 2.5.5] and Lemma 4.4.4) \( \square \)

### Lemma 4.5.4

Let \( f : X \to \check{X} \) and \( g : Z \to \check{Z} \) be morphisms of locally noetherian fs log adic spaces whose underlying morphisms of adic spaces are isomorphisms, and let \( \iota : Z \to X \) and \( \check{\iota} : \check{Z} \to \check{X} \) be strict immersions compatible with \( f \) and \( g \). Then, for any abelian sheaf \( F \) on \( X_{\text{két}} \), and for each \( i \geq 0 \), we have a canonical isomorphism

\[
R^{i} g_{\text{két},*} i_{\text{két}}^{-1}(F) \cong i_{\text{két}}^{-1} R^{i} f_{\text{két},*}(F).
\]

This applies, in particular, to the case where \( \check{X} \) and \( \check{Z} \) are the underlying adic spaces of \( X \) and \( Z \), respectively, equipped with their trivial log structures, in which case \( X_{\text{két}} \cong X_{\text{ét}} \) and \( Z_{\text{két}} \cong Z_{\text{ét}} \), and so \( f_{\text{két}} : X_{\text{két}} \to \check{X}_{\text{két}} \) and \( g_{\text{két}} : Z_{\text{két}} \to \check{Z}_{\text{két}} \)

\[
\text{can be identified with the natural morphisms of sites } X_{\text{két}} \\
\text{and } Z_{\text{két}} \to \check{X}_{\text{ét}} \text{ and } \check{Z}_{\text{ét}}.
\]

**Proof.** Up to compatibly replacing \( X \) and \( \check{X} \) with open subspaces, we may assume that \( \iota \) and \( \check{\iota} \) are compatible strict closed immersions. By Lemma 4.5.3(2), and by applying \( i_{\text{két},*} \) to \( (4.5.5) \), it suffices to show that we have a canonical isomorphism

\[
R^{i} f_{\text{két},*} i_{\text{két}}^{-1}(F) \cong i_{\text{két}}^{-1} R^{i} f_{\text{két},*}(F).
\]

Let \( j : W \to X \) and \( \check{j} : \check{W} \to \check{X} \) denote the complementary open immersions. By Lemma 4.5.3(1), we have a long exact sequence

\[
\cdots \to R^{i} f_{\text{két},*} j_{\text{két}}^{-1}(F) \to R^{i} f_{\text{két},*}(F) \to R^{i} f_{\text{két},*} i_{\text{két}}^{-1}(F) \to \cdots .
\]

By the definition of \( j_{\text{két}}^{-1} \) and \( j_{\text{két},*} \), and by comparing stalks at log geometric points as in Lemma 4.4.4, we obtain \( R^{i} f_{\text{két},*} j_{\text{két}}^{-1}(F) \cong j_{\text{két}}^{-1} R^{i} f_{\text{két},*}(F) \), which induces the desired (4.5.6), by Lemma 4.5.3(1) again. \( \square \)
Lemma 4.5.7. Let \( \iota : Z \to X \) be a strict closed immersion of locally noetherian fs log adic spaces over \( \Spa(\mathbb{Q}_p, \mathbb{Z}_p) \).

1. For \( ? = \text{an}, \text{ét}, \text{or két} \), the canonical morphism \( \iota^{-1}_? (\mathcal{O}_{X, ?}^+ / p) \to \mathcal{O}_{Z, ?}^+ / p \) is an isomorphism.
2. For any \( \mathbb{F}_p \)-sheaf \( \mathcal{F} \) on \( \Z_{\text{két}} \), the canonical morphism
   \[
   (\iota_{Z, \text{két}, *}(\mathcal{F})) \otimes (\mathcal{O}_{X, \text{két}}^+ / p) \to \iota_{Z, \text{két}, *}(\mathcal{F} \otimes (\mathcal{O}_{X, \text{két}}^+ / p))
   \]
   is an isomorphism.

Proof. The case where \( ? = \text{an} \) or \( \text{ét} \) is already in [Sch13] Lem. 3.14. As for \( ? = \text{két} \), the proof is similar, which we explain as follows. At each log geometric point \( \xi = (\Spa(l, l^+), M_x) \to X \), the stalk of \( \mathcal{O}_{X, \xi}^+ / p \) at \( s \) is isomorphic to \( l^+/p \) by construction, because \( \ker(\mathcal{O}_{X, \xi} \to l^+) \), which is the same as \( \ker(\mathcal{O}_{X, \xi} \to l) \), is \( p \)-divisible. The analogous statement for \( \mathcal{O}_{Z, \xi}^+ / p \) is true. Hence, we can finish the proof by comparing stalks, by Lemma 4.4.4.

Lemma 4.5.8. Let \( g : Y \to Z \) be a morphism of locally noetherian fs log adic spaces over \( \Spa(\mathbb{Q}_p, \mathbb{Z}_p) \) such that its underlying morphism of adic spaces is an isomorphism. Suppose moreover that \( g^\#: \mathcal{M}_{Z, g(\mathfrak{y})} \to \mathcal{M}_{Y, \mathfrak{y}} \) is injective and splits, for each geometric point \( \mathfrak{y} \) of \( Y \). Then, for any \( \mathbb{F}_p \)-sheaf \( \mathcal{F} \) on \( \Y_{\text{két}} \), the canonical morphism
   \[
   R^ig_{\text{két}, *}(\mathcal{F}) \otimes_{\mathbb{F}_p} (\mathcal{O}_{Z, \text{két}}^+ / p) \to R^ig_{\text{két}, *}(\mathcal{F} \otimes_{\mathbb{F}_p} (\mathcal{O}_{Y, \text{két}}^+ / p))
   \]
   is an isomorphism.

Proof. By Lemma 4.4.4 it suffices to show that, for each log geometric point \( \tilde{\mathfrak{z}} : (\Spa(l, l^+), M) \to Z \) as in Construction 4.4.3, the induced morphism
   \[
   (\iota_{Z, \text{két}, *}(\mathcal{F}))_{\tilde{\mathfrak{z}}} \otimes_{\mathbb{F}_p} (\mathcal{O}_{Z, \text{két}}^+ / p)_{\tilde{\mathfrak{z}} \to \mathcal{O}_{Y, \text{két}}^+ / p)_{\mathfrak{z}}
   \]
   is an isomorphism. Since \( g \) induces an isomorphism of the underlying adic spaces, \( \mathfrak{z} \) uniquely lifts to a geometric point \( \Spa(l, l^+) \to X \), which we denote by \( \tilde{\mathfrak{y}} \). For convenience, let us equip \( \tilde{\mathfrak{y}} \) and \( \mathfrak{z} \) with the log structures pulled back from \( Y \) and \( Z \), respectively, and view them as log points. By assumption, \( g^\#: \mathcal{M}_{Z, \tilde{\mathfrak{y}}} \to \mathcal{M}_{Y, \mathfrak{y}} \) is injective and splits, and hence we have \( \mathcal{M}_{Y, \mathfrak{y}} \cong \mathcal{M}_{Z, \tilde{\mathfrak{y}}} \cong N \), for some fs monoid \( N \). Therefore, we can lift \( \tilde{\mathfrak{z}} \) to some (saturated by not fine) log point \( (\Spa(l, l^+), M') \to Y \) such that \( M' \cong M \cong N \). By taking fiber products with \( \Spa(l, l^+)[N] \) over \( \Spa(l, l^+)[N] \) and \( \Spa(l, l^+)[N] \), by taking reduced subspaces, and by taking the limit with respect to \( \mathfrak{z} \) (cf. Construction 4.4.3), we can further lift this log point to a log geometric point \( \tilde{\mathfrak{y}} \) of \( Y \) above \( \tilde{\mathfrak{y}} \). This, by a limit argument similar to the one in the proof of Lemma 4.4.27 and by Proposition 4.4.7 and Lemma 4.5.7, we may identify (4.5.9) with \( H^i(\Gamma, \mathcal{F}) \otimes_{\mathbb{F}_p} (l^+/p) \to H^i(\Gamma, l^+/p) \), where \( \Gamma := \ker(\pi_1^{\text{két}}(\mathfrak{y}, \tilde{\mathfrak{y}}) \to \pi_1^{\text{két}}(\mathfrak{x}, \tilde{\mathfrak{x}})) \cong \text{Hom}(\mathbb{N}^{sp} \hat{\mathbb{Z}}(1)(l)) \). Since \( H^i(\Gamma, \mathcal{F}) \) is computed by some bounded complex of free \( \mathbb{F}_p \)-modules, and since \( H^i(\Gamma, l^+/p) \) is computed by the tensor product of this bounded complex with the flat \( \mathbb{F}_p \)-module \( l^+/p \), we see that (4.5.9) is an isomorphism, as desired.

Proposition 4.5.10. Let \( f : Y \to X \) be a morphism of locally noetherian fs log adic spaces over \( \Spa(\mathbb{Q}_p, \mathbb{Z}_p) \) such that its underlying morphism of adic spaces is a closed immersion. Suppose moreover that \( (f^*(\mathcal{M}_X))_{\mathfrak{y}} \to \mathcal{M}_{Y, \mathfrak{y}} \) is injective and splits, for each geometric point \( \mathfrak{y} \) of \( Y \). Then, for any \( \mathbb{F}_p \)-sheaf \( \mathcal{F} \) on \( \Y_{\text{két}} \), the canonical morphism
   \[
   R^if_{\text{két}, *}(\mathcal{F}) \otimes_{\mathbb{F}_p} (\mathcal{O}_{X, \text{két}}^+ / p) \to R^if_{\text{két}, *}(\mathcal{F} \otimes_{\mathbb{F}_p} (\mathcal{O}_{Y, \text{két}}^+ / p))
   \]
   is an isomorphism.
Proof. In this case, let $Z$ denote the underlying adic space of $Y$ equipped with the log structure pulled back from $X$. Then $f : Y \to X$ factors as the composition of a morphism $g : Y \to Z$ as in Lemma 4.5.8 and a strict closed immersion $i : Z \to X$ as in Lemma 4.5.7. Then this lemma follows from Lemmas 4.5.3, 4.5.7, and 4.5.8. \end{proof}

4.6. Purity of torsion local systems. We have the following purity result for torsion Kummer étale local systems. 

**Theorem 4.6.1.** Let $X$, $D$, and $k$ be as in Example 2.3.17. Let $U := X - D$, and let $j : U \to X$ denote the canonical open immersion. Suppose moreover that $\text{char}(k) = 0$ and $k^+ = \mathcal{O}_k$. Let $\mathbb{L}$ be a torsion local system on $U_{\text{ét}}$. Then $j_{\text{ét},*}(\mathbb{L})$ is a torsion local system on $X_{\text{két}}$, and $R^i j_{\text{két},*}(\mathbb{L}) = 0$ for all $i > 0$.

Let us start with some preparations.

**Lemma 4.6.2.** In the setting of Theorem 4.6.1 consider the commutative diagram:

$$
(4.6.3) \quad \xymatrix{ U_{\text{két}} \ar[r]^-{j_{\text{két}}} & X_{\text{két}} \ar[d]^-{\varepsilon_{\text{ét}}} \ar@{=}[r] & \ar[l]^-{\varepsilon_{\text{ét}}} U_{\text{ét}} \ar[r]^-{j_{\text{ét}}} & X_{\text{ét}}. }
$$

Then the canonical morphism

$$
(4.6.4) \quad R\varepsilon_{\text{ét},*}(\mathbb{Z}/n) \to Rj_{\text{ét},*}(\mathbb{Z}/n)
$$

is an isomorphism; and $R^i j_{\text{ét},*}(\mathbb{Z}/n) \cong (\wedge^i(M_X^{\text{gp}}/(nM_X^{\text{gp}}))(-i), for every $i \geq 0$.

**Proof.** By Lemma 4.4.29 it suffices to show that the composition

$$
(\wedge^i(M_X^{\text{gp}}/(nM_X^{\text{gp}}))(-i) \to R\varepsilon_{\text{ét},*}(\mathbb{Z}/n) \to Rj_{\text{ét},*}(\mathbb{Z}/n)
$$

(induced by 4.4.28 and 4.6.4) is an isomorphism. Since this assertion is étale local on $X$, we may assume that $D \subset X$ is the analytification of a normal crossings divisor on a smooth scheme over $k$, and further reduce the assertion to its classical analogue for schemes by [Hub96] Prop. 2.1.4 and Thm. 3.8.1, which is known (see, for example, [Ill02] Thm. 7.2).

**Lemma 4.6.5.** In the setting of Lemma 4.6.2, the canonical morphism

$$
(4.6.6) \quad \mathbb{Z}/n \to Rj_{\text{két},*}(\mathbb{Z}/n)
$$

is an isomorphism.

**Proof.** Let $C$ be the cone of (4.6.6) (in the derived category). It suffices to show that $H^0(W_{\text{két}}, C) = 0$, for each $W \to X$ that is the composition of an étale covering and a standard Kummer étale covering of $X$. Note that the complement of $U \times_X W$ in $W$ is a normal crossings divisor, which induces the fs log structure on $W$ as in Example 2.3.17. Consider the diagram (4.6.3), with $U \to X$ replaced by $U \times_X W \to W$. Since the corresponding morphism (4.6.4) for this new diagram is an isomorphism by Lemma 4.6.2 and since (4.6.4) is obtained from (4.6.6) by applying $\varepsilon_{\text{ét}}$ to both sides, we have $R\varepsilon_{\text{ét},*}(C)|_{W_{\text{két}}} = 0$ on $W_{\text{ét}}$, and so $H^0(W_{\text{két}}, C) = 0$, as desired. \end{proof}

**Proof of Theorem 4.6.1.** Let $V \to U$ be a finite étale cover trivializing $\mathbb{L}$. By Proposition 4.2.1, it extends to a finite Kummer étale cover $f : Y \to X$, where $Y$ is a normal rigid analytic variety with its log structures defined by the preimage of $D$. Then $j_{\text{két},*}(\mathbb{L})|_Y$ is constant, and $j_{\text{két},*}(\mathbb{L})$ is a torsion local system on $X_{\text{két}}$. 


Given this $f : Y \to X$, up to étale localization on $X$, we have some $X \overset{\pi}{\to} X$ as in Lemma 4.2.5. Then the underlying adic space of $Z := Y \times_X X \overset{\pi}{\to} X$ is a smooth rigid analytic variety, its fs log structure is defined by some normal crossings divisor as in Example 2.3.17 and the induced morphism $Z \to X$ is Kummer étale, because these are true for $X \overset{\pi}{\to} X$. (Alternatively, we can construct $Z \to X$, as in the proof of Proposition 4.2.1 by using Lemma 4.2.3 and the last assertion of Lemma 4.2.2.) Thus, in order to show that $R^i \mathbb{j}_{1*}(\mathbb{L}) = 0$, for all $i > 0$, up to replacing $X$ with $Z$, we may assume that $\mathbb{L} = \mathbb{Z}/n$ is constant, in which case Lemma 4.6.5 applies. □

**Corollary 4.6.7.** Let $k$ and $j : U \hookrightarrow X$ be as in Theorem 4.6.1. Let $\mathbb{L}$ be an étale $\mathbb{F}_p$-local system on $U_{\text{ét}}$. Then $\mathbb{L} := \mathbb{j}_{k*}(\mathbb{L})$ is a Kummer étale $\mathbb{F}_p$-local system extending $\mathbb{L}$, and $H^i(U_{\text{ét}}, \mathbb{L}) \cong H^i(X_{k*}, \mathbb{L})$, for all $i \geq 0$.

**Proof.** This follows immediately from Theorem 4.6.1. □

5. PRO-KUMMER ÉTALE TOPOLOGY

5.1. The pro-Kummer étale site. In this subsection, we define the pro-Kummer étale site on log adic spaces, a log analogue of Scholze’s pro-étale site in Sch13a.

For any category $C$, by Sch13a Prop. 3.2, the category pro-$C$ is equivalent to the category whose objects are functors $F : I \to C$ from small cofiltered index categories and whose morphisms are given by $\text{Hom}(F, G) = \varprojlim_{f : j \to i} \text{Hom}(F(i), G(j))$, for each $F : I \to C$ and $G : J \to C$. We shall use this equivalent description in what follows. For each $F : I \to C$ as above, we shall denote $F(i)$ by $F_i$, for each $i \in I$, and denote the corresponding object in pro-$C$ as $\varprojlim_{i \in I} F_i$.

Let $X$ be a locally noetherian fs log adic space. Consider the category pro-$X_{k*}$. Then any object in pro-$X_{k*}$ is of the form $U = \varprojlim_{i \in I} U_i$, where each $U_i \to X$ is Kummer étale, with underlying topological space $|U| := \varprojlim_i |U_i|$.

**Definition 5.1.1.** (1) A morphism $U \to V$ in pro-$X_{k*}$ is Kummer étale (resp. finite Kummer étale, resp. étale, resp. finite étale) if it is the pullback under some morphism $V \to V_0$ in pro-$X_{k*}$ of some Kummer étale (resp. finite Kummer étale, resp. strictly étale, resp. strictly finite étale) morphism $U_0 \to V_0$ in $X_{k*}$.

(2) A morphism $U \to V$ in pro-$X_{k*}$ is pro-Kummer étale if it can be written as a cofiltered inverse limit $U = \varprojlim_{i \in I} U_i$ of objects $U_i \to V$ Kummer étale over $V$ such that $U_j \to U_i$ is finite Kummer étale and surjective for all sufficiently large $i$ (i.e., all $i \geq i_0$, for some $i_0 \in I$). Such a presentation $U = \varprojlim_{i \in I} U_i \to V$ is called a pro-Kummer étale presentation.

(3) A morphism $U \to V$ as in (2) is pro-finite Kummer étale if all $U_i \to V$ are finite Kummer étale.

**Definition 5.1.2.** The pro-Kummer étale site $X_{\text{prok*}}$ has as underlying category the full subcategory of pro-$X_{k*}$ consisting of objects that are pro-Kummer étale over $X$, and each covering of an object $U \in X_{\text{prok*}}$ is given by a family of pro-Kummer étale morphisms $\{f_i : U_i \to U\}_{i \in I}$ such that $|U| = \cup_{i \in I} f_i(|U_i|)$ and such that each $f_i : U_i \to U$ can be written as an inverse limit $U_i = \varprojlim_{\mu < \lambda} U_{i\mu} \to U$ satisfying the following conditions (cf. Sch16):

(1) Each $U_{i\mu} \in X_{\text{prok*}}$, and $U = U_0$ is an initial object in the limit.

(2) The limit runs through the set of ordinals $\mu$ less than some ordinal $\lambda$. 

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(3) For each \( \mu < \lambda \), the morphism \( U_\mu \to U_{< \mu} := \lim \left< U_{\mu'} \right\rangle \) is the pullback of a Kummer étale morphism in \( X_{\text{két}} \), and is the pullback of a surjective finite Kummer étale morphism in \( X_{\text{két}} \) for all sufficiently large \( \mu \).

**Remark 5.1.3.** There is another version of pro-étale site introduced in [SW17]. But we will not try to introduce the corresponding version of pro-Kummer étale site in this paper.

This definition is justified by the following analogue of [Sch13a] Lem. 3.10:

**Lemma 5.1.4.**

1. Let \( U, V, \) and \( W \) be objects in pro-\( X_{\text{két}} \). Suppose that \( U \to V \) is a Kummer étale (resp. finite Kummer étale, resp. étale, resp. finite étale, resp. pro-Kummer étale, resp. pro-finite Kummer étale) morphism and \( W \to V \) is any morphism. Then the fiber product \( U \times_V W \) exists, and \( U \times_V W \to W \) is Kummer étale (resp. finite Kummer étale, resp. étale, resp. finite étale, resp. pro-Kummer étale, resp. pro-finite Kummer étale). Moreover, the induced map \(|U \times_V W| \to |U| \times_V |W|\) is surjective.

2. A composition of two Kummer étale (resp. finite Kummer étale, resp. étale, resp. finite étale) morphisms in pro-\( X_{\text{két}} \) is still Kummer étale (resp. finite Kummer étale, resp. étale, resp. finite étale).

3. Let \( U \) be an object in pro-\( X_{\text{két}} \), and let \( W \subseteq |U| \) be a quasi-compact open subset. Then there exists an object \( V \) in pro-\( X_{\text{két}} \) with an étale morphism \( V \to U \) such that \(|V| \to |U|\) induces a homeomorphism \(|V| \sim W\). If, in addition, \( U \) is an object in \( X_{\text{prokét}} \), then there exists \( V \) as above that, for any morphism \( V' \to U \) in \( X_{\text{prokét}} \) such that \(|V'| \to |U|\) factors through \( W \), the morphism \( V' \to U \) also factors through \( V \).

4. Pro-Kummer étale morphisms in pro-\( X_{\text{két}} \) are open (i.e., they induce open maps between the underlying topological spaces).

5. Let \( V \) be an object in \( X_{\text{prokét}} \). A surjective Kummer étale (resp. surjective finite Kummer étale) morphism \( U \to V \) in pro-\( X_{\text{két}} \) is the pullback under some morphism \( V \to V_0 \) in pro-\( X_{\text{két}} \) of a surjective Kummer étale (resp. surjective finite Kummer étale) morphism \( U_0 \to V_0 \) in \( X_{\text{két}} \).

6. Let \( W \) be an object in \( X_{\text{prokét}} \), and let \( U \to V \to W \) be pro-Kummer étale (resp. pro-finite Kummer étale) morphisms in pro-\( X_{\text{két}} \). Then \( U \) and \( V \) are also objects in \( X_{\text{prokét}} \), and the composition \( U \to W \) is pro-Kummer étale (resp. pro-finite Kummer étale).

7. Arbitrary finite projective limits exist in \( X_{\text{prokét}} \).

8. Any base change of a covering in \( X_{\text{prokét}} \) is also a covering.

**Proof.** The statements [1]–[7] follow from essentially the same arguments as in the proof of [Sch13a] Lem. 3.10, with inputs from Propositions 2.3.32, 2.3.23 and 2.3.27, Corollary 4.1.9, and Proposition 4.1.14 here.

As for the statement remaining [8], suppose that \( \{U_i \to U\}_{i \in I} \) is a covering of \( U \in X_{\text{prokét}} \), and that \( V \to U \) is any morphism in \( X_{\text{prokét}} \). We need to show that \( \{U_i \times_U V \to V\}_{i \in I} \) is also a covering. Firstly, if \( U_i = \lim_{\mu < \lambda} U_\mu \to U \) is an inverse limit satisfying the conditions in Definition 5.1.2, then so is \( U_i \times_U V = \lim_{\mu < \lambda} (U_\mu \times_U V) \to V \). As for the surjectivity, by working locally \( U \), we may assume that \( U \) is quasi-compact and that \( \{U_i \to U\}_{i \in I} \) is a finite covering. By taking the disjoint union of \( U_i \to U \), we may further reduce to the case where \( I = \{i_0\} \) is a singleton, in which case \(|U_{i_0} \times_U V| \to |V|\) is surjective, by [1]. \( \square \)
Let us also record the following analogue of [Sch13a, Prop. 3.12].

**Proposition 5.1.5.** Let \( X \) be a locally noetherian fs log adic space.

1. Let \( U = \varprojlim U_i \rightarrow X \) be a pro-Kummer étale presentation of \( U \in X_{\text{proké}} \) such that all \( U_i \) are affinoid. Then \( U \) is quasi-compact and quasi-separated.
2. Objects \( U \) as in [1] generates \( X_{\text{proké}} \), and are stable under fiber products.
3. The topos \( X_{\text{proké}} \) is algebraic.
4. An object \( U \in X_{\text{proké}} \) is quasi-compact (resp. quasi-separated) if and only if \( |U| \) is quasi-compact (resp. quasi-separated).
5. Suppose that \( U \rightarrow V \) is an inverse limit of finite Kummer étale surjective morphisms in \( X_{\text{proké}} \). Then \( U \) is quasi-compact (resp. quasi-separated) if and only if \( V \) is.
6. A morphism \( U \rightarrow V \) in \( X_{\text{proké}} \) is quasi-compact (resp. quasi-separated) if and only if \( |U| \rightarrow |V| \) is quasi-compact (resp. quasi-separated).
7. The sheaf \( X_{\text{proké}} \) is quasi-separate (resp. coherent) if and only if \( |X| \) is quasi-separated (resp. coherent).

**Proof.** By Lemma 5.1.4[1], pro-Kummer étale morphisms are open. Hence, the same arguments as in the proof of [Sch13a, Prop. 3.12] also work here. \( \square \)

Let \( \nu : X_{\text{proké}} \rightarrow X_{\text{ké}} \) be the natural projection of sites. We have induced functors \( \nu^{-1} : \text{Sh}(X_{\text{ké}}) \rightarrow \text{Sh}(X_{\text{proké}}) \) and \( \nu_* : \text{Sh}(X_{\text{proké}}) \rightarrow \text{Sh}(X_{\text{ké}}) \).

**Proposition 5.1.6.** Let \( F \) be any abelian sheaf on \( X_{\text{ké}} \), and \( U = \varprojlim U_i \) any qcqs object in \( X_{\text{proké}} \). Then \( H^j(U_{\text{proké}}, \nu^{-1}(F)) = \varprojlim H^j(U_i, \text{ké}, F) \), for all \( j \geq 0 \).

**Proof.** This follows from essentially the same argument as in the proof of [Sch13a, Lem. 3.16], by using Proposition 5.1.5[1] and (2) here. \( \square \)

**Proposition 5.1.7.** For any abelian sheaf \( F \) on \( X_{\text{ké}} \), the canonical morphism \( F \rightarrow R\nu_* \nu^{-1}(F) \) is an isomorphism.

**Proof.** For each \( j \geq 0 \), the sheaf \( R^j \nu_* \nu^{-1}(F) \) on \( X_{\text{ké}} \) is associated with the presheaf \( U \mapsto H^j(U_{\text{proké}}, \nu^{-1}(F)) \). If \( j = 0 \), then \( F \rightarrow \nu_* \nu^{-1}(F) \), because \( H^0(U_{\text{ké}}, F) \rightarrow H^0(U_{\text{proké}}, \nu^{-1}(F)) \), for all qcqs objects \( U \) in \( X_{\text{ké}} \), by Proposition 5.1.6. If \( j > 0 \), essentially by definition, the cohomology \( H^j(U, F) \) vanishes locally in the Kummer étale topology, and hence the associated sheaf \( R^j \nu_* \nu^{-1}(F) \) is zero, as desired. \( \square \)

**Corollary 5.1.8.** The functor \( \nu^{-1} : \text{Sh}_{\text{Ab}}(X_{\text{ké}}) \rightarrow \text{Sh}_{\text{Ab}}(X_{\text{proké}}) \) is fully faithful.

For technical purposes, let us also define the pro-finite Kummer étale site.

**Definition 5.1.9.** The pro-finite Kummer étale site \( X_{\text{proké}} \) has as underlying category the category pro-\( X_{\text{ké}} \), and each covering of \( U \in X_{\text{proké}} \) is given by a family of pro-finite Kummer étale morphisms \( \{ f_i : U_i \rightarrow U \}_{i \in I} \) such that \( |U| = \cup_{i \in I} f_i(|U_i|) \) and such that each \( f_i : U_i \rightarrow U \) can be written as an inverse limit \( U_i = \varprojlim_{\mu < \lambda} U_\mu \rightarrow U \) satisfying the following conditions:

1. Each \( U_\mu \in X_{\text{proké}} \), and \( U = U_0 \) is an initial object in the limit.
2. The limit runs through the set of ordinals \( \mu \) less than some ordinal \( \lambda \).
3. For each \( \mu < \lambda \), the morphism \( U_\mu \rightarrow U_{\mu'} = \varprojlim_{\mu' < \mu} U_{\mu'} \) is the pullback of a finite Kummer étale morphism in \( X_{\text{ké}} \), and is the pullback of a surjective finite Kummer étale morphism for all sufficiently large \( \mu \).
Definition 5.1.10. For each pro-finite group \( G \), the site \( G\text{-PFSets} \) has as underlying category the category of pro-finite sets with continuous actions of \( G \), and each covering of \( S \in G\text{-PFSets} \) is given by a family of continuous \( G \)-equivariant maps \( \{ f_i : S_i \to S \}_{i \in I} \) such that \( S = \bigcup_{i \in I} f_i(S_i) \) and such that each \( f_i : S_i \to S \) can be written as an inverse limit \( S_i = \lim_{\mu < \lambda} S_{\mu} \to S \) satisfying the following conditions:

1. Each \( S_{\mu} \in G\text{-PFSets} \), and \( S = S_0 \) is an initial object in the limit.
2. The limit runs through the set of ordinals \( \mu \) less than some ordinal \( \lambda \).
3. For each \( \mu < \lambda \), the map \( S_{\mu} \to S_{<\mu} := \lim_{\mu' < \mu} S_{\mu'} \) is the pullback of a surjective map of finite sets.

Remark 5.1.11. Since a pro-finite set with a continuous action of a pro-finite group \( G \) is equivalent to an inverse limit of finite sets with continuous \( G \)-actions, we have a canonical equivalence of categories \( G\text{-PFSets} \cong \text{pro}(G\text{-FSets}) \).

Proposition 5.1.12. Let \( X \) be a connected locally noetherian fs log adic space, and let \( \zeta \) be a log geometric point of \( X \). Then there is an equivalence of sites

\[ X_{\text{prokét}} \cong \pi^1_{\text{két}}(X, \zeta)\text{-PFSets} \]

sending \( U = \lim_{i} U_i \to X \) to \( S(U) := \lim_{i} \text{Hom}_X(\zeta, U_i) \).

Proof. By Corollary 4.4.18, \( X_{\text{prokét}} \cong \pi^1_{\text{két}}(X, \zeta)\text{-FSets} \), and hence the composition \( X_{\text{prokét}} = \text{pro} \circ X_{\text{prokét}} \cong \text{pro}(\pi^1_{\text{két}}(X, \zeta)\text{-FSets}) \cong \pi^1_{\text{két}}(X, \zeta)\text{-PFSets} \) sends \( U \) to \( S(U) \). By comparing definitions, this isomorphism also matches the coverings. \( \square \)

5.2. Localization and base change functors. For any morphism \( f : Y \to X \) of locally noetherian fs log adic spaces, by the same explanations as in Section 4.5 we have a morphism of topoi \( (f^{-1}, f_*): Y_{\text{prokét}} \to X_{\text{prokét}} \).

Proposition 5.2.1. Let \( f : Y \to X \) be a qcqs morphism of locally noetherian fs log adic spaces. Let \( \nu^{-1}_X : \text{Sh}(X_{\text{két}}) \to \text{Sh}(X_{\text{prokét}}) \) and \( \nu^{-1}_Y : \text{Sh}(Y_{\text{két}}) \to \text{Sh}(Y_{\text{prokét}}) \) denote the natural functors. Then, for any abelian sheaf \( F \) on \( Y_{\text{két}} \), we have a natural isomorphism \( \nu^{-1}_X Rf_*(F) \sim Rf_* \nu^{-1}_Y (F) \).

Proof. This is because, for each \( i \geq 0 \), the \( i \)-th cohomology of both sides can be identified with the sheafification of the presheaf sending a qcqs object \( U = \lim_{j} U_j \) in \( X_{\text{prokét}} \) to \( \lim_{j} H^i(U_j \times_Y X, F) \). \( \square \)

When \( Y \in X_{\text{prokét}} \), let \( X_{\text{prokét}}/Y \) denote the localized site. Then we have the following natural functors:

1. The inverse image (or pullback) functor
\[ f^{-1} : \text{Sh}(X_{\text{prokét}}) \to \text{Sh}(X_{\text{prokét}}/Y) : F \mapsto (U \mapsto f^{-1}(F)(U) := F(U)) \]
2. The base change functor
\[ f_* : \text{Sh}(X_{\text{prokét}}/Y) \to \text{Sh}(X_{\text{prokét}}) : F \mapsto (U \mapsto f_*(F)(U) := F(U \times_Y Y)) \]
3. The localization functor
\[ f_l : \text{Sh}(X_{\text{prokét}}/Y) \to \text{Sh}(X_{\text{prokét}}) \]

sending \( F \in \text{Sh}(X_{\text{prokét}}/Y) \) to the sheafification of the presheaf \( f_l(F) : U \to \coprod_{U \to Y} F(U, \alpha) \), where the coproduct is over all pro-Kummer étale morphisms \( h : U \to Y \) over \( X \). We also denote by \( f_l : \text{Sh}_{\text{Ab}}(X_{\text{prokét}}/Y) \to \)
Sh_{\text{Ab}}(X_{\text{prokét}}) the induced functor between the categories of abelian sheaves, in which case the above coproduct becomes a direct sum.

It is formal that $f_!$ is left adjoint to $f^{-1}$, and hence $f_*$ is right exact.

**Remark 5.2.2.** If $Y \rightarrow X$ is Kummer étale, then naturally $X_{\text{prokét}}/Y \cong Y_{\text{prokét}}$.

**Lemma 5.2.3.** Let $f : V \rightarrow W$ be a finite Kummer étale morphism between objects in $X_{\text{prokét}}$. If $f$ has a section $g : W \rightarrow V$, then there exists a finite Kummer étale morphism $W' \rightarrow W$ between objects in $X_{\text{prokét}}$, and an isomorphism $h : V \xrightarrow{\sim} W \coprod W'$ such that the composition $h \circ g$ is the natural inclusion $W \hookrightarrow W \coprod W'$.

**Proof.** By Lemma 5.1.4, we may assume that $f : V \rightarrow W$ is the pullback of some finite Kummer étale morphism $V_0 \rightarrow W_0$ in $X_{\text{prokét}}$. Let $W = \varprojlim V_i$ be a pro-Kummer étale presentation. Without loss of generality, we may assume that all transition morphisms $W_j \rightarrow W_0$ are finite Kummer étale, and that $W \rightarrow W_0$ factors through $W_i \rightarrow W_0$ for all $i$, so that $V \cong \varprojlim_i (W_i \times W_0)$. Then we may replace $W_0$ with some $W_i$ and assume that $W \rightarrow W_0$ and hence $V \rightarrow W_0$ are pro-finite Kummer étale. We may also assume that $W_0$ is connected. Let $G := \pi_{1,\text{prokét}}(W_0)$ (see Remark 4.4.21). By Proposition 5.1.12 $V \rightarrow W_0$ and $W \rightarrow W_0$ correspond to pro-finite sets $S$ and $S_0$ with continuous $G$-actions, and the morphism $f : V \rightarrow W$ and the splitting $g : W \rightarrow V$ gives rise to a $G$-equivariant decomposition $S \cong S_0 \coprod S'$ and hence the desired $h : V \xrightarrow{\sim} W \coprod W'$ (cf. the proof of Lemma 4.1.1).

**Proposition 5.2.4.** Let $f : Y \rightarrow X$ be a finite Kummer étale morphism of locally noetherian fs log adic spaces. Then we have a natural isomorphism $f_! \xrightarrow{\sim} f_* : \text{Sh}_{\text{Ab}}(Y_{\text{prokét}}) \rightarrow \text{Sh}_{\text{Ab}}(X_{\text{prokét}})$. Consequently, both functors are exact.

**Proof.** For each $U \in X_{\text{prokét}}$, any pro-Kummer étale morphism $h : U \rightarrow Y$ over $X$ induces a splitting $U \rightarrow Y \times_X U$ of the finite Kummer étale morphism $Y \times_X U \rightarrow U$, and hence we have a decomposition $Y \times_X U \cong U \coprod U'$, by Lemma 5.2.3. Then we can finish the proof by the same arguments as in the proof of Proposition 4.5.

### 5.3. Log affinoid perfectoid objects.

Recall that affinoid perfectoid objects form a basis for the pro-étale topology of any locally noetherian adic space over $\text{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ (see [Sch13a] Def. 4.3 and Prop. 4.8 and [Sch17] Lem. 15.3). We would like to establish a suitable log analogue of this fact.

**Definition 5.3.1.** Let $X$ be an analytic locally noetherian fs log adic space over $\text{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)$. An object $U$ in $X_{\text{prokét}}$ is called log affinoid perfectoid if it admits a pro-Kummer étale presentation

$$U = \varprojlim U_i = \varprojlim \text{Spa}(R_i, R_i^+, \mathcal{M}_i, \alpha_i) \rightarrow X$$

satisfying the following conditions:

1. There is an initial object $0 \in I$.
2. Each $U_i$ admits a global sharp fs chart $P_i$ such that each transition morphism $U_j \rightarrow U_i$ is modeled on a Kummer chart $P_i \rightarrow P_j$.
3. The affinoid algebra $(R, R^+) := (\varprojlim_{i \in I} (R_i, R_i^{u+}))^{\wedge}$, where each $(R_i, R_i^{u+})$ is the uniformization of $(R_i, R_i^+)$ as in [KL15] Def. 2.8.13 (i.e., $R_i^{u}$ is the completion of $R_i$ for the spectral seminorm and $R_i^{u+}$ is the completion of the image of $R_i^+$ in $R_i^{u}$) and where the completion is as in [KL15] Def. 2.6.1, is a perfectoid affinoid algebra.
(4) The monoid \( P := \lim_{\leftarrow i} P_i \) is \( n \)-divisible, for all \( n \geq 1 \).

In this situation, we say that \( U = \lim_{\leftarrow i} U_i \) is a perfectoid presentation of \( U \).

The following remark provides an equivalent form of Definition 5.3.1 when \( X \) is defined over \( \text{Spa}(\mathbb{Q}_p, \mathbb{Z}_p) \), and explains the compatibility of the definitions of affinoid perfectoid objects in [Sch13a Def. 4.3] and [KL15 Def. 9.2.4].

Remark 5.3.2. In Definition 5.3.1 suppose that \( X \) is defined over \( \text{Spa}(\mathbb{Q}_p, \mathbb{Z}_p) \). In this case, note that \( p \) is invertible in \( \ker(R_i^+ \to R_i^{u+}) \). It follows that \( R_i^+/p^n \cong R_i^{u+}/p^n \), for each \( n \geq 1 \). Then the completion \( (R,R^+) = (\lim_{\leftarrow i} (R_i^u,R_i^{u+}))^\wedge \) is simply the \( p \)-adic completion of \( \lim_{\leftarrow i} (R_i,R_i^+) \).

Remark 5.3.3. By Proposition 5.1.5 a log affinoid perfectoid object \( U \) as in Definition 5.3.1 is qcqs. By abuse of language, we shall sometimes say that \( U \) is modeled on \( P \). Since \( P \) is sharp and saturated, the condition (4) in Definition 5.3.1 is equivalent to the condition that \( P \) is uniquely \( n \)-divisible, for all \( n \geq 1 \).

Lemma 5.3.4. Let \( P \) be a sharp fs monoid. Suppose that \( X \) is an analytic locally noetherian adic space over \( \text{Spa}(\mathbb{Z}_p, \mathbb{Z}_p) \) equipped with the trivial log structure as in Example 2.2.7 and that \( Y = X(P) \) as in Example 2.2.19. Suppose that \( \lim_{\leftarrow i} U_i \) is an affinoid perfectoid object in \( X_{\text{prokrét}} \), which exists by [Sch17 Lem. 15.3], where all \( U_i \) are equipped with the trivial log structures as well. Let us equip \( I \times \mathbb{Z}_{\geq 1} \) with the partial ordering such that \((i,m) \geq (j,n)\) exact when \( i \geq j \) and \( n|m \). Then \( \lim_{\leftarrow ((i,n) \in I \times \mathbb{Z}_{\geq 1})} U_i(\frac{1}{n}P) \) is a log perfectoid affinoid object in \( Y_{\text{prokrét}} \), which gives a pro-Kummer étale (resp. pro-finite Kummer étale) cover of \( Y \) when \( \lim_{\leftarrow i} U_i \) is a pro-étale (resp. pro-finite étale) cover of \( X \).

Proof. This follows from Lemmas 2.2.19 and 5.1.4 and Definition 5.3.1. \( \square \)

Remark 5.3.5. Let \( U \in X_{\text{prokrét}} \) be a log affinoid perfectoid object as in Definition 5.3.1. Then \( \hat{U} = \text{Spa}(R,R^+) \) is an affinoid perfectoid space, called the associated affinoid perfectoid space, or simply the associated perfectoid space. In this case, we write \( \hat{U} \sim \lim_{\leftarrow i} U_i \). The assignment \( U \mapsto \hat{U} \) defines a functor from the category of log affinoid perfectoid objects to the category of affinoid perfectoid spaces. We emphasize that \( \hat{U} \) does not live in \( X_{\text{prokrét}} \). Thanks to the following Lemma 5.3.6 we can identify the underlying topological spaces of \( \hat{U} \) and \( \lim_{\leftarrow i} U_i \).

Lemma 5.3.6. Let \( U = \lim_{\leftarrow i} U_i \in X_{\text{prokrét}} \) be a log affinoid perfectoid object, and let \( \hat{U} \) be the associated affinoid perfectoid space, as in Remark 5.3.5. Then the natural map of topological spaces \( \hat{U} \to \lim_{\leftarrow i} U_i \) is a homeomorphism.

Proof. The map is bijective because a continuous valuation on \( R \) is equivalent to a compatible system of continuous valuations on \( R_i \)'s. By [KL15 Lem. 2.6.5], each rational subset of \( \hat{U} \) comes from the pullback of a rational subset of some \( U_i \), and hence the topologies also agree, as desired. \( \square \)

Lemma 5.3.7. Let \( \iota : Z \to X \) be a strict closed immersion (see Definition 2.2.23) of analytic locally noetherian fs log adic spaces over \( \text{Spa}(\mathbb{Z}_p, \mathbb{Z}_p) \). Then, for each log affinoid perfectoid \( U = \lim_{\leftarrow i \in I} U_i \in X_{\text{prokrét}} \), the pullback \( V := U \times_X Z := \lim_{\leftarrow i \in I} (U_i \times_X Z) \) is a log affinoid perfectoid object in \( Z_{\text{prokrét}} \). Moreover, the natural morphism \( \hat{V} \to \hat{U} \) is a closed immersion of adic spaces.
Proof. By definition and by Proposition 4.1.14, the conditions (1), (2), and (4) in Definition 5.3.1 are satisfied. It remains to verify the condition (3). Let \((R, R^+)\) be the completion of \(\lim_{\longleftarrow i \in I} (R^i, R^{i+})\), which is perfectoid by assumption. For each \(i \in I\), write \(\text{Spa}(R_i, R^{i+}) \times_X Z = \text{Spa}(S_i, S^{i+})\). Then the induced homomorphism \(R^i_0 \to S^i_0\) is surjective, because \(i : Z \to X\) is strict and hence the underlying adic space of the fiber product coincides with the fiber product of the underlying adic spaces. Since \(R^i_0\) and \(S^i_0\) are uniform, the quotient norm on \(S^i_0\) induced by the one on \(R^i_0\) is just the (spectral) norm on \(S^i_0\). Let \((S, S^+)\) be the completion of \(\lim_{\longleftarrow i \in I} (S^i_0, S^{i+})\). Then \(S\) is uniform and admits a surjective bounded homomorphism \(R \to S\). In this case, \(S\) is perfectoid, by [KL15 Thm. 3.6.17(b)], and the natural morphism \(\text{Spa}(S, S^+) \to \text{Spa}(R, R^+)\) is a closed immersion, as desired. \(\square\)

Lemma 5.3.8. Let \(U = \lim_{\longleftarrow i \in I} U_i \in X_{\text{prokét}}\) be a log affinoid perfectoid object, with associated perfectoid space \(\tilde{U}\), as in Remark 5.3.5. Suppose that \(V \to U\) is a Kummer étale (resp. finite Kummer étale) morphism in \(X_{\text{prokét}}\) that is the pullback of some Kummer étale (resp. finite Kummer étale) morphism \(V_0 \to U_0\) between affinoid log adic spaces in \(X_{két}\). Then \(V \to U\) is étale (resp. finite étale), and \(V\) is log affinoid perfectoid. The induced morphism \(\tilde{V} \to \tilde{U}\) is étale (resp. finite étale), and the construction \(V \mapsto \tilde{V}\) induces an equivalence of topoi \(\tilde{U}_{\text{prokét}} \cong X^\sim_{\text{prokét}/U}\).

Proof. We may assume that \(0 \in I\) and that \(0\) is an initial object (up to replacing \(I\) with a cofinal subcategory). Let \(U_0 = \lim_{\longleftarrow i \in I} U_i\) be as in Lemma 4.2.5 such that \(V_0 \times_{U_0} U_i^{\sim} \to U_0^{\sim}\) is strictly étale (resp. strictly finite étale). Since \(P = \lim_{\longleftarrow i \in I} P_i\) is \(m\)-divisible, there is some \(i \in I\) such that \(P_0 \to P_i\) factors as \(P_0 \to \frac{1}{n} P_0 \to P_i\). Then \(V_0 \times_{U_0} U_i \to U_i\) is strictly étale (resp. strictly finite étale). We may replace \(I\) with the cofinal full subcategory of objects that receive morphisms from \(i\). Then \(V := (V_0 \times_{U_0} U_i) \times_{U_i} U \to U\) is strictly étale (resp. strictly finite étale), and hence so is \(\tilde{V} \to \tilde{U}\). This shows that we have a well-defined morphism of sites \(\tilde{U}_{\text{prokét}} \to X_{\text{prokét}/U}\). This induces an equivalence of topos, because every étale morphism \(W \to \tilde{U}\) that is a composition of rational localizations and finite étale morphisms arises in the above way, by [KL15 Lem. 2.6.5 and Prop. 2.6.8]. \(\square\)

Corollary 5.3.9. Let \(U = \lim_{\longleftarrow i \in I} U_i\) be an object in \(X_{\text{prokét}}\) as in Definition 5.3.1 such that \(U_i \to X\) is a composition of rational localizations and finite Kummer étale morphisms, for all sufficiently large \(i\). Then \(U \times_X V\) is a log affinoid perfectoid of \(X_{\text{prokét}},\) for each log affinoid perfectoid object \(V\) of \(X_{\text{prokét}}\).

Proof. By Lemma 5.3.8, \(U_i \times_X V\) is log affinoid perfectoid, for all sufficiently large \(i\). Hence, \(U \times_X V \cong \lim_{\longleftarrow i \in I} (U_i \times_X V)\) is also log affinoid perfectoid, because the \(p\)-adic completion of a direct limit of perfectoid affinoid algebras is again perfectoid. \(\square\)

Lemma 5.3.10. Let \(U\) and \(V\) be log affinoid perfectoid objects as in Definition 5.3.1, with a morphism \(V \to U\), in \(X_{\text{prokét}}\). Suppose that \(U = \lim_{\longleftarrow i \in I} U_i\) is a pro-Kummer étale presentation with \(U_i\) and \(U\) modeled on \(P_i\) and \(P = \lim_{\longleftarrow i \in I} P_i\), respectively. Then \(V\) admits a pro-Kummer étale presentation \(V = \lim_{\longleftarrow j \in J} V_j\) with each \(V_h\) modeled on some \(P_i\), so that \(V\) is also modeled on \(P\).

Proof. Let \(V = \lim_{\longleftarrow h \in H} V_h\) be a pro-Kummer étale presentation. For each \(i\), the morphism \(V \to U_i\) in \(X_{\text{prokét}}\) factors through some morphism \(V_h \to U_i\) in \(X_{két}\), for
all sufficiently large $h$. For each such $(i, h)$, by the argument in the proof of Lemma 5.3.8 there is some $i, h \in I$ such that $V_h \times_{U_h} U_h \to U_t$ is étale for all $t \geq i, h$, in which case $V_h \times_{U_h} U_h$ is modeled on $P_t$. Hence, we obtain the desired presentation $V = \varprojlim_{j \in J} V_j$ by considering the index category $J$ formed by $j = (i, h, t)$ such that $t \geq i, h$, with the partial ordering such that $(i, h, t) \geq (i', h', t')$ in $J$ exactly when $i \geq i'$, $h \geq h'$, and $t \geq t'$; and by taking $V_j := V_h \times_{U_h} U_t$ for each $j = (i, h, t) \in J$. □

**Proposition 5.3.11.** Let $X$ be an analytic locally noetherian fs log adic space over $\text{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)$. Then the subcategory of log affinoid perfectoid objects in $X_{\text{prokét}}$ is stable under fiber products.

**Proof.** Let $U$, $V$, and $W$ be log affinoid perfectoid objects, with morphisms $V \to U$ and $W \to U$, in $X_{\text{prokét}}$. Let $U = \varprojlim_i U_i$ be as in Lemma 5.3.10. By Lemma 5.3.10, $V$ admits a pro-Kummer étale presentation $V = \varprojlim_i V_j$ such that each $V_j$ is modeled on some $P_i$, and the same is true for $W$. Consequently, $V \times_U W$ also admits a pro-Kummer étale presentation of this kind, and hence is log affinoid perfectoid, with associated perfectoid space $V \times_U W \cong \hat{V} \times_\hat{U} \hat{W}$. (The last fiber product is indeed a perfectoid space by [Sch12, Prop. 6.18].) □

**Proposition 5.3.12.** Let $X$ be an analytic locally noetherian fs log adic space over $\text{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)$. Then the log affinoid perfectoid objects in $X_{\text{prokét}}$ form a basis.

**Proof.** We need to show that, for each $U = \varprojlim_{i \in I} U_i \in X_{\text{prokét}}$, étale locally on $U$ and $X$, there exists a pro-Kummer étale cover of $U$ by log affinoid perfectoid objects; and we may assume that it has a final object $U_0$ such that $U_i \to U_0$ is finite Kummer étale, for all $i \in I$, and that $U_0 \to X$ is a composition of rational localizations and finite Kummer étale morphisms. By Lemma 5.1.4 and [Sch13a, Prop. 4.8], we may assume that $X = \text{Spa}(R, R^+)$ is affinoid and that its underlying adic space admits a pro-étale cover by an affinoid perfectoid object $\varprojlim_j U_j$ in $X_{\text{proét}}$. Also, we may assume that $X$ admits a sharp fs chart $P_X \to \hat{M}$, which induces a strict closed immersion $X \hookrightarrow Y := X(P)$ as in Remark 2.3.3. Consider the pro-Kummer étale cover of $Y$, as in Lemma 5.3.4, given by the log affinoid perfectoid object $\varprojlim_{j \in J} U_j \langle \frac{1}{n} \rangle P$ in $Y_{\text{prokét}}$, whose pullback to $X$ gives a pro-Kummer étale cover of $X$ by a log perfectoid affinoid object $V$ of $X_{\text{prokét}}$, by Lemma 5.3.7. Thus, $U \times_X V \to U$ is a desired pro-Kummer étale cover of $U$ by a log affinoid perfectoid object in $X_{\text{prokét}}$, by Lemma 5.1.4 and Corollary 5.3.9. □

**Proposition 5.3.13.** Let $X$ be an analytic locally noetherian fs log adic space over $\text{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)$. Then $X_{\text{prokét}}$ has a basis $B$ such that $H^i(X_{\text{prokét}}, \nu^{-1}(L)) = 0$, for all $V \in B$, all $p$-torsion locally constant sheaf $\nu$ on $X_{\text{két}}$, and all $i > 0$.

**Proof.** Let $U$ be a log affinoid perfectoid object of $X_{\text{prokét}}$, and let $\hat{U} = \text{Spa}(A, A^+)$ denote the associated affinoid perfectoid space. By passing to a covering, we may assume that $A$ is integral. Let $(A_\infty, A_\infty^+)$ be a universal cover of $A$ (i.e., $A_\infty$ is the union of all finite étale extensions $A_j$ of $A$ in a fixed algebraic closure of the fractional field of $A$, and $A_\infty^+$ is the integral closure of $A^+$ in $A_\infty$). Let $(\hat{A}_\infty, \hat{A}_\infty^+) := (\varprojlim_j (A_j, A_j^+))^\wedge$, and $\hat{U}_\infty := \text{Spa}(\hat{A}_\infty, \hat{A}_\infty^+)$. Then this $\hat{U}_\infty$ is affinoid perfectoid. By the argument in the proof of Lemma 5.3.8, there is some $V \to U$ in $X_{\text{prokét}}$, with $V = \varprojlim_j V_j$ log affinoid perfectoid, such that $\hat{V} \cong \hat{U} \to \hat{U}$. Note that $L|_V$ is a trivial local system because, for any finite Kummer étale cover $Y \to$
X trivializing \( \mathbb{L} \), the pullback \( W := Y \times_X V \to V \) and the induced morphism \( \tilde{W} \to \tilde{V} \) are strictly finite étale by Lemma \[5.3.8\] and so \( \tilde{W} \to \tilde{V} \) has a section by assumption. Consequently, we have \( H^i(X_{\text{pro-két}}/V, \nu^{-1}(\mathbb{L})) \cong H^i(\tilde{V}_{\text{pro-két}}, \nu^{-1}(\mathbb{L})) \cong H^i(\tilde{V}_0, \mathbb{L}) = 0 \), for all \( i > 0 \), where the first and second isomorphisms follow from Lemma \[5.3.8\] and \[Sch13a\, Cor. 3.17(i)\] (note that the locally noetherian assumption there on \( X \) is not needed), respectively, and the last equality follows essentially verbatim from the last paragraph of the proof of \[Sch13a\, Thm. 4.9\]. \( \square \)

### 5.4. Structure sheaves.

**Definition 5.4.1.** Let \( X \) be a locally noetherian fs log adic space over \( \text{Spa}(\mathbb{Q}_p, \mathbb{Z}_p) \).

We define the following sheaves on \( X_{\text{pro-két}} \).

1. **The integral structure sheaf** is \( \mathcal{O}^+_X := \nu^{-1}(\mathcal{O}^+_X) \), and the structure sheaf is \( \mathcal{O}_{X_{\text{pro-két}}} := \nu^{-1}(\mathcal{O}_{X_{\text{két}}}) \).
2. **The completed integral structure sheaf** is \( \tilde{\mathcal{O}}^+_X := \varprojlim \{ \mathcal{O}^+_X/p^n \} \), and the completed structure sheaf is \( \tilde{\mathcal{O}}_{X_{\text{pro-két}}} := \varprojlim \{ \mathcal{O}_{X_{\text{pro-két}}}/p^n \} \).
3. **The tilted structure sheaves** are \( \tilde{\mathcal{O}}^+_X := \varprojlim \{ \mathcal{O}^+_X \} \) and \( \tilde{\mathcal{O}}_{X_{\text{pro-két}}} := \varprojlim \{ \mathcal{O}_{X_{\text{pro-két}}} \} \), where the transition morphisms \( \Phi \) are given by \( x \mapsto x^p \). When the context is clear, we shall simply write \( \tilde{\mathcal{O}}^+ \) instead of \( \varprojlim \{ \mathcal{O}^+_X \} \).
4. We have \( \alpha : \mathcal{M}_{X_{\text{pro-két}}} := \nu^{-1}(\mathcal{M}_{X_{\text{két}}}) \to \mathcal{O}_{X_{\text{pro-két}}} \) and \( \alpha^p : \mathcal{M}_{X_{\text{pro-két}}} := \lim_{\varprojlim} \mathcal{M}_{X_{\text{pro-két}}} \to \tilde{\mathcal{O}}_{X_{\text{pro-két}}} \). When the context is clear, we shall simply write \( \mathcal{M} \) instead of \( \lim_{\varprojlim} \mathcal{M}_{X_{\text{pro-két}}} \) and \( \mathcal{M}^p \) instead of \( \mathcal{M}_{X_{\text{pro-két}}} \), respectively.

**Proposition 5.4.2.** In Definition \[5.4.1\] we have \( \mathcal{M}_{X_{\text{pro-két}}}(U) = \lim_{\varprojlim} \mathcal{M}_U(U_i) \), for any pro-Kummer étale presentation \( U = \lim_{\varprojlim} U_i \in X_{\text{pro-két}} \).

**Proof.** The proof is similar to the one of Proposition \[5.1.6\]. Note that \( \nu^{-1}(\mathcal{M}_{X_{\text{pro-két}}}) \) is the sheaf associated with the presheaf \( \mathcal{M} \) sending \( U = \lim_{\varprojlim} U_j \) to \( \lim_{\varprojlim} \mathcal{M}_U(U_j) \). Also, quasi-compact objects form a basis of \( X_{\text{pro-két}} \). Hence, it suffices to prove that, for any quasi-compact \( U \) and any finite covering \( \{ V_h \to U \}_h \) by quasi-compact objects in \( X_{\text{pro-két}} \), the complex \( 0 \to \mathcal{M}(U) \to \prod_h \mathcal{M}(V_h) \to \prod_{h,b} \mathcal{M}(V_h \times_U V_b) \) is exact. By the same argument as in the proof of Proposition \[5.1.6\], this is reduced to the case of a single Kummer étale cover \( V \to U \), and hence to the exactness of \( 0 \to \mathcal{M}_X(U_0) \to \mathcal{M}_X(V_0) \to \mathcal{M}_X(V_0 \times_{U_0} V_0) \), for some Kummer étale cover \( V_0 \to U_0 \) in \( X_{\text{két}} \), which finally follows from Proposition \[4.3.4\]. \( \square \)

The following result is an analogue of \[Sch13a\, Thm. 4.10\].

**Theorem 5.4.3.** Let \( X \) be a locally noetherian fs log adic space over \( \text{Spa}(\mathbb{Q}_p, \mathbb{Z}_p) \). Let \( U \in X_{\text{pro-két}} \) be a log affinoid perfectoid object, with associated perfectoid space \( \tilde{U} = \text{Spa}(\mathbb{R}, \mathbb{R}^+) \). Let \( (\mathbb{R}^+, \mathbb{R}^+) \) be its tilt.

1. For each \( n > 0 \), we have \( \mathcal{O}^+_{X_{\text{pro-két}}}(U)/p^n \cong R^+/p^n \), and it is canonically almost isomorphic to \( (\mathcal{O}^+_{X_{\text{pro-két}}}/p^n)(U) \).
2. For each \( n > 0 \), we have \( H^i(U, \mathcal{O}^+_{X_{\text{pro-két}}}/p^n)^a = 0 \), for all \( i > 0 \). Consequently, \( H^i(U, \tilde{\mathcal{O}}^+_{X_{\text{pro-két}}})^a = 0 \).
(3) We have $\hat{\mathcal{O}}^+_{X_{prokét}}(U) \cong R^+$ and $\hat{\mathcal{O}}_{X_{prokét}}(U) \cong R$, and the ring $\hat{\mathcal{O}}^+_{X_{prokét}}(U)$ is canonically isomorphic to the $p$-adic completion of $\mathcal{O}^+_{X_{prokét}}(U)$.

(4) We have $\hat{\mathcal{O}}^+_{X_{prokét}}(U) \cong R^+$ and $\hat{\mathcal{O}}_{X_{prokét}}(U) \cong R^\circ$.

(5) We have $H^i(U, \hat{\mathcal{O}}^+_{X_{prokét}})^a = 0$, for all $i > 0$.

Proof. Let us temporarily write the subscripts “prokét” etc from $\mathcal{O}^+_{X_{prokét}}$ etc.

Let us first prove (1) and (2). By definition, $\mathcal{O}^+_X(U)/p^n \cong R^+/p^n$. By Proposition 5.3.12, giving a sheaf on $X_{prokét}$ is equivalent to giving a presheaf on the full subcategory of log affinoid perfectoid objects $U$ in $X_{prokét}$, satisfying the sheaf property for pro-Kummer étale coverings by such objects. Consider such a presheaf of almost $R^+$-algebras $\mathcal{F}$ given by assigning $\mathcal{F}(U) := (\mathcal{O}^+_X(U)/p^n)^a$ to each such $U$. We claim that $\mathcal{F}$ is a sheaf with cohomology vanishing above degree zero. By arguing as in the proof of Proposition 5.1.6, it suffices to verify the exactness of the Čech complex $0 \to \mathcal{F}(U) \to \mathcal{F}(V) \to \mathcal{F}(V \times_U V) \to \mathcal{F}(V \times_U V \times_U V) \to \cdots$ for some Kummer étale cover $V \to U$ in $X_{prokét}$ that is the pullback of a Kummer étale cover $V_0 \to U_0$ in $X_{két}$. Furthermore, we may assume that $V_0 \to U_0$ is a composition of finite Kummer étale morphisms and rational localizations. By Lemma 3.1.8, $V$ is log affinoid perfectoid, and $V$ is étale over $U$. Moreover, $\mathcal{F}(U) = (\mathcal{O}^+_X(U)/p^n)^a \cong (\mathcal{O}^+_X(U)/p^n)^{a}$ and $\mathcal{F}(V \times_U V) = (\mathcal{O}^+_X(V \times_U V)/p^n)^a \cong (\mathcal{O}^+_X(V \times_U V)/p^n)^{a}$.

Hence, (1) and the first statement of (2) follows from the almost exactness of $0 \to \mathcal{O}^+_U(U)/p^n \to \mathcal{O}^+_U(V)/p^n \to \mathcal{O}^+_U(V \times_U V)/p^n \to \cdots$, by [Sch12, Thm. 7.13] and the $p$-torsionfreeness of $\mathcal{O}^+_U$. From these, by [Sch13a, Lem. 3.18], the remaining statement of (2) also follows.

As for (3), we first show that, for any $n > m$, the image of $(\mathcal{O}^+_X/p^n)(U) \to (\mathcal{O}^+_X/p^m)(U)$ is equal to $R^+/p^m$ (under the canonical isomorphisms). For each $f \in (\mathcal{O}^+_X/p^n)(U)$, there exists some $g \in R^+$ such that $p^{n-m}f = g$ in $(\mathcal{O}^+_X/p^m)(U)$. Let $h = g/p^{n-m} \in R^+$. Since the multiplication by $p^{n-m}$ induces an injection $(\mathcal{O}^+_X/p^n)(U) \to (\mathcal{O}^+_X/p^m)(U)$, it follows that $f = h$ in $(\mathcal{O}^+_X/p^m)(U)$. Therefore, the image of $f$ under $(\mathcal{O}^+_X/p^m)(U) \to (\mathcal{O}^+_X/p^m)(U)$ lands in $R^+/p^m$, as desired. Consequently, we have $\mathcal{O}^+_X(U) \cong \varprojlim_{\mathbb{Z}/p} (\mathcal{O}^+_X/p^n)(U) = \varprojlim_{\mathbb{Z}/p} R^+/p^n \cong R^+$, and hence $\hat{\mathcal{O}}_{X_{prokét}}(U) = R$.

Next, let us prove (5) and an almost version of (4). Let $\mathcal{G} := \mathcal{O}^+_X/p$. By Proposition 5.4.3, $H^i(U_{prokét}, G)^a = 0$, for all log affinoid perfectoid $U \in X_{prokét}$ and $i > 0$. Moreover, $\mathcal{G}(U)^a \cong (\mathcal{O}^+_X(U)/p)^a$, for any such $U$. By definition, $\hat{\mathcal{O}}^+_X(U) \cong \lim_{\mathbb{Z}/p} \mathcal{G}$. Let $\mathcal{B}$ be the basis of $X_{prokét}$ formed by log affinoid perfectoid objects. By applying [Sch13a, Lem. 3.18] to the sheaf $\mathcal{G}$ and the basis $\mathcal{B}$, we know that $R^j\lim_{\mathbb{Z}/p} \mathcal{G}$ is almost zero, for all $j > 0$, and there are almost isomorphisms $\hat{\mathcal{O}}^+_X(U) \cong (\varprojlim_{\mathbb{Z}/p} \mathcal{G}(U)) \cong \varprojlim_{\mathbb{Z}/p} (\mathcal{G}(U)) \cong \varprojlim_{\mathbb{Z}/p} (R^+/p) \cong R^\circ$. By [Sch13a, Lem. 3.18] again, $H^i(U_{prokét}, \hat{\mathcal{O}}^+_X)^a \cong H^i(U_{prokét}, \varprojlim_{\mathbb{Z}/p} \mathcal{G})^a = 0$, for all $i > 0$.

Finally, let us prove (4). Consider the sheaf associated with the presheaf $\mathcal{H}$ on $X_{prokét}$ determined by $\mathcal{H}(U) = \mathcal{O}^+_U(\hat{U})$, for each $U \in \mathcal{B}$. It suffices to show that $\mathcal{H}$ satisfies the sheaf property for coverings by objects in $\mathcal{B}$. Let $U$ and $V$ be log affinoid perfectoid objects in $X_{prokét}$, and let $V \to U$ be a pro-Kummer étale cover. Let $R$, $S$, and $T$ be the perfectoid algebras associated with $U$, $V$, and $U \times_U V$, respectively. Then it suffices to show the exactness of $0 \to R^\circ \to S^\circ \to T^\circ$.
Note that this is the inverse limit (along Frobenius) of $0 \to R^+/p \to S^+/p \to T^+/p$, and this last sequence is exact by the fact that $\mathcal{O}_{\tilde{U}_{\ker}}^{+}$ is a sheaf and $p$-torsion free. Thus, the desired exactness follows from the vanishing of $R^1 \varprojlim (R^+/p)$. □

The following proposition is an analogue of [KL15, Thm. 9.2.15].

**Theorem 5.4.4.** Let $X$ be a locally noetherian fs log adic space over $\text{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. Let $U$ be a log affinoid perfectoid object of $X_{\prokét}$. The functor $\mathcal{H} \mapsto H := \mathcal{H}(\tilde{X})$ is an equivalence from the category of finite locally free $\hat{\mathcal{O}}_{X_{\prokét} \mid U}$-modules on $X_{\prokét \mid U}$ to the category of finite projective $\hat{\mathcal{O}}_{X_{\prokét} \mid U}$-modules, with a quasi-inverse given by $H \mapsto \mathcal{H}(V) := H \otimes_{\hat{\mathcal{O}}_{X_{\prokét} \mid U}} \hat{\mathcal{O}}_{X_{\prokét} \mid U}(V)$, for each log affinoid perfectoid object $V$ in $X_{\prokét}$ over $U$. Moreover, for each finite locally free $\hat{\mathcal{O}}_{X_{\prokét} \mid U}$-module $\mathcal{H}$, we have $H^i(X_{\prokét \mid U}, \mathcal{H}) = 0$, for all $i > 0$.

**Proof.** By Lemma 5.3.8, the first statement follows from [KL15, Thm. 9.2.15]. The proof of the second statement is similar to that of [LZ17, Prop. 2.3], with the input of [KL15, Lem. 2.6.5(a)] replaced with Lemma 5.1.49 here. □

Let us also record the following consequence of Lemma 5.3.7 and Theorem 5.4.3:

**Proposition 5.4.5.** Let $i : Z \to X$ be a strict closed immersion of locally noetherian fs log adic spaces over $\text{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. Then the natural morphism $\hat{\mathcal{O}}_{X_{\prokét}} \to i_{\prokét \ast} (\hat{\mathcal{O}}_{Z_{\prokét}})$ is surjective. More precisely, its evaluation at every log affinoid perfectoid object $U$ in $X_{\prokét}$ is surjective.

6. Kummer étale cohomology

6.1. Toric charts revisited. Let $V = \text{Spa}(S_1, S_1^+)$ be a log smooth affinoid fs log adic space over $\text{Spa}(k, k^+)$. Then $k$ is a sharp fs monoid. The goal of this subsection is to prove the following:

**Proposition 6.1.1.** In the above setting, assume moreover that $k$ is characteristic zero and contains all roots of unity. Let $V$ be a toric chart as above, and let $L$ be an $\mathbb{F}_p$-local system on $V_{\ker}$. Then we have the following:

1. $H^i(V_{\ker}, L \otimes \mathbb{F}_p (O^+_V/p))$ is almost zero for $i > n = \dim(V)$.
2. Let $V' \subseteq V$ be a rational subset such that $V'$ is strictly contained in $V$ (i.e., the closure of $V'$ in $V$ is contained in $V$). Then the image of the canonical morphism $H^i(V_{\ker}, L \otimes_{\mathbb{F}_p} (O^+_V/p)) \to H^i(V'_{\ker}, L \otimes_{\mathbb{F}_p} (O^+_V/p))$ is an almost finitely generated $k^+$-module, for each $i \geq 0$.

In order to prove Proposition 6.1.1 we need some preparations. Let us first introduce an explicit pro-finite Kummer étale cover of $E$. For each $m \geq 1$, consider

$$E_m := \text{Spa}(k(\frac{1}{m}P), k^+(\frac{1}{m}P)) = \text{Spa}(R_m, R_m^+)$$

and the log affinoid object

$$\tilde{E} := \varprojlim_m E_m \in E_{\prokét},$$
Lemma 6.1.6. (1) $k^+[P_{Q_{\geq 0}}]_1 = k^+[P]$ for the trivial character $\chi = 1$.

(2) Each direct summand $k^+[P_{Q_{\geq 1}}]_\chi$ is a finite $k^+[P]$-module.
Lemma 6.1.7. Fix a finite subset \( \{a_1, \ldots, a_r\} \) of \( \mathbb{Z} \) such that \( \pi^{-1}(\chi) \subset \frac{1}{m} P \), so that \( \pi^{-1}(\chi) = S_{\chi} + P \). Let \( \sigma \) be a convex subset of \( P^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q} \) of the form \( \sigma = \mathbb{Q}_{\geq 0} b_1 + \cdots + \mathbb{Q}_{\geq 0} b_s \) for some subset \( \{b_1, \ldots, b_s\} \) of \( P \) that forms a \( \mathbb{Q} \)-basis of \( P^{\text{gp}} = P^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q} \) (cf. [KKMSD75, Ch. I, Sec. 1, pp. 6-7]). Take a sufficiently large integer \( N \geq 1 \) such that \( P \subset \frac{1}{N} \sum_{j=1}^{s} \mathbb{Z}_{\geq 0} b_j \). Since \( \{b_1, \ldots, b_s\} \) is a \( \mathbb{Q} \)-basis of \( P^{\text{gp}} \), if \( a \in \pi^{-1}(\chi) \), then we can write \( a = \sum_{j=1}^{s} c_j b_j \), for some \( c_j \in \frac{1}{m} \mathbb{Z} \geq 0 \), for all \( j \). This shows that \( \pi^{-1}(\chi) \subset \frac{1}{m} P \) with \( m' = m \), as desired.

Lemma 6.1.8. Let \( M \) be any \( k^+/p^n \)-module on which \( \Gamma \) acts via a primitive character \( \chi : \Gamma \to \mu_{m} \). Then \( H^i(\Gamma, M) \) is annihilated by \( \zeta_m - 1 \), where \( \zeta_m \in \mu_m \) is any primitive \( m \)-th root of unity, for each \( i \geq 0 \). Moreover, if there exist a finite extension \( k_0 \) of \( \mathbb{Q}_p \)(\( \mu_m \)) in \( k \) with ring of integers \( k_0^+ = \mathcal{O}_{k_0} \), a finitely generated \( k_0^+/p^n \)-algebra \( T_0 \), and a finite (and hence finitely presented) \( T_0 \)-module \( M_0 \) such that \( M \cong M_0 \otimes_{k_0^+/p^n} (k^+/p^n) \) as \( \Gamma \)-modules over \( T := T_0 \otimes_{k_0^+/p^n} (k^+/p^n) \), then \( H^i(\Gamma, M) \) is a finitely presented \( T \)-module, for each \( i \geq 0 \).

Proof. By choosing a \( \mathbb{Z} \)-basis of \( P^{\text{gp}} \), there is a group isomorphism \( \Gamma \cong \hat{\mathbb{Z}}(1)^n \), where \( n = \text{rk}_2(P^{\text{gp}}) \) (see (6.1.4)). Then the lemma follows from a direct computation using the Koszul complex of \( \Gamma \) (as in the proof of [Sch13a, Lem. 5.5]) and (for the last assertion of the lemma) using the flatness of \( K^+/p^n \) over \( K_0^+/p^n \) (and the compatibility with flat base change in the formation of Koszul complexes).

Remark 6.1.9. Since \( R_1^+/p \cong (k^+/p)[P] \) and \( R^+/p \cong (k^+/p)[P_{\mathcal{Q}^{\geq 0}}] \), by Lemmas 6.1.6 and 6.1.7, the natural morphism \( H^i(\Gamma, (R_1^+/p)) \to H^i(\Gamma, (R^+/p)) \) is injective, with cokernel annihilated by \( \zeta_p - 1 \), for each \( i \geq 0 \). Moreover, the \( R_1^+/p \)-module \( H^i(\Gamma, (R^+/p)) \) is almost finitely presented, because, for each \( \epsilon > 0 \) such that \( p', \epsilon \)-torsion makes sense, there are only finitely many \( \chi \) such that the finitely presented \( R^+/p \)-module direct summand \( H^i(\Gamma, (k^+/p)[P_{\mathcal{Q}^{\geq 0}}]) \) is nonzero and not \( p', \epsilon \)-torsion. By the use of Koszul complexes as in the proof of Lemma 6.1.7 for any composition of rational localizations and finite étale morphisms \( \text{Spa}(S_1, S_1^+) \to \text{Spa}(R_1, R_1^+) \), we have \( H^i_{\text{cont}}(\Gamma, (S_1^+/p) \otimes_{R_1^+/p} (R^+/p)) \cong (S_1^+/p) \otimes_{R_1^+/p} H^i_{\text{cont}}(\Gamma, (R^+/p)) \).

By the same argument as in the proof of [Sch13a, Lem. 4.5], we obtain the following:

Lemma 6.1.10. Let \( X \) be a locally noetherian fs log adic space over \( \text{Spa}(k, k^+) \). Let

\[
U = \lim_{i \in I} U_i = \lim_{i \in I} (\text{Spa}(R_i, R_i^+), \mathcal{M}_i)
\]

be a log affinoid perfectoid object in \( X_{\text{prokét}} \), and let \( (R, R^+) := \left( \lim_{i \in I} (R_i, R_i^+) \right)^\wedge \), so that \( U = \text{Spa}(R, R^+) \) is the associated affinoid perfectoid space.

Suppose that, for some \( i \), there is a strictly étale morphism \( V_i = \text{Spa}(S_i, S_i^+) \to U_i \) that is a composition of rational localizations and strictly finite étale morphisms.
For each $j \geq i$, let $V_j := V_i \times_{U_j} U_j = \text{Spa}(S_j, S_j^+)$, and let
\[
V := V_i \times_{U_i} U \cong \varprojlim_j V_j \in X_{\text{prokét}}.
\]

Let $(S, S^+) := (\varprojlim_j (S_j, S_j^+))^\wedge$. Let $T_j$ be the $p$-adic completion of the $p$-torsion free quotient of $S_j^+ \otimes_{R_j^+} R^+$. Then we have the following:

1. $(S, S^+)$ is a perfectoid affinoid $(k, k^+)$-algebra, and $V$ is a log affinoid perfectoid object in $X_{\text{prokét}}$ with associated perfectoid space $\hat{V} = \text{Spa}(S, S^+)$. Moreover, $\hat{V} = V_j \times_{U_j} \hat{U}$ in the category of adic spaces.
2. For each $j \geq i$, we have $S \cong T_j[1/p]$, and the cokernel of $T_j \to S^+$ is annihilated by some power of $p$.
3. For each $\varepsilon \in \mathbb{Q}_{>0}$, there exists some $j \geq i$ such that the cokernel of $T_j \to S^+$ is annihilated by $p^\varepsilon$.

Remark 6.1.10. Lemma 6.1.9 is applicable, in particular, to the log affinoid object $U = \varprojlim_{m \in \mathbb{Z}_{\geq 1}} E_m$ in $E_{\text{prokét}}$ and any strictly étale morphism $V = \text{Spa}(S_1, S_1^+) \to E$ (for $m = 1$) giving a toric chart.

Lemma 6.1.11. Let $X$ be a locally noetherian fs log adic space over $\text{Spa}(k, k^+)$. Suppose that $U$ is a log affinoid perfectoid object of $X_{\text{prokét}}$, with associated perfectoid space $\hat{U} = \text{Spa}(R, R^+)$. Let $\mathbb{L}$ be an $\mathbb{F}_p$-local system on $U_{\text{két}}$. Then:

1. $H^i(U_{\text{két}}, L \otimes_{\mathbb{F}_p}(\mathcal{O}_X^+/p))$ is almost zero, for all $i > 0$.
2. $L(U) := H^0(U_{\text{két}}, L \otimes_{\mathbb{F}_p}(\mathcal{O}_X^+/p))$ is an almost finitely generated projective $R^+/p$-module (see [GR03, Def. 2.4.4]). Moreover, for any morphism $U' \to U$ in $X_{\text{prokét}}$, where $U'$ is a log affinoid perfectoid object of $X_{\text{prokét}}$, with associated perfectoid space $\hat{U}' = \text{Spa}(R', R'^+)$, we have $L(U') \cong L(U) \otimes_{R^+/p}(R'^+/p)$.

Proof. We may assume that $X$ is connected. Choose any finite Kummer étale Galois cover $Y \to X$ trivializing $\mathbb{L} \cong \mathbb{F}_p$. By Lemma 5.3.8, $W := U \times_X Y \to U$ is finite étale, and $W$ is log affinoid perfectoid, with associated perfectoid space $\tilde{W} = \text{Spa}(T, T^+)$. For each $j \geq 1$, let $W_{j/U}$ denote the $j$-fold fiber product of $W$ over $U$. By Proposition 5.1.7 and Theorem 5.4.3, $H^i(W_{j/U}^\text{két}, L \otimes_{\mathbb{F}_p}(\mathcal{O}_W^+/p))$ is almost zero, for all $i > 0$ and $j$, and $H^0(W_{j/U}^\text{két}, L \otimes_{\mathbb{F}_p}(\mathcal{O}_W^+/p))$ is almost isomorphic to $(\mathcal{O}_{W_{j/U}^\text{két}}(W_j(U))/p)^r$. By the faithful flatness of $T^+/p \to R^+/p$, the desired results follow from almost faithfully flat descent (see [GR03, Sec. 3.4]).

Now we are ready for the following:

Proof of Proposition 6.1.1. Consider the Galois cover $\tilde{V} \to V = \text{Spa}(S_1, S_1^+)$ with Galois group $\Gamma$, with $\tilde{V} = \text{Spa}(S, S^+)$, as above. Since $\tilde{V}^{j/V} \cong \tilde{V} \times \Gamma^{-1}$ is a log affinoid perfectoid object in $V_{\text{prokét}}$, for each $j \geq 1$, we have
\[
H^i(\tilde{V}_{j/\text{két}}^{j/V}, L \otimes_{\mathbb{F}_p}(\mathcal{O}_{\tilde{V}}^+/p)) \cong \text{Hom}_{\text{cont}}(\Gamma^{-1}, L),
\]
where $L := H^0(\tilde{V}_{\text{két}}, L \otimes_{\mathbb{F}_p}(\mathcal{O}_{\tilde{V}}^+/p))^\wedge$ is an almost finitely generated projective $S^+/p$-module, equipped with the discrete topology, by Propositions 5.1.6 and 5.1.7.
By Proposition 5.1.7 again, and by Lemma 6.1.11 and the Cartan–Leray spectral sequence (see [AGV73, V, 3.3]), we have an almost isomorphism

\[ H^i(V_{\text{ét}}, \mathbb{L} \otimes \mathcal{F}_p (\mathcal{O}_\mathcal{V}^+/p)) \cong H^i((\mathcal{V} \rightarrow V), \mathbb{L} \otimes \mathcal{F}_p (\mathcal{O}_\mathcal{V}^+/p)) \cong H^i_{\text{cont}}(\Gamma, L), \]

where the last isomorphism follows from Proposition 5.1.12 and [Sch13a, Prop. 3.7(iii)] (and the correction in [Sch16]). Hence, the statement (1) of Proposition 6.1.1 follows from the fact that \( \Gamma \cong \mathbb{Z}(1)^n \) has cohomological dimension \( n \).

As for the statement (2), let us write \( V' = \text{Spa}(S'_1, S'_j^+) \) and \( \mathcal{V}' = \text{Spa}(S', S'^+) \). We need to show that the image of \( H^i_{\text{cont}}(\Gamma, L) \rightarrow H^i_{\text{cont}}(\Gamma, L \otimes_{S^+/p} (S'^+/p)) \) is an almost finitely generated \( k^+ \)-module. Since \( L \) is an almost finitely generated projective \( S^+/p \)-module, it suffices to show that the image of

\[ H^i_{\text{cont}}(\Gamma, S^+/p) \rightarrow H^i_{\text{cont}}(\Gamma, S'^+/p) \]

is almost finitely generated. Choose \( n + 2 \) rational subsets \( V^{(n+2)} = V' \subset \cdots \subset V^{(1)} = V \) such that \( V^{(j+1)} \) is strictly contained in \( V^{(j)} \), for each \( j \). Write \( V^{(j)} = \text{Spa}(S^{(j)}_1, S^{(j)}_j^+) \), and let \( V_m^{(j)} := V^{(j)} \times_{\mathcal{E}_m} \mathcal{E}_m = \text{Spa}(S^{(j)}_m, S^{(j)}_m^+) \). Then \( \mathcal{V}^{(j)} := \lim_m V_m^{(j)} \) is a log affinoid perfectoid object in \( V_{\text{prokÉt}} \), with associated perfectoid space \( \mathcal{V}^{(j)} = \text{Spa}(S^{(j)}, S^{(j)}_j^+) \). By Lemma 6.1.9 and Remark 6.1.10 it suffices to show that the image of

\[ H^i_{\text{cont}}(\Gamma, (S^{(1)}_m + \otimes R^+_m R^+)/(p)) \rightarrow H^i_{\text{cont}}(\Gamma, (S^{(n+2)}_m + \otimes R^+_m R^+)/(p)) \]

is almost finitely generated, for all \( m \in \mathbb{Z}_{\geq 1} \). Note that \( m\Gamma \) acts trivially on \( S^{(j)}_m^+ \), and we have the Hochschild–Serre spectral sequence

\[ H^i(\Gamma/m, H^{i+2}_{\text{cont}}(m\Gamma, (S^{(j)}_m + \otimes R^+_m R^+)/(p))) \Rightarrow H^{i+j+2}_{\text{cont}}(\Gamma, (S^{(j)}_m + \otimes R^+_m R^+)/(p)). \]

By [Sch13a, Lem. 5.4] and Remark 6.1.8 it suffices to show that the image of

\[ (S^{(j)}_m + \otimes R^+_m/p) H^i_{\text{cont}}(m\Gamma, R^+/(p)) \rightarrow (S^{(j+1)}_m + \otimes R^+_m/p) H^i_{\text{cont}}(m\Gamma, R^+/(p)) \]

is almost finitely generated, for all \( j = 1, \ldots, n+1 \) and \( m \geq 1 \). Since the image of \( S^{(j)}_m + \otimes R^+_m + \otimes R^+_m/p \) is an almost finitely generated \( k^+ \)-module, it suffices to note that \( H^i_{\text{cont}}(m\Gamma, R^+/(p)) \) is almost finitely generated over \( R^+_m/p \), by Remark 6.1.8 (up to replacing \( (R_1, R_1^+), \Gamma, \text{etc} \) with \( (R_m, R_m^+), \Gamma_m, \text{etc} \)).

6.2. Primitive comparison theorem. The main goal of this subsection is to prove the following primitive comparison theorem, with the finiteness of cohomology as a byproduct, generalizing the strategy in [Sch13a, Sec. 5]:

Theorem 6.2.1. Let \( (k, k^+) \) be an affinoid field, where \( k \) is algebraically closed and of characteristic zero, and let \( X \) be a proper log smooth fs log adic space over \( \text{Spa}(k, k^+) \) (see Definitions 2.2.26 and 3.1.1). Let \( \mathbb{L} \) be an \( \mathbb{F}_p \)-local system on \( X_{\text{ét}} \). Then we have the following:

1. \( H^i(X_{\text{ét}}, \mathbb{L} \otimes_{\mathbb{F}_p} (\mathcal{O}_X^+/p)) \) is an almost finitely generated \( k^+ \)-module (see [GR03, Def. 2.3.8]) for each \( i \geq 0 \), and is almost zero for \( i \gg 0 \).
2. There is a canonical almost isomorphism

\[ H^i(X_{\text{ét}}, \mathbb{L} \otimes_{\mathbb{F}_p} (k^+/p)) \cong H^i(X_{\text{ét}}, \mathbb{L} \otimes_{\mathbb{F}_p} (\mathcal{O}_X^+/p)) \]

of \( k^+ \)-modules, for each \( i \geq 0 \).
Consequently, \( H^i(X_{k\et}, L) \) is a finite-dimensional \( \mathbb{F}_p \)-vector space for each \( i \geq 0 \), and \( H^i(X_{k\et}, L) = 0 \) for \( i > 0 \). In addition, if \( X \) is as in Example \[2.3.17\] then \( H^i(X_{k\et}, L) = 0 \) for \( i > 2 \dim(X) \).

**Remark 6.2.2.** Recall that there is no general finiteness results for the étale cohomology of \( \mathbb{F}_p \)-local systems on non-proper rigid analytic varieties over \( k \), as is well known (via Artin–Schreier theory) that \( H^1(D, \mathbb{F}_p) \) is infinite.

Nevertheless, we have the following:

**Corollary 6.2.3.** Let \( U \) be a smooth rigid analytic variety that is Zariski open in a proper rigid analytic variety over \( k \). Then \( H^i(U_{\et}, L) \) is a finite-dimensional \( \mathbb{F}_p \)-vector space, for every \( \mathbb{F}_p \)-local system \( L \) on \( U_{\et} \) and every \( i \geq 0 \). Moreover, \( H^i(U_{\et}, L) = 0 \) for \( i > 2 \dim(U) \).

**Proof.** By resolution of singularities (as in [BM97]), we may assume that we have a smooth compactification \( U \rightarrow X \) such that \( U = X - D \) for some normal crossings divisor \( D \) of \( X \). Now apply Theorems [4.6.1 and 6.2.1].

**Lemma 6.2.4.** Let \( X \) be a proper log smooth fs log adic space over \( \text{Spa}(k, \mathcal{O}_k) \). For each integer \( N \geq 2 \), we can find \( N \) affinoid étale coverings of \( X \)

\[ \{ V^{(N)}_h \}_{h=1}^m, \{ V^{(1)}_h \}_{h=1}^m \]

satisfying the following properties:

- \( V^{(N)}_h \subset \cdots \subset V^{(1)}_h \) is a chain of rational subsets, for each \( h = 1, \ldots, m \).
- \( V^{(j+1)}_h \subset V^{(j)}_h \subset V^{(j)}_j \) for all \( h = 1, \ldots, m \) and \( j = 1, \ldots, N - 1 \).
- \( V^{(1)}_h \times_X V^{(1)}_{h_2} \rightarrow V^{(1)}_{h_1} \) is a composition of rational localizations and finite étale morphisms, for \( 1 \leq h_1, h_2 \leq m \).
- Each \( V^{(1)}_h \) admits a toric chart \( V^{(1)}_h \rightarrow \text{Spa}(k[P_h], \mathcal{O}_k[P_h]) \), for some sharp fs monoid \( P_h \).

**Proof.** By Proposition [3.1.10] and the same argument as in the proof of [Sch13a, Lem. 5.3], there exist \( N \) affinoid analytic open coverings of \( X \)

\[ \{ U^{(N)}_h \}_{h=1}^m, \{ U^{(1)}_h \}_{h=1}^m \]

satisfying the following properties:

- \( U^{(N)}_h \subset \cdots \subset U^{(1)}_h \) is a chain of rational subsets, for each \( h = 1, \ldots, m \).
- \( U^{(j+1)}_h \subset U^{(j)}_h \subset U^{(j)}_j \) for all \( h = 1, \ldots, m \) and \( j = 1, \ldots, N - 1 \).
- \( U^{(1)}_h \cap U^{(1)}_{h_2} \subset U^{(1)}_{h_1} \) is a rational subset, for \( 1 \leq h_1, h_2 \leq m \).
- There exist finite étale covers \( U^{(1)}_h \rightarrow U^{(1)}_h \) such that each \( V^{(1)}_h \) admits a toric chart \( V^{(1)}_h \rightarrow \mathbb{E}_j = \text{Spa}(k[P_h], \mathcal{O}_k[P_h]) \) (which is, in particular, a composition of rational localizations and finite étale morphisms) for some sharp fs monoid \( P_h \).

Then it suffices to take \( V^{(j)}_h := V^{(1)}_h \times_{U^{(1)}_h} U^{(j)}_h \), for all \( h \) and \( j \).

**Proof of Theorem 6.2.1.** Consider \( X' := X \times_{\text{Spa}(k, k^+)} \text{Spa}(k, \mathcal{O}_k) \subset X \). Consider any covering \( \{ U_h \} \) of \( X \) by log affinoid perfectoid objects in \( X_{\text{prok\acute{e}t}} \), whose pullback \( \{ U_h \times_X X' \} \) is a covering of \( X' \) by log affinoid perfectoid objects in \( X'_{\text{prok\acute{e}t}} \). By Lemma \[6.1.11\] \( H^i(U_{h, \text{k\acute{e}t}}, \mathbb{L} \otimes_{\mathbb{F}_p} (\mathcal{O}_{X'}^+/p)^a \sim H^i(U_{k, \text{k\acute{e}t}} \times_X X', \mathbb{L} \otimes_{\mathbb{F}_p} (\mathcal{O}_{X'}^+/p)^a, \)
for all \( i \geq 0 \) and all \( h \). By Proposition 5.1, and by comparing the spectral sequences associated with the coverings, we obtain \( H^i(X_{\text{prok}}, \mathbb{L} \otimes_{\mathbb{F}_p} (\mathcal{O}_X^+/p)^a) \cong H^i(X'_{\text{prok}}, \mathbb{L} \otimes_{\mathbb{F}_p} (\mathcal{O}_X^+/p)^a) \), for each \( i \geq 0 \). Hence, up to replacing \( X \) with \( X' \), we may assume that \( k^+ = \mathcal{O}_k \) in what follows.

Let \( \{ V_h^{(1)} \}_{h=1}^{m} \) as in Lemma 6.2.4. For each subset \( H = \{ h_1, \ldots, h_s \} \subset \{ 1, \ldots, m \} \), let \( V_H := V_{h_1} \times_X \cdots \times_X V_{h_s}^\flat \). For each \( j = 1, \ldots, N \), we have a spectral sequence
\[
E^{i,j}_{v_{H}} = \oplus_{|H|=i+j+1} H^{i+j} \big( V_H^{\flat}, \mathbb{L} \otimes_{\mathbb{F}_p} (\mathcal{O}_X^+/p) \big) \Rightarrow H^{i+j} \big( X_{\text{prok}}, \mathbb{L} \otimes_{\mathbb{F}_p} (\mathcal{O}_X^+/p) \big)
\]
For \( j = 1, \ldots, N - 1 \), we also have natural morphisms between spectral sequences \( E^{i,j}_{v_{H}} \to E^{i,j+1}_{v_{H}} \). Then the desired finiteness result follows from Proposition 6.1.1 and [Sch13a, Lem. 5.4]. Moreover, by Proposition 6.1.1 and the spectral sequence (6.2.5), we have
\[
H^{i} \big( X_{\text{prok}}, \mathbb{L} \otimes_{\mathbb{F}_p} (\mathcal{O}_X^+/p)^a \big) = 0 \quad \text{for } i \gg 0.
\]

**Proof of Theorem 6.2.1(2).** Consider the Artin–Schreier sequence
\[
0 \to \mathbb{L} \to \mathbb{L} \otimes_{\mathbb{F}_p} \mathcal{O}_{X_{\text{prok}}}^\Lambda \to \mathbb{L} \otimes_{\mathbb{F}_p} \mathcal{O}_{X_{\text{prok}}}^\Lambda \to 0,
\]
where \( \sigma = \text{Id} \otimes (\Phi - \text{Id}) \) and \( \Phi \) is the Frobenius morphism (induced by \( x \mapsto x^p \)). The exactness of (6.2.5) can be checked locally on log affinoid perfectoid \( U \in X_{\text{prok}} \) over which \( \mathbb{L} \) is trivial, which then follows (by using Lemma 5.3.8) from the same argument in the proof of [Sch13a, Thm. 5.1].

Choose any \( \varpi \in k^p \) such that \( \varpi^p = p \). By Theorem 6.2.1(1) and [Sch13a, Lem. 2.12], there exists some \( r \geq 0 \) such that we have almost isomorphisms
\[
H^i(X_{\text{prok}}, \mathbb{L} \otimes_{\mathbb{F}_p} (\mathcal{O}_{X_{\text{prok}}}^+/\varpi^m))^a \cong (\mathcal{O}_{k^p}^+/\varpi^m)^r, \quad \text{for all } m,
\]
which are compatible with each other and with the Frobenius morphism. By [Sch13a, Lem. 3.18], we have
\[
\lim_{\varpi^m \to 0} \big( \mathbb{L} \otimes_{\mathbb{F}_p} (\mathcal{O}_{X_{\text{prok}}}^+/\varpi^m) \big)^a \cong (\mathcal{O}_{k^p})^r, \quad \text{and consequently}
\]
\[
H^i(X_{\text{prok}}, \mathbb{L} \otimes_{\mathbb{F}_p} \mathcal{O}_{X_{\text{prok}}}^+) \cong (k^p)^r, \quad \text{by inverting } \varpi,
\]
which are still compatible with the Frobenius morphisms.

Thus, by considering the long exact sequence associated with (6.2.5), and by Proposition 5.1.7, we see that \( H^i(X_{\text{prok}}, \mathbb{L}) \cong H^i(X_{\text{prok}}, \mathbb{L} \otimes_{\mathbb{F}_p} \mathcal{O}_{X_{\text{prok}}}^+)^{p-1} \cong \mathbb{F}_p \)
and that \( H^i(X_{\text{prok}}, \mathbb{L} \otimes_{\mathbb{F}_p} (k^p)^{a}) \cong H^i(X_{\text{prok}}, \mathbb{L} \otimes_{\mathbb{F}_p} (\mathcal{O}_{X_{\text{prok}}}^+/p)^a) \), as desired.

**Proof of the remaining statements of Theorem 6.2.1.** We still need to show that, if \( X \) is as in Example 3.1.13, then \( H^i(X_{\text{prok}}, \mathbb{L}) = 0 \) for \( i > 2 \dim(X) \). By Theorem 6.2.1(2), it suffices to show that \( H^i(X_{\text{prok}}, \mathbb{L} \otimes_{\mathbb{F}_p} (\mathcal{O}_X^+/p)^a) = 0 \) for \( i > 2 \dim(X) \). Note that, in Example 3.1.13, \( X \) analytic locally admits smooth toric charts \( X \to \mathbb{A}^n \). Hence, by the same argument as in the proof of [Sch13a, Lem. 5.3], all the étale covering \( \{ V^{\flat}_m \}_{m=1} \) in Lemma 6.2.4 can be chosen to be analytic coverings. Let \( \lambda : X_{\text{prok}} \to X_{\text{an}} \) denote the natural projection of sites. By Proposition 6.1.1, \( R^j \lambda_*(\mathbb{L} \otimes_{\mathbb{F}_p} (\mathcal{O}_X^+/p)) = 0 \) for \( j > \dim(X) \). Since the cohomological dimension of \( X_{\text{an}} \) is bounded by \( \dim(X) \), by [dJvdP96, Prop. 2.5.8], the desired vanishing follows. (This is the same argument as in the proof of [Sch13a, Lem. 5.9].)

6.3. \( p \)-adic local systems.

**Definition 6.3.1.** Let \( X \) be a locally noetherian fs log adic space.

1. A \( \mathbb{Z}_p \)-local system (or lisse \( \mathbb{Z}_p \)-sheaf) on \( X_{\text{prok}} \) is an inverse system of \( \mathbb{Z}/p^n \)-modules \( \mathbb{L} = (\mathbb{L}_n)_{n \geq 1} \) on \( X_{\text{prok}} \) such that each \( \mathbb{L}_n \) is a locally constant
sheaf which are locally (on $X_{k\acute{e}t}$) associated with finitely generated $\mathbb{Z}/p^n$-modules, and such that the inverse system is isomorphic to an inverse system in which $\mathbb{L}_{n+1}/p^n \cong \mathbb{L}_n$.

(2) A $\mathbb{Q}_p$-local system (or lisse $\mathbb{Q}_p$-sheaf) on $X_{k\acute{e}t}$ is an object of the stack associated with the fibered category of isogeny lisse $\mathbb{Z}_p$-sheaves.

**Definition 6.3.2.** Let $X$ be a locally noetherian fs log adic space. Let $\widehat{\mathbb{Z}}_p := \lim_{\leftarrow n} \left( \mathbb{Z}/p^n \right)$ as a sheaf of rings on $X_{pro\acute{e}t}$, and let $\widehat{\mathbb{Q}}_p := \mathbb{Z}_p[1/p]$. A $\widehat{\mathbb{Z}}_p$-local system on $X_{pro\acute{e}t}$ is a sheaf of $\widehat{\mathbb{Z}}_p$-modules on $X_{pro\acute{e}t}$ that is locally (on $X_{pro\acute{e}t}$) isomorphic to $L \otimes_{\mathbb{Z}_p} \widehat{\mathbb{Z}}_p$ for some finitely generated $\mathbb{Z}_p$-modules $L$. The notion of $\widehat{\mathbb{Q}}_p$-local system on $X_{pro\acute{e}t}$ is defined similarly.

**Lemma 6.3.3.** Let $X$ be a locally noetherian fs log adic space over $\text{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. Let $\nu : X_{pro\acute{e}t} \to X_{k\acute{e}t}$ denote the natural projection of sites.

(1) The functor sending $L = (\mathbb{L}_n)_{n \geq 1}$ to $\widehat{L} := \lim_{\leftarrow n} \nu^{-1}(\mathbb{L}_n)$ is an equivalence of categories from the category of $\mathbb{Z}_p$-local systems on $X_{k\acute{e}t}$ to the category of $\widehat{\mathbb{Z}}_p$-local systems on $X_{pro\acute{e}t}$. Moreover, $\widehat{L} \otimes_{\mathbb{Z}_p} \widehat{\mathbb{Q}}_p$ is a $\widehat{\mathbb{Q}}_p$-local system.

(2) For all $i > 0$, we have $R^i \lim_{\leftarrow n} \nu^{-1}(\mathbb{L}_n) = 0$.

**Proof.** Apply Proposition 6.3.13 and [Sch13a, Lem. 3.18]. \qed

**Corollary 6.3.4.** Let $k$, $X$, and $U$ be as in Theorem 4.6.1. Let $\mathbb{L}$ be an étale $\mathbb{Z}_p$-local system on $U_{k\acute{e}t}$. Then $\mathbb{L} := \mathcal{L}_{k\acute{e}t,*}(\mathbb{L})$ is a Kummer étale $\mathbb{Z}_p$-local system extending $\mathbb{L}$, and there are canonical isomorphisms

$$H^i(U_{k\acute{e}t}, \mathbb{L}) \cong H^i(X_{k\acute{e}t}, \mathbb{L}) \cong H^i(X_{pro\acute{e}t}, \widehat{\mathbb{L}})$$

of finite $\mathbb{Z}_p$-modules, for each $i \geq 0$, where $\widehat{\mathbb{L}}$ denotes any algebraic closure of $\mathbb{L}$.

**Proof.** The first isomorphism follows from Corollary 4.6.7 by taking limits of $\mathbb{Z}_p/p^n$-local systems over $m \in \mathbb{Z}_{\geq 1}$, which is justified by the finite-dimensionality of the cohomology of $\mathbb{F}_p$-local systems on $X_{k\acute{e}t}$, shown in Theorem 6.2.1. The second isomorphism follows from Proposition 5.1.7 and Lemma 6.3.3(2). The finiteness of these isomorphic $\mathbb{Z}_p$-modules follows, again, from Theorem 6.2.1. \qed

**Corollary 6.3.5.** Let $f : X \to Y$ be a log smooth morphism of log adic spaces whose log structures are defined by normal crossings divisors $D \to X$ and $E \to Y$, respectively, as in Example 2.3.17. Assume that the underlying morphisms of adic spaces of $f$ and $f|_{X-D} : X-D \to Y-E$ are both proper. Let $\mathbb{L}$ be any $\mathbb{Z}_p$-local system on $X_{k\acute{e}t}$. Then $R^i f_{k\acute{e}t,*}(\mathbb{L})$ is a $\mathbb{Z}_p$-local system on $Y_{k\acute{e}t}$, for each $i$.

**Proof.** This follows from [SW17, Thm. 10.5.1] and Corollary 6.3.4. \qed

We end this subsection with the following lemma describing the pullback of $\widehat{\mathbb{Q}}_p$-local systems under strict closed immersions:

**Lemma 6.3.6.** Let $i : Z \to X$ be a strict closed immersion of locally noetherian fs log adic spaces over $\text{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. Let $\mathbb{L}$ be a $\widehat{\mathbb{Z}}_p$-local system on $X_{pro\acute{e}t}$. Then

$$i^{-1}(\widehat{L}) \otimes_{\widehat{\mathbb{Q}}_p} \widehat{\mathbb{O}}_{Z_{pro\acute{e}t}}(U \times X Z) \cong (\widehat{L} \otimes_{\widehat{\mathbb{Q}}_p} \widehat{\mathbb{O}}_{X_{pro\acute{e}t}}(U)) \otimes_{\widehat{\mathbb{Q}}_p} \widehat{\mathbb{O}}_{Z_{pro\acute{e}t}}(U \times X Z),$$

for every log affinoid perfectoid object $U$ of $X_{pro\acute{e}t}$.\qed
\textbf{Proof.} By Lemma \[5.3.7\], \(U \times_X \hat{Z}\) is a log affinoid perfectoid object of \(\hat{Z}_{\prok\et}\), and the natural morphism \(\hat{O}_{X_{\prok\et}} \to \hat{t}_{\prok\et,*}(\hat{O}_{\hat{Z}_{\prok\et}})\) induces a surjective homomorphism 
\(\hat{O}_{X_{\prok\et}}(U) \to \hat{O}_{\hat{Z}_{\prok\et}}(U \times_X \hat{Z})\). By Theorem \[5.4.4\], it suffices to prove the lemma by replacing \(U\) with some log affinoid perfectoid object \(V\) of \(X_{\prok\et}\) over \(U\) such that \(\hat{L}|_V\) is trivial, in which case the assertion is clear. \(\square\)

6.4. \textbf{Quasi-unipotent nearby cycles.} In this subsection, as an application of our results, we reformulate Beilinson’s ideas (see [Rei87]; cf. [Rei10]) and define the unipotent and quasi-unipotent nearby cycles in the rigid analytic setting.

Let \(\mathbb{G}_m := \text{Spec}(k[z, z^{-1}])\) be the multiplicative group scheme over a field \(k\) of characteristic zero. Then \(\pi_1(\mathbb{G}_m, 1) \cong \pi_1(\mathbb{G}_m, \overline{k}, 1) \rtimes \text{Gal}(\overline{k}/k)\), and \(\pi_1(\mathbb{G}_m, \overline{k}, 1) \cong \hat{\mathbb{Z}}(1)\) as \(\text{Gal}(\overline{k}/k)\)-modules. For each \(r \geq 1\), let \(\mathbb{J}_r\) denote the rank \(r\) unipotent étale \(\mathbb{Z}_p\)-local system on \(\mathbb{G}_m\) such that a topological generator \(\gamma \in \pi_1(\mathbb{G}_m, \overline{k}, 1)\) acts as a principal unipotent matrix \(J_r\) and such that \(\text{Gal}(\overline{k}/k)\) acts on \(\ker(J_r - 1)\) trivially. There is a natural inclusion \(\mathbb{J}_r \hookrightarrow \mathbb{J}_{r+1}\), together with a projection \(\mathbb{J}_{r+1} \twoheadrightarrow \mathbb{J}_r(\overline{k}, -1)\) such that the composition \(\mathbb{J}_r \twoheadrightarrow \mathbb{J}_r(\overline{k}, -1)\) is given by the monodromy action. For each \(m \geq 1\), let \([m]\) denote the \(m\)-th power homomorphism \([m]\) of \(\mathbb{G}_m\), and let \(\mathbb{K}_m := [m]_*(\mathbb{Z}_p)\). When \(m | m'\), there is a natural inclusion \(\mathbb{K}_m \hookrightarrow \mathbb{K}_{m'}\) (defined by adjunction).

Now let \(k\) be a nontrivial nonarchimedean field, and let \(k^+ = \mathcal{O}_k\). We shall denote the analytifications of the above objects and morphisms to \(\mathbb{G}_m^{an}\) and their further pullbacks to \(\mathbb{D}^\times = \mathbb{D} - \{0\}\), by the same symbols.

Let \(X\) be a rigid analytic variety over \(k\). Let \(f : X \to \mathbb{D}\) be a morphism over \(k\), which induces an open immersion \(j : U := f^{-1}(\mathbb{D}^\times) \to X\) and a closed immersion \(i : f^{-1}(0) \hookrightarrow X\) such that \(D := f^{-1}(0)_{\text{red}}\) (the reduced subspace) is a normal crossings divisor, so that \(X\) is equipped with the fs log structure defined by \(D \rightarrow X\), as in Example \[2.3.17\]. Note that \((f^{-1}(0))_{\et} \cong D_{\et}\). Let \(U\) be equipped with the trivial log structure, with an open immersion \(j : U \to X\). Let \(D^\partial\) be the adic space \(D_{\et}\) equipped with the log structure pulled back from \(X_{\et}\), with a strict closed immersion \(i^\partial : D^\partial \rightarrow X\) and a canonical morphism \(\varepsilon^\partial : D^\partial \rightarrow D\).

\textbf{Definition 6.4.1.} In the above setting, for each \(\mathbb{Q}_p\)-local system \(L\) on \(U_{\et} \cong U_{\kjet}\), its sheaf of \textit{unipotent nearby cycles} (with respect to \(f\)) is defined as

\[
R\Psi^u_f(L) := R\varepsilon^\partial_{\et,*} \left( \lim_{r} \Delta^1_{k_{\kjet}} \mathbb{J}_{k_{\kjet,*}} \left( L \otimes_{\mathbb{Z}_p} f^\partial_{\et,*} (\mathbb{J}_{f}(r)) \right) \right),
\]

and its sheaf of \textit{quasi-unipotent nearby cycles} is defined as

\[
R\Psi^{qu}_f(L) := R\varepsilon^\partial_{\et,*} \left( \lim_{m,r} \Delta^1_{k_{\kjet}} \mathbb{J}_{k_{\kjet,*}} \left( L \otimes_{\mathbb{Z}_p} f^\partial_{\et,*} (\mathbb{K}_m) \otimes_{\mathbb{Z}_p} f^\partial_{\et,*} (\mathbb{J}_{f}(r)) \right) \right).
\]

Suppose that \(\{D_j\}_{j \in I} \) is the set of irreducible components of \(D\) (see [Con99]), so that \(f^{-1}(0) = \sum_{j \in I} n_j D_j\) (as Cartier divisors on \(X\); see [SW17] Sec. 5.3, especially Prop. 5.3.4), for some integers \(n_j \geq 1\) giving the multiplicities of \(D_j\). For each \(J \subset I\), consider the fs log adic spaces \(U_J\) and \(U^\partial_J\) introduced in Example \[2.3.18\] together with a canonical morphism \(\varepsilon^\partial_J : U^\partial_J \rightarrow U_J\) (whose underlying morphism of adic spaces is a canonical isomorphism) and a strict immersion \(i^\partial_J : U^\partial_J \rightarrow X\). Note that the log structure of \(U_J\) is trivial, while the one of \(U^\partial_J\) is pulled back from \(X_J\). We shall simply denote the underlying adic space of \(U^\partial_J\) by \(U_J\). By construction, \(X\) is set-theoretically the disjoint union of such locally closed subspaces \(U_J\).
At each geometric point $\xi = \Spec(l, l^+)$ of $U_J$ (and hence also of $U^0_J$), by projection to factors of polydisk as in Examples \ref{example:sub} and \ref{example:polydisk}, we have locally a morphism from $U^0_J$ to $s = (\Spec(k, \mathcal{O}_k), Z^\gamma_{\geq 0})$ as in Example \ref{example:gamma}, such that the log structure of $U^0_J$ is pulled back from $s$, and so that we have $M_{U^0_J, \xi} \cong Z^\gamma_{\geq 0}$ and $\pi^\kappa_{j}(U^0_J(\xi), \mathcal{F}_J) \cong \Gamma^J := \widehat{\mathbb{Z}}(1) \cong \Hom(M_{U^0_J, \xi}, \widehat{\mathbb{Z}}(1))$ (whose operations will be denoted multiplicatively). Hence, any $\mathbb{Z}_p$-local system $\mathcal{M}$ on $U^0_J(\xi)_{\acute{e}t}$ is equivalent to a (trivial) $\mathbb{Z}_p$-local system on $U_J(\xi)_{\acute{e}t}$ with $\Gamma^J$-action. Let $\mathcal{E}^0_{j, \xi} : U^0_J(\xi) \to U_J(\xi)$ denote the pullback of $\mathcal{E}^0_j$ to $U_J(\xi)$. Then, by Lemma \ref{lemma:pullback}, we have a canonical isomorphism $R^i\mathcal{E}^0_{j, \xi, \kappa}(\mathcal{M}) \cong H^i(\Gamma^J, \mathcal{M})$, for each $i \geq 0$.

Let $0^\partial$ and $\partial^\theta$ be as in Example \ref{example:gamma} together with $(\mathbb{Z}/n)$-local systems $\mathbb{J}^\partial_{r, n}$ and $\mathbb{K}^\partial_{m, n}$ over $0^\partial$ defined by representations of $\pi^\kappa_{j}(0^\partial, \partial^\theta) \cong \widehat{\mathbb{Z}}(1) \times \Gal(\overline{\mathbb{F}}/k)$. By taking limits over $n \in \mathbb{N}$, we obtain $\mathbb{Z}_p$-local systems $\mathbb{J}^\partial$ and $\mathbb{K}^\partial$ over $0^\partial$. The pullback under $f : X \to \mathbb{D}$ (as a morphism between fs log adic spaces) induces, for any $\xi$ and $\tilde{\xi}$ as in the last paragraph, a canonical morphism $f_{\xi} : U^0_J(\xi) \to 0^\partial$, and the homomorphism $\pi^\kappa_{j}(U^0_J(\xi), \tilde{\xi}) \to \pi^\kappa_{j}(0^\partial, \partial^\theta)$ induced by $f_{\xi}$ can be identified with the composition of $\Gamma^J \cong \widehat{\mathbb{Z}}(1) \to \widehat{\mathbb{Z}}(1) : (x_j)_{j \in J} \mapsto \sum_{j \in J} n_j x_j$ with the canonical homomorphism $\widehat{\mathbb{Z}}(1) \to \widehat{\mathbb{Z}}(1) \times \Gal(\overline{\mathbb{F}}/k)$. Let $\gamma_j$ be any element of the $j$-th factor of $\Gamma^J \cong \widehat{\mathbb{Z}}(1)$ that is mapped to $n_j \gamma$ in $\widehat{\mathbb{Z}}(1)$. Then $(f_{\xi}^{-1}(\mathbb{J}^\partial_{\xi}))|_{U^0_J(\xi)} \cong f_{\xi}^{-1}(\mathbb{J}^\partial_{\xi})$ is the rank $r$ local system such that $\gamma_j$ acts by $\mathbb{J}^\partial_{\xi}$.

For each $\mathbb{Q}_p$-local system $\mathcal{M}$ on $U^0_J(\xi)_{\acute{e}t}$, let us denote by $W$ a formal variable on which $\gamma_j^{-1}$ acts by $W \mapsto W + n_j$, and write $\mathcal{M}[W] = \lim_{\leftarrow r \geq 1} (\mathcal{M}[W])^{\leq r}$, where the superscript “$\leq r$” means “up to degree $r - 1$.” Note that, by matching a standard basis of $\mathbb{J}^\partial$ with binomial monomials up to degree $r - 1$ in $W$, as in the proof of \cite[Lem. 2.10]{LZ17}, we have $\mathcal{M}[W]^{\leq r - 1} \cong \mathcal{M} \otimes_{\mathbb{Z}_p} f_{\xi}^{-1}(\mathbb{J}^\partial_{\xi})$.

**Lemma 6.4.2.** Suppose that either $J = \emptyset$ or there exists some $j_0 \in J$ such that $\gamma_{j_0}$ acts quasi-unipotently (i.e., a positive power of $\gamma_{j_0}$ acts unipotently) on $\mathcal{M}$. Then the local systems $H^i(\Gamma^J, \mathcal{M}[W])$ stabilize as $r \to \infty$, and hence the direct limit $H^i(\Gamma^J, \mathcal{M}[W])$ exists as a $\mathbb{Q}_p$-local system, for each $i \geq 0$. When $J = \{j_0\}$ is a singleton, $H^i(\Gamma_{\{j_0\}}, \mathcal{M}[W])$ vanishes when $i \neq 0, 1$; is canonically isomorphic to the maximal subsheaf of $\mathcal{M}$ on which $\Gamma_{\{j_0\}}$ acts unipotently, when $i = 0$; is finite-dimensional when $i = 1$; and is zero when $i = 1$ and $\gamma_j$ acts unipotently.

**Proof.** We may assume that $J \neq \emptyset$. By the Hochschild–Serre spectral sequence, by first considering the action of $[m] \Gamma^J(\mathcal{M}[W])$ for some $m$, and then the induced action of the finite quotient $\Gamma^J(\mathcal{M}[W])/\mathcal{M}[\Gamma^J(\mathcal{M}[W])] \cong \mathbb{Z}/m\mathbb{Z}$, and then the induced action of $\Gamma^J/\Gamma_{\{j_0\}} \cong \Gamma^{J - \{j_0\}}$, it suffices to treat the special case where $J = \{j_0\}$ is a singleton and $\gamma_{j_0}$ acts unipotently. Then the lemma is reduced to its last statement, which follows from the same argument as in the proof of \cite[Lem. 2.10]{LZ17}, by matching a basis of $\mathbb{J}^\partial$ with binomial monomials up to degree $r - 1$ in $W$.

**Lemma 6.4.3.** Let $\mathbb{L}$ be a $\mathbb{Q}_p$-local system on $X_{\acute{e}t}$ such that $\mathbb{L}|_{U_{\acute{e}t}}$ has quasi-unipotent monodromy along (irreducible components of) $D$. For each $m \geq 1$, consider the canonical morphism $[m] : \mathbb{D} \to \mathbb{D}$, whose pullback under $f : X \to \mathbb{D}$ is a finite Kummer étale cover $g_m : X_m \to X$, which induces $f_m : X_m \mathbb{Z}/m \mathbb{D}$ by composition. Let $D_m$ denote the reduced subspace of $X_m \times_X D$ (in the category
of adic spaces), which is canonical isomorphic to \(D\) via the second projection, and let \(U_m := f_m^{-1}(X) = g_m^{-1}(U) = X_m - D_m\). Then there exists \(m_0 \geq 1\) such that \(R\psi_f^\eta_*(L_U) \cong R\psi_f^\eta_*(g_m^{-1}(L)|_{U_m})\) over \(D_{et}\), whenever \(m_0|m\).

Proof. For each \(m \geq 1\), let \(D_m^\eta\) denote the adic space \(D_m\) equipped with the log structure pulled back from \(X_m\). Let \(j_m : U_m \rightarrow X_m\), \(\gamma_m : D_m^\eta \rightarrow D_m\), and \(\varepsilon_m : D_m \rightarrow D_m\) denote the canonical morphisms. Then \[\mathcal{L}|_U \otimes_{Z_p} f_e^{-1}(\mathbb{K}_m) \cong (g_m|_{U_m})_{et,*}(\mathcal{L}|_{U_m})\]
over \(U_{et}\), and \[R\psi_f^\eta|_{et,*} \gamma_{m,ket,*}^{-1}(\mathcal{L}|_U \otimes_{Z_p} f_e^{-1}(\mathbb{K}_m) \otimes_{Z_p} f_e^{-1}(\mathbb{J}_r)) \cong R\psi_f^\eta|_{m,et,*} \gamma_{m,ket,*}^{-1}(\mathcal{L}|_{X_m} \otimes_{Z_p} f_e^{-1}(\mathbb{J}_r))\]
over \(D_{et}\), by Proposition 4.5.4 and Lemma 4.5.4. Since \(L\) has quasi-unipotent monodromy along \(D\), there exists some \(m_0 \geq 1\) such that \(L|_{U_m}\) has unipotent monodromy along \(D_m\), whenever \(m_0|m\).

We claim that, when \(m_0|m\), the canonical morphism \[R\psi_f^\eta|_{m,et,*} \gamma_{m,ket,*}^{-1}(L|_{X_m} \otimes_{Z_p} f_e^{-1}(\mathbb{J}_r)) \rightarrow R\psi_f^\eta|_{m,et,*} \gamma_{m,ket,*}^{-1}(L|_{X_m} \otimes_{Z_p} f_e^{-1}(\mathbb{J}_r))\]
induced by \(\mathbb{K}_{m_0} \rightarrow \mathbb{K}_m\) is an isomorphism for all sufficiently large \(r\) (depending on \(m_0\) and \(m\)). Given this claim, for all \(m\) divisible by \(m_0\), we have \[R\psi_f^\eta|_{m,et,*} \gamma_{m,ket,*}^{-1}(L|_{U_m} \otimes_{Z_p} f_e^{-1}(\mathcal{L}|_{X_m}) \otimes_{Z_p} f_e^{-1}(\mathbb{J}_r)) \cong R\psi_f^\eta|_{m,et,*} \gamma_{m,ket,*}^{-1}(L|_{U_m} \otimes_{Z_p} f_e^{-1}(\mathbb{J}_r))\]
for all sufficiently large \(r\), and the lemma follows.

In order to verify the claim, we may pullback to \(U_j^\beta(\xi)\), for all nonempty \(J \subset I\) and all geometric point \(\xi\) of \(U_j^\beta\). By Lemmas 4.5.4 and 6.4.2, it suffices to show that the canonical morphism \[H^i(|m_0|\Gamma^J, L|_{U_j^\beta(\xi)}|W]) \rightarrow H^i(|m|\Gamma^J, L|_{U_j^\beta(\xi)}|W])\]
is an isomorphism. By the Hochschild–Serre spectral sequence, we may first compare the cohomology of \([m_0]\Gamma^J(\xi_0)\) and \([m]\Gamma^J(\xi_0)\), for some \(\xi_0 \in J\), which is concentrated in degree zero and gives the full \(L|_{U_j^\beta(\xi)}\) in both cases, because \(\gamma^{m_0}\) and \(\gamma^m\) act unipotently, by assumption. Then we compare the cohomology groups of \([m_0]\Gamma^J(\xi_0)(\xi_0)\) and \([m]\Gamma^J(\xi_0)(\xi_0)\), which coincide as they are related by a Hochschild–Serre spectral sequence in terms of the cohomology of \(Q_p\)-modules with unipotent actions of the finite group \([m_0]\Gamma^J(\xi_0)(\xi_0)/[m]\Gamma^J(\xi_0)(\xi_0)).\]

\(\square\)

**Proposition 6.4.4.** Let \(L\) be a \(Q_p\)-local system on \(X_{ket}\) such that \(L|_{U_{et}}\) has quasi-unipotent monodromy along (irreducible components of) \(D\). Let \(m \geq 1\) be any integer such that \(L|_{U_m}\) has unipotent monodromy along \(D_m\), where \(U_m\) and \(D_m\) are as in Lemma 6.4.3. Then, for each nonempty \(J \subset I\) and each \(i \geq 0\), and for each geometric point \(\xi\) of \(U_j^\beta\), we have \(R^i\psi_f^\eta(L_U|_{U_j^\beta(\xi)} \cong H^i(\Gamma^J, L|_{U_j^\beta(\xi)}|W])\) and \(R^i\psi_f^\eta(L_U|_{U_j^\beta(\xi)} \cong H^i(|m|\Gamma^J, L|_{U_j^\beta(\xi)}|W])\) as \(Q_p\)-local systems on \(U_j(\xi)_{et}\).

Consequently, if \(D = (f^{-1}(0))_{red}\) is smooth over \(k\) and if \(L|_U\) has quasi-anipotent monodromy along \(D\), then \(R^i\psi_f^\eta(L|_U)\) is concentrated in degree zero and can be identified with the subsheaf \(L^{|U|}\) of \(L|_{D^\eta}\) whose pullback to \(D^\eta(\xi)\) is the maximal subsheaf on which \(\pi^1_{ket}(D^\eta(\xi), \xi) \cong \mathbb{Z}(1)\) (as in Example 4.4.26) acts unipotently, for
each log geometric point $\tilde{\xi}$ of $D^0$ above each geometric point $\xi$ of $D$; and $R\Psi_f^\triangleright(L|_U)$ (which is the same $R\Psi_f^\triangleright(L|_U)$ as above when $L|_U$ has unipotent monodromy along $D$) is also concentrated in degree zero and can be identified with the whole $L|_{D^0}$.

**Proof.** Combine Lemmas 4.5.4, 6.4.2, and 6.4.3.

---

**Appendix A. Kiehl’s property for coherent sheaves**

In this appendix, by adapting the gluing argument in [KL15, Sec. 2.7] and by using [Hub94, Thm. 2.5], we establish Kiehl’s property for coherent sheaves on (possibly nonanalytic) noetherian adic spaces. By combining this with results in [KL15, Sec. 8.2] and [Ked19, Sec. 1.3–1.4], we also state some versions of Tate’s sheaf property and Kiehl’s gluing property for adic spaces that are either locally noetherian or analytic and étale sheafy.

Recall the following definition from [KL15, Def. 1.3.7]:

**Definition A.1.** By a **gluing diagram**, we will mean a commuting diagram of ring homomorphisms

$$
\begin{array}{ccc}
R & \longrightarrow & R_1 \\
\downarrow & & \downarrow \\
R_2 & \longrightarrow & R_{12}
\end{array}
$$

such that the $R$-module sequence

$$0 \to R \to R_1 \oplus R_2 \to R_{12} \to 0,$$

in which the last nontrivial arrow is the difference between the given homomorphisms, is exact. By a **gluing datum** over this diagram, we mean a datum consisting of modules $M_1$, $M_2$, and $M_{12}$ over $R_1$, $R_2$, and $R_{12}$, respectively, equipped with isomorphisms $\psi_1 : M_1 \otimes_R R_{12} \iso M_{12}$ and $\psi_2 : M_2 \otimes_R R_{12} \iso M_{12}$. We say such a gluing datum is **finite** if the modules are finite over the respective rings.

Given a gluing datum as above, let $M := \ker(\psi_1 - \psi_2 : M_1 \oplus M_2 \to M_{12})$. There are natural morphisms $M \to M_1$ and $M \to M_2$ of $R$-modules, which induce maps $M \otimes_R R_1 \to M_1$ and $M \otimes_R R_2 \to M_2$, respectively.

The following is [KL15, Lem. 1.3.8]:

**Lemma A.2.** Consider a finite gluing datum for which $M \otimes_R R_1 \to M_1$ is surjective. Then the following are true.

1. The morphism $\psi_1 - \psi_2 : M_1 \oplus M_2 \to M_{12}$ is surjective.
2. The morphism $M \otimes_R R_2 \to M_2$ is also surjective.
3. There exists a finitely generated $R$-submodule $M_0$ of $M$ such that, for $i = 1, 2$, the morphism $M_0 \otimes_R R_i \to M_i$ is surjective.

**Lemma A.3.** In the above setting, suppose in addition that $R_i$ is noetherian and that $R_i \to R_{12}$ is flat, for $i = 1, 2$. Suppose that, for every finite gluing datum, the map $M \otimes_R R_1 \to M_1$ is surjective. Then, for any finite gluing datum, $M$ is a finitely presented $R$-module, and $M \otimes_R R_1 \to M_1$ and $M \otimes_R R_2 \to M_2$ are bijective.

**Proof.** Choose $M_0$ as in Lemma A.2. Choose a surjection $F \to M_0$ of $R$-modules, with $F$ finite free, and put $F_1 = F \otimes_R R_1$, $F_2 = F \otimes_R R_2$, $F_{12} = F \otimes_R R_{12}$,
$N = \ker(F \to M), N_1 = \ker(F_1 \to M_1), N_2 = \ker(F_2 \to M_2),$ and $N_{12} = \ker(F_{12} \to M_{12}).$ By Lemma A.2, we have a commutative diagram

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & N & N_1 \oplus N_2 & N_{12} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & F & F_1 \oplus F_2 & F_{12} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & M & M_1 \oplus M_2 & M_{12} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0
\end{array}
\]

with exact rows and columns, excluding the dotted arrows. Since $R_{12}$ is flat over $R_i$, the sequence

\[
0 \to N_i \otimes_{R_i} R_{12} \to F_{12} \to M_{12} \to 0
\]

is exact, and hence $N_i \otimes_{R_i} R_{12} \equiv N_{12}$. By hypothesis, $R_i$ is noetherian, and so $N_i$ is finite over $R_i$. Consequently, $N_1$, $N_2$, and $N_{12}$ form a finite gluing datum as well. By Lemma A.2 again, the dotted horizontal arrow in (A.4) is surjective. By diagram chasing, the dotted vertical arrow in (A.4) is also surjective; that is, we may add the dotted arrows to (A.4) while preserving exactness of the rows and columns. In particular, $M$ is a finitely generated $R$-module. It follows that $N$ is finitely generated. This implies that $M$ is finitely presented.

For $i = 1, 2$, we obtain a commutative diagram

\[
\begin{array}{cccc}
N \otimes_{R_i} R_i & F_i & M \otimes_{R_i} R_i & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & N_i & F_i & M_i \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0
\end{array}
\]

with exact rows—the first one is derived from the left column of (A.4) by tensoring over $R$ with $R_i$, while the second one is derived from the middle column of (A.4). Since the left vertical arrow is surjective, by the five lemma, the right vertical arrow is injective. It follows that the map $M \otimes_{R_i} R_i \to M_i$ is a bijection, as desired. \[\Box\]

**Definition A.5.** We call a homomorphism of Huber rings $f : A \to B$ **strict adic** if, for one (and hence every) choice of an ideal of definition $I \subset A$, the image $f(I)$ is an ideal of definition of $B$. It is clear that a strict adic morphism is adic.

The following is modeled on [KL15 Lem. 2.7.2].

**Lemma A.6.** Let $R_1 \to S$ and $R_2 \to S$ be homomorphisms of complete Huber rings such that their sum $\psi : R_1 \oplus R_2 \to S$ is strict adic. Then, for any ideal of definition $I_S$ of $S$, there exists some $l > 0$ such that, for every $n > 0$, every $U \in \text{GL}_n(S)$ with $U - 1 \in M_n(I_S^l)$ is of the form $\psi(U_1) \psi(U_2)$ for some $U_i \in \text{GL}_n(R_i)$.

**Proof.** Since $\psi$ is strict adic, for any ideals of definition $I_1 \subset R_1$ and $I_2 \subset R_2$, we have an ideal of definition $I_S^l := \psi(I_1 \oplus I_2) \subset S$. Choose $l > 0$ such that
Choose sets of generators that $\psi(U_j) \psi(U_2)$ for some $U_i \in \text{GL}_n(R_i)$. Given $U \in \text{GL}_n(S)$ with $U - 1 \in M_n(I_S^m)$ for some $m > 0$, put $V = U - 1$. By assumption, we may lift $V$ to a pair $(X, Y) \in M_n(I^n_1) \times M_n(I^n_2)$. Then it is straightforward that the matrix $U' = \psi(1 - X) U \psi(1 - Y)$ satisfies $U' - 1 \in M_n(I_S^{2m})$. Hence, we may construct the desired matrices by iterating this construction. □

The following is modeled on [KL15, Lem. 2.7.4].

**Lemma A.7.** In the context of Definition A.1 and the paragraph following it, suppose in addition that

1. the Huber rings $R_1$, $R_2$ and $R_{12}$ are complete;
2. $R_1 \oplus R_2 \rightarrow R_{12}$ is strict adic; and
3. the map $R_2 \rightarrow R_{12}$ has a dense image.

Then, for $i = 1, 2$, the natural map $M \hat{\otimes}_R R_i \rightarrow M_i$ is surjective.

**Proof.** Choose sets of generators $\{m_{1,1}, \ldots, m_{n,1}\}$ and $\{m_{1,2}, \ldots, m_{n,2}\}$ of $M_1$ and $M_2$, respectively, of the same cardinality. Then there exist $A, B \in M_n(R_{12})$ such that $\psi_2(m_{j,1}) = \sum_i A_{ij} \psi_1(m_{i,2})$ and $\psi_1(m_{j,2}) = \sum_i B_{ij} \psi_2(m_{i,1})$, for all $j$. Since $R_2 \rightarrow R_{12}$ has a dense image, by Lemma A.6 there exists $B' \in M_n(R_2)$ such that $1 + A(B' - B) = C_1 C_2^{-1}$ for some $C_i \in \text{GL}_n(R_i)$. For each $j = 1, \ldots, n$, let $x_j := (x_{j,1}, x_{j,2}) = (\sum_i (C_1)_{ij} m_{i,1}, \sum_i (B'C_2)_{ij} m_{i,2}) \in M_1 \times M_2$. Then $\psi_2(x_{j,1}) - \psi_2(x_{j,2}) = \sum_i (C_1 - AB'C_2)_{ij} \psi_1(m_{i,1}) = \sum_i ((1 - AB)C_2)_{ij} \psi_1(m_{i,2}) = 0$, and hence $x_j \in M$. Since $C_i \in \text{GL}_n(R_i)$, for $i = 1, 2$, it follows that $\{x_{j,1}\}_{j=1}^n$ generates $M_i$ over $R_i$ as well. Hence, $M \hat{\otimes}_R R_i \rightarrow M_i$ is surjective, as desired. □

**Theorem A.8.** Let $X = \text{Spa}(R, R^+)$ be a noetherian affinoid adic space. The categories of coherent sheaves on $X$ and finitely generated $R$-modules are equivalent via the global sections functor.

**Proof.** By [KL15, Lem. 2.4.20], it suffices to verify Kiehl’s gluing property for any simple Laurent covering $\{\text{Spa}(R_i, R^+_i) \rightarrow X\}_{i=1,2}$. Write $\text{Spa}(R_{12}, R^+_{12}) = \text{Spa}(R_1, R^+_1) \times_X \text{Spa}(R_2, R^+_2)$, with all Huber pairs completed by our convention. By the noetherian hypothesis, and by [Hub94, Thm. 2.5], $R$, $R_i$, and $R_{12}$ form a gluing diagram. Moreover, $R_i \rightarrow R_{12}$ is flat with a dense image, for $i = 1, 2$. Thus, we can conclude the proof by applying Lemmas A.3 and A.7 as desired. □

Thus, we have the following version of Tate’s sheaf property and Kiehl’s gluing property (see [KL15, Def. 2.7.6]) over certain affinoid adic spaces:

**Proposition A.9.** Let $X = \text{Spa}(R, R^+)$ be a noetherian (resp. analytic) affinoid adic space, and let $M$ be a finite (resp. finite projective) $R$-module. Let $\tilde{M}$ denote the presheaf on $X$ defined by setting $\tilde{M}(U) = M \hat{\otimes}_R \mathcal{O}_X(U)$, for each open subset $U \subset X$. Then the following are true:

1. The presheaf $\tilde{M}$ is a sheaf. Moreover, the sheaf $\tilde{M}$ is acyclic in the sense that $H^i(U, \tilde{M}) = 0$ for every rational subset $U \subset X$ and every $i > 0$.
2. The functor $M \mapsto \tilde{M}$ defines an equivalence of categories from the category of finite (resp. finite projective) $R$-modules to the category of coherent sheaves (resp. vector bundles) on $X$, with a quasi-inverse given by $\mathcal{F} \mapsto \mathcal{F}(X)$. 
Proof. When $X$ is noetherian, (1) is [Hub94, Thm. 2.5], while (2) is Theorem A.8. When $X$ is analytic, these follow from [Ked19, Thm. 1.4.2 and 1.3.4]. □

By [KL15, Prop. 8.2.20], we have the following analogue of Proposition A.9:

**Proposition A.10.** Let $X = \text{Spa}(R, R^+)$ be a noetherian (resp. analytic étale sheafy) affinoid adic space. Let $B$ be a basis of $X_{\text{ét}}$ as in our notation and conventions on page 3, which exists because $X$ is étale sheafy. Let $M$ be a finite (resp. finite projective) $R$-module. Let $\tilde{M}$ denote the presheaf on $X_{\text{ét}}$ defined by setting $\tilde{M}(U) = M \hat{\otimes}_R \mathcal{O}_X(U)$, for each $U \in X_{\text{ét}}$. Then the following are true:

1. The presheaf $\tilde{M}$ is a sheaf. Moreover, $\tilde{M}$ is acyclic on $B$; i.e., for every $Y \in B$, we have $H^0(Y, \tilde{M}) = \tilde{M}(U)$ and $H^i(Y, \tilde{M}) = 0$, for all $i > 0$.
2. The functor $M \mapsto \tilde{M}$ defines an equivalence of categories from the category of finite (resp. finite projective) $R$-modules to the category of coherent sheaves (resp. vector bundles) on $X_{\text{ét}}$, with a quasi-inverse given by $F \mapsto F(X)$.

**Corollary A.11.** For any $X$ in Proposition A.10, the presheaf $\mathcal{O}_{X_{\text{ét}}}$ is a sheaf.

**References**


