VANISHING THEOREMS FOR TORSION AUTOMORPHIC SHEAVES ON COMPACT PEL-TYPE SHIMURA VARIETIES

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Abstract. Given a compact PEL-type Shimura variety, a sufficiently regular weight (defined by mild and effective conditions), and a prime number \( p \) unramified in the linear data and larger than an effective bound given by the weight, we show that the (Betti) cohomology with \( \mathbb{Z}_p \)-coefficients of the given weight vanishes away from the middle degree, and hence has no \( p \)-torsion. We do not need any other assumption (such as ones on the images of the associated Galois representations).

Introduction

The cohomology of Shimura varieties (with coefficients in algebraic representations of the associated reductive groups) has been an important tool for studying the relation between the theory of automorphic forms and arithmetic. In this article, we try to answer a basic question:

Question. Let \( p \) be a prime number. When is the (Betti) cohomology of the Shimura variety with (possibly non-trivial) integral coefficients \( p \)-torsion free?

Certainly, when we fix both the level and the coefficient system, the answer is in the affirmative for all sufficiently large \( p \). But to the best of our knowledge, there has been no known, effective bound that applies to general Shimura varieties. Moreover, it is a priori unclear whether such a bound can be found that is insensitive to raising the level, even if we focus only on neat and prime-to-\( p \) levels.

The main results of this article provide the following (partial) answer: Consider a compact PEL-type Shimura variety at a neat level, a weight \( \mu \) that is “sufficiently regular” (a mild and effective condition which, in the unitary case, coincides with the usual notion of regularity), and a prime number \( p \) that is unramified in the linear data defining the Shimura variety. If the level is maximal hyperspecial at \( p \) and if \( p \) is larger than an effective bound that is a function of \( \mu \) (but is independent of the prime-to-\( p \) level), then the Betti cohomology of the variety with coefficients

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\end{itemize}

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in the $\mathbb{Z}_p$-module corresponding to $\mu$ is concentrated in the middle degree, and has no $p$-torsion. (See Theorem 8.12 and Corollary 8.13 for more precise statements. Variants in other cohomology theories are also given in Section 8.)

We stress that all the conditions we need are explicit and can be verified easily in practice. We do not make any assumptions such as the ones on the images of the associated Galois representations (which are often far from effective). (See for example the remark in §9, line 6] on their “residually large image” assumption (RLI).)

Our approach to this problem is to use the de Rham cohomology of the good reduction modulo $p$ of the Shimura variety in question. The main technical inputs are Illusie’s vanishing theorem, Faltings’s dual BGG construction, and a new observation relating the (geometric) “Kodaira type” conditions on the coefficient systems to the (representation-theoretic) “sufficient regularity” conditions.

Although all the techniques we use have been known for many years, their simple combination (when the level is neat and prime-to-$p$) has not been implemented in any special cases. By base extension to $\mathbb{C}$, we also obtain the first purely algebraic proof of certain vanishing results that had only been proved by transcendental methods.

We remark that closely related questions on (the absence of) $p$-torsion in the cohomology of Lubin–Tate towers have been considered in the work of Boyer. Our approach differs fundamentally from his, and does not subsume the results there.

Here is an outline of the article. In Sections 1 and 2, we review the basic setups in geometry and representation theory, which are standard but necessary. In Sections 3–4, we explain the realization of automorphic bundles and their cohomology using fiber products of the universal abelian scheme over our Shimura variety, following [13, pp. 234–235], [19, III.2], and [37, II.2]. In Section 5, we explain how the comparison among different cohomology theories with automorphic coefficients can be reduced to the standard results with constant coefficients. (We work out these sections in detail, sometimes with steps not readily available in the literature, because we want to pin down optimal bounds on the sizes of $p$.) In Section 6, we introduce Illusie’s vanishing theorem [22] and its reformulations using Faltings’s dual BGG construction. Then we explain our key observation (mentioned above) in Section 7, with an analysis on ample automorphic line bundles with weights of “minimal size”. This is the most crucial part of this article. The main results will be presented in Section 8 including our vanishing theorems for cohomology with automorphic coefficients, and their obvious implications to questions of torsion-freeness and liftability.

The ideas in this article can be generalized to all PEL-type cases (including non-compact ones), which we have carried out in the article [31]. See the introduction there for more details.

The results in this article on torsion-freeness and liftability have potential applications to the study of $p$-adic modular forms and Taylor–Wiles systems. (For example, Michael Harris has applied our results to the study of Taylor–Wiles systems. See [18].) After all, very little has been known (or even conjectured) about the torsion in the cohomology of Shimura varieties. We naturally expect more of such interesting results and applications to appear in the future.

We shall follow [29, Notations and Conventions] unless otherwise specified.
1. **Geometric setup**

1.1. **Linear algebraic data.** Let \((\mathcal{O}, \star, L, \langle \cdot, \cdot \rangle, h_0)\) be an integral PEL datum in the following sense:

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1. **Geometric setup**

1.1. **Linear algebraic data.** Let \((\mathcal{O}, \star, L, \langle \cdot, \cdot \rangle, h_0)\) be an integral PEL datum in the following sense:
(1) \( \mathcal{O} \) is an order in a (nonzero) semisimple algebra, finite-dimensional over \( \mathbb{Q} \),

together with a positive involution \( * \).

(2) \( (L, \langle \cdot, \cdot \rangle, h_0) \) is a PEL-type \( \mathcal{O} \)-lattice as in [29] Def. 1.2.1.3]. (In [29] Def. 1.2.1.3] \( h_0 \) was denoted by \( h \).)

We shall denote the center of \( \mathcal{O} \otimes \mathbb{Q} \) by \( F \). (Then \( F \) is a product of number fields.)

**Definition 1.1** (cf. [29] Def. 1.2.1.5]). Let \( \mathcal{O} \) and \( (L, \langle \cdot, \cdot \rangle) \) be given as above.

Then we define for any \( \mathcal{O} \)-module \( g,r \) such that

\( h(g,r) = r \)

The assignment is functorial in \( \mathcal{O} \) and defines a group functor \( G \) over \( \text{Spec}(\mathbb{Z}) \). The pairing to the second factor \( (g,r) \mapsto r \) defines a homomorphism \( v : G \to \mathbb{G}_m \),

which we call the **similitude character**. For simplicity, we shall often denote elements \( (g,r) \) in \( G \) only by \( g \), and denote by \( v(g) \) the value of \( r \) when we need it.

(This is an abuse of notation, because \( r \) is not always determined by \( g \).)

The homomorphism \( h_0 : \mathbb{C} \to \text{End}_{\mathcal{O} \otimes \mathbb{R}}(L \otimes \mathbb{R}) \) defines a Hodge structure of weight \(-1\) on \( L \), with Hodge decomposition

\[
(1.2) \quad L \otimes \mathbb{C} = V_0 \oplus V_0^c, 
\]

such that \( h_0(z) \) acts as \( 1 \otimes z \) on \( V_0 \), and as \( 1 \otimes z^c \) on \( V_0^c \). Let \( F_0 \) be the reflex field (see [29] Def. 1.2.5.4]) defined by the \( \mathcal{O} \otimes \mathbb{C} \)-module \( V_0 \).

By abuse of notation, we shall denote the ring of integers in \( F \) (resp. \( F_0 \)) by \( \mathcal{O}_F \) (resp. \( \mathcal{O}_{F_0} \)). This is in conflict with the notation of the order \( \mathcal{O} \) in the integral PEL datum, but the precise interpretation will be clear from the context.

Let \( \text{Diff}^{-1} \) be the inverse different of \( \mathcal{O} \) over \( \mathbb{Z} \) (see [29] Def. 1.1.1.11]), and let \( \text{Disc} = [\text{Diff}^{-1} : \mathcal{O}] \mathbb{Z} \) be the discriminant of \( \mathcal{O} \) over \( \mathbb{Z} \) (see [29] Def. 1.1.1.6 and Prop. 1.1.1.12]). We say that a rational prime number \( p > 0 \) is good if it satisfies the following conditions (cf. [29] §5 and [29] Def. 1.4.1.1):

(1) \( p \) is unramified in \( \mathcal{O} \), in the sense that \( p \nmid \text{Disc} \) (as in [29] Def. 1.1.1.14]).

(2) \( p \neq 2 \) if \( \mathcal{O} \otimes \mathbb{Q} \) involves simple factors of type D (as in [29] Def. 1.2.1.15]).

(3) The pairing \( \langle \cdot, \cdot \rangle \) is perfect after base change to \( \mathbb{Z}_p \).

Let us fix any choice of a good prime \( p > 0 \).

**Lemma 1.3.** There exists a finite extension \( F'_0 \) of \( F_0 \) in \( \mathbb{C} \), unramified at \( p \), together with an \( \mathcal{O} \otimes \mathcal{O}_{F'_0/(p)} \)-module \( L_0 \) such that \( L_0 \otimes \mathcal{O} \otimes \mathbb{C} \cong V_0 \) as \( \mathcal{O} \otimes \mathbb{C} \)-modules.

See [29] Lem. 1.2.5.9 in the revision] for a proof. For each fixed \( F'_0 \), the choice of \( L_0 \) is unique up to isomorphism because \( \mathcal{O} \otimes \mathcal{O}_{F'_0/(p)} \)-modules are uniquely determined by their multi-ranks. (See [29] Lem. 1.1.3.4]. We will review the notion of multi-ranks in Section 2.1)

Let us denote by \( \langle \cdot, \cdot \rangle_{\text{can.}} : (L_0 \oplus L'_0(1)) \times (L_0 \oplus L'_0(1)) \to \mathcal{O}_{F'_0/(p)}(1) \) (cf. [29] Lem. 1.4.1.16]) the alternating pairing \( \langle (x_1, f_1), (x_2, f_2) \rangle_{\text{can.}} := f_2(x_1) - f_1(x_2) \). The natural right action of \( \mathcal{O} \) on \( L'_0(1) \) defines a natural left action of \( \mathcal{O} \) by composition with the involution \( * : \mathcal{O} \to \mathcal{O} \). Then (1.2) canonically induces an isomorphism \( L'_0(1) \otimes \mathbb{C} \cong V'_0 \) of \( \mathcal{O} \otimes \mathbb{C} \)-modules.
Definition 1.4. For any $\mathcal{O}_{F_0^{\prime}(p)}$-algebra $R$, set

$$
G_0(R) := \left\{ (g,r) \in \text{GL}_{\mathcal{O}} \otimes_{\mathbb{Z}} (L_0 \oplus L_0^\vee(1)) \otimes \mathcal{O}_{F_0^{\prime}(p)} \right\},
$$

$$
P_0(R) := \left\{ (g,r) \in G_0(R) : g(L_0^\vee(1) \otimes \mathcal{O}_{F_0^{\prime}(p)}) = L_0^\vee(1) \otimes \mathcal{O}_{F_0^{\prime}(p)} \right\},
$$

$$
M_0(R) := \text{GL}_{\mathcal{O}} \otimes_R (L_0^\vee(1) \otimes \mathcal{O}_{F_0^{\prime}(p)}) \times G_m(R),
$$

where we view $M_0(R)$ canonically as a quotient of $P_0(R)$ by

$$
P_0(R) \rightarrow M_0(R) : (g,r) \mapsto (g|_{L_0^\vee(1) \otimes \mathcal{O}_{F_0^{\prime}(p)}} \otimes R, r).
$$

The assignments are functorial in $R$, and define group functors $G_0$, $P_0$, and $M_0$ over $\text{Spec}(\mathcal{O}_{F_0^{\prime}(p)})$.

By [29, Prop. 1.1.1.17, Cor. 1.2.5.7, and Cor. 1.2.3.10], there exists a discrete valuation ring $R_1$ over $\mathcal{O}_{F_0^{\prime}(p)}$ satisfying the following conditions:

1. The maximal ideal of $R_1$ is generated by $p$, and the residue field $\kappa_1$ of $R_1$ is a finite field of characteristic $p$. In this case, the $p$-adic completion of $R_1$ is isomorphic to the Witt vectors $W(\kappa_1)$ over $\kappa_1$.

2. The ring $\mathcal{O}_{F}$ is split over $R_1$, in the sense that $\Upsilon := \text{Hom}_{\mathbb{Z}, \text{alg}}(\mathcal{O}_{F}, R_1)$ has cardinality $[F : \mathbb{Q}]$. Then there is a canonical isomorphism

$$\mathcal{O}_F \otimes_{\mathbb{Z}} R_1 \cong \prod_{\tau \in \Upsilon} \mathcal{O}_{F, \tau}$$

where each $\mathcal{O}_{F, \tau}$ can be identified as the $\mathcal{O}_{F}$-algebra $R_1$ via $\tau$.

3. There exists an isomorphism

$$L \otimes_{\mathbb{Z}} R_1 \cong (L_0 \oplus L_0^\vee(1), (\cdot, \cdot)_{\text{can.}}) \otimes \mathcal{O}_{F_0^{\prime}(p)}$$

inducing an isomorphism $G \otimes_{\mathbb{Z}} R_1 \cong G_0 \otimes_{\mathcal{O}_{F_0^{\prime}(p)}} R_1$ realizing $P_0 \otimes_{\mathcal{O}_{F_0^{\prime}(p)}} R_1$ as a subgroup of $G \otimes_{\mathbb{Z}} R_1$. (The existence of the isomorphism (1.6) follows from [29, Cor. 1.2.3.10] by comparing multi-ranks.)

Remark 1.7. For the purpose of studying questions such as the vanishing or freeness of cohomology with torsion coefficients, it is harmless (and helpful) to enlarge the coefficient rings.

From now on, let us fix the choices of $R_1$ and the isomorphism (1.6), and set

$$G_{F,1} := \mathcal{O}_F \otimes_{\mathbb{Z}} R_1, \quad \mathcal{O}_1 := \mathcal{O} \otimes_{\mathbb{Z}} R_1, \quad L_1 := L \otimes_{\mathbb{Z}} R_1, \quad L_{0,1} := L_0 \otimes_{\mathbb{Z}} R_1, \quad G_1 := \mathcal{O}_{F_0^{\prime}(p)} \otimes_{\mathbb{Z}} R_1, \quad P_1 := P_0 \otimes_{\mathcal{O}_{F_0^{\prime}(p)}} R_1,$$

and $M_1 := M_0 \otimes_{\mathcal{O}_{F_0^{\prime}(p)}} R_1$.

1.2. PEL-type Shimura varieties. Let $\mathcal{H}$ be a neat open compact subgroup of $G(\mathbb{Z}_p)$. (See [11, 0.6] or [29, Def. 1.4.1.8] for the definition of neatness.)

By [29, Def. 1.4.1.4] (with $\square = \{ p \}$ there), the data of $(L, (\cdot, \cdot), h_0)$ and $\mathcal{H}$ define a moduli problem $M_\mathcal{H}$ over $\mathcal{S}_0 = \text{Spec}(\mathcal{O}_{F_0^{\prime}(p)})$, parameterizing tuples $(A, \lambda, i, \alpha_\mathcal{H})$ over schemes $S$ over $\mathcal{S}_0$ of the following form:
(1) $A \to S$ is an abelian scheme.
(2) $\lambda : A \to A^\vee$ is a polarization of degree prime to $p$.
(3) $i : \mathcal{O} \to \text{End}_S(A)$ is an $O$-endomorphism structure as in \cite{29} Def. 1.3.3.1.
(4) $\text{Lie}_{A/S}$ with its $\mathcal{O} \otimes \mathbb{Z}(p)$-module structure given naturally by $i$ satisfies the
determinantal condition in \cite{29} Def. 1.3.4.2 given by $(L \otimes \mathbb{R}, \langle \cdot, \cdot \rangle, h_0)$.
(5) $\alpha_H$ is an (integral) level-$H$ structure of $(A, \lambda, i)$ of type $(L \otimes \mathbb{Z}(p), \langle \cdot, \cdot \rangle)$ as
in \cite{29} Def. 1.3.7.8].

**Remark.** The definition (by isomorphism classes) can be identified with the one in
\cite{26} [5] (by prime-to-$p$ quasi-isogeny classes) by \cite{29} Prop. 1.4.3.3].

By \cite{29} Thm. 1.4.1.12 and Cor. 7.2.3.10, $M_H$ is representable by a (smooth) quasi-projective scheme over $S_0$ (under the assumption that $H$ is neat).

Consider the (real analytic) set $X = G(\mathbb{R})h_0$ of $G(\mathbb{R})$-conjugates $h : \mathbb{C} \to \text{End}_{\mathcal{O} \otimes \mathbb{R}}(L \otimes \mathbb{R})$ of $h_0 : \mathbb{C} \to \text{End}_{\mathcal{O} \otimes \mathbb{R}}(L \otimes \mathbb{R})$. Let $H^p := H$ and $H_p := G(\mathbb{Z}_p)$
be open compact subgroups of $G(\hat{\mathbb{Z}})$ and $G(\mathbb{Q}_p)$, respectively, and let $H$ be the open compact subgroup $H^p H_p$ of $G(\hat{\mathbb{Z}})$. It is well known (see \cite{26} \S 8 or \cite{27} \S 2)]
that there exists a quasi-projective variety $\text{Sh}_H$ over $F_0$, together with a canonical open and closed immersion $\text{Sh}_H \hookrightarrow M_H \otimes_{O_{F_0}(p)} F_0$ (because $H$ is neat), such that the
analytization of $\text{Sh}_H \otimes \mathbb{C}$ can be canonically identified with the double coset space
$G(\mathbb{Q}) \backslash X \times G(A^\infty)/H$. (Note that $\text{Sh}_H \hookrightarrow M_H \otimes_{O_{F_0}(p)} F_0$ is not an isomorphism in
general, due to the so-called “failure of Hasse’s principle”. See for example \cite{26} \S 8 and \cite{29} Rem. 1.4.3.11).]

Let $M_{H,0}$ denote the schematic closure of $\text{Sh}_H$ in $M_H$. Then $M_{H,0}$ is smooth over
$S_0$. In this article, we shall maintain from now on the following:

**Assumption 1.8.** The double coset space $G(\mathbb{Q}) \backslash X \times G(A^\infty)/H$, with its real analytic
structure inherited from $X$, is compact.

**Theorem 1.9** (see [28] \S 4]). Under Assumption 1.8, $M_{H,0}$ is proper (and hence projective) over $S_0$.

**Remark** 1.10. The dimension of $X$ as a complex manifold, and hence the relative dimension of any component of the smooth scheme $M_{H,0}$ over $S_0$, can be calculated easily because $X$ is embedded as an open subset of $G_0(\mathbb{C})/P_0(\mathbb{C})$ (by sending any
$h \in X$ to the Hodge filtration it defines).

Let $S_1 := \text{Spec}(R_1)$, and let $M_{H,1} := M_{H,0} \times S_1$. By abuse of notation, we denote
the pullback of the universal object over $M_H$ to $M_{H,1}$ by $(A, \lambda, i, \alpha_H) \to M_{H,1}$.

Consider the relative de Rham cohomology $H^1_{\text{dR}}(A/M_{H,1})$ and the relative
de Rham homology $H^1_{\text{dR}}(A/M_{H,1}) := \text{Hom}_{M_{H,1}}(H^1_{\text{dR}}(A/M_{H,1}), \mathcal{O}_{M_{H,1}})$. We
have the canonical pairing $\langle \cdot, \cdot \rangle_{\lambda} : H^1_{\text{dR}}(A/M_{H,1}) \times H^1_{\text{dR}}(A/M_{H,1}) \to \mathcal{O}_{M_{H,1}}(1)$
defined as the composite of $(\text{Id} \times \lambda)$, followed by the perfect pairing $H^1_{\text{dR}}(A/M_{H,1}) \times H^1_{\text{dR}}(A^\vee/M_{H,1}) \to \mathcal{O}_{M_{H,1}}(1)$ defined by the first Chern class of the Poincaré line bundle over $A \times A^\vee$. (See for example \cite{10} 1.5.) Under the assumption that $\lambda$ has degree prime-to-$p$, we know that
\( \lambda \) is separable, that \( \lambda * \) is an isomorphism, and hence that the pairing \( \langle \cdot, \cdot \rangle _{\lambda} \) above is perfect. Let \( \langle \cdot, \cdot \rangle _{\lambda} \) also denote the induced pairing on \( H^1_{\text{dR}}(A/M_{\mathcal{H},1}) \times H^1_{\text{dR}}(A/M_{\mathcal{H},1}) \) by duality. By Lemma 2.53, we have canonical short exact sequences \( 0 \rightarrow \text{Lie}^\vee_{M_{\mathcal{H},1}} \rightarrow H^1_{\text{dR}}(A/M_{\mathcal{H},1}) \rightarrow \text{Lie}^\lambda_{A/M_{\mathcal{H},1}} \rightarrow 0 \) and \( 0 \rightarrow \text{Lie}^\vee_{M_{\mathcal{H},1}} \rightarrow H^1_{\text{dR}}(A/M_{\mathcal{H},1}) \rightarrow \text{Lie}^\lambda_{A/M_{\mathcal{H},1}} \rightarrow 0 \). The submodules \( \text{Lie}^\vee_{A/M_{\mathcal{H},1}} \) and \( \text{Lie}^\lambda_{A/M_{\mathcal{H},1}} \) are maximal totally isotropic with respect to \( \langle \cdot, \cdot \rangle _{\lambda} \).

Let \( \tilde{M}_{\mathcal{H},1} \) be the first infinitesimal neighborhood of the diagonal image of \( M_{\mathcal{H},1} \) in \( M_{\mathcal{H},1} \times M_{\mathcal{H},1} \), and let \( \text{pr}_1, \text{pr}_2 : \tilde{M}_{\mathcal{H},1} \rightarrow M_{\mathcal{H},1} \) be the two projections. Then we have by definition the canonical morphism \( O_{M_{\mathcal{H},1}} \rightarrow \mathcal{O}^1_{M_{\mathcal{H},1}/S_1} := \text{pr}_1 \ast \text{pr}_2 \ast (O_{M_{\mathcal{H},1}}) \), where \( \mathcal{O}^1_{M_{\mathcal{H},1}/S_1} \) is the sheaf of principal parts of order one. The isomorphism \( s : \tilde{M}_{\mathcal{H},1} \rightarrow \tilde{M}_{\mathcal{H},1} \) over \( M_{\mathcal{H},1} \) swapping the two components of the fiber product then defines an automorphism \( s^* \) of \( \mathcal{O}^1_{M_{\mathcal{H},1}/S_1} \). The kernel of the structural morphism \( \text{str}^* : \mathcal{O}^1_{M_{\mathcal{H},1}/S_1} \rightarrow \mathcal{O}_{M_{\mathcal{H},1}/S_1} \) canonically isomorphic to \( \Omega^1_{M_{\mathcal{H},1}/S_1} \) by definition, is spanned by the image of \( s^* - \text{Id}^* \) (induced by \( \text{pr}_1^* - \text{pr}_2^* \)).

An important property of the relative de Rham cohomology of any smooth morphism like \( A \rightarrow M_{\mathcal{H},1} \) is that, for any two smooth lifts \( \tilde{A}_1 \rightarrow M_{\mathcal{H},1} \) and \( \tilde{A}_2 \rightarrow M_{\mathcal{H},1} \) of \( A \rightarrow M_{\mathcal{H},1} \), there is a canonical isomorphism \( H^1_{\text{dR}}(\tilde{A}_2/\tilde{M}_{\mathcal{H},1}) \simeq H^1_{\text{dR}}(\tilde{A}_1/\tilde{M}_{\mathcal{H},1}) \) lifting the identity morphism on \( H^1_{\text{dR}}(A/M_{\mathcal{H},1}) \). (See for example [29 Prop. 2.1.6.4].)

If we consider \( \tilde{A}_1 := pr_1^* A \) and \( \tilde{A}_2 := pr_2^* A \), then we obtain a canonical isomorphism \( pr_2^* H^1_{\text{dR}}(A/M_{\mathcal{H},1}) \cong H^1_{\text{dR}}(\text{pr}_2^* A/\tilde{M}_{\mathcal{H},1}) \simeq H^1_{\text{dR}}(pr_1^* A/\tilde{M}_{\mathcal{H},1}) \cong pr_1^* H^1_{\text{dR}}(A/M_{\mathcal{H},1}) \), which we denote by \( \text{Id}^* \) by abuse of notation. On the other hand, the pullback by the swapping automorphism \( s : \tilde{M}_{\mathcal{H},1} \rightarrow \tilde{M}_{\mathcal{H},1} \) defines another canonical isomorphism \( s^* : pr_2^* H^1_{\text{dR}}(A/M_{\mathcal{H},1}) \cong H^1_{\text{dR}}(pr_2^* A/\tilde{M}_{\mathcal{H},1}) \simeq H^1_{\text{dR}}(pr_1^* A/\tilde{M}_{\mathcal{H},1}) \cong pr_1^* H^1_{\text{dR}}(A/M_{\mathcal{H},1}) \).

**Definition 1.11.** The Gauss–Manin connection \( \nabla : H^1_{\text{dR}}(A/M_{\mathcal{H},1}) \rightarrow H^1_{\text{dR}}(A/M_{\mathcal{H},1}) \otimes \Omega^1_{M_{\mathcal{H},1}/S_1} \) on \( H^1_{\text{dR}}(A/M_{\mathcal{H},1}) \) is the composition

\[
H^1_{\text{dR}}(A/M_{\mathcal{H},1}) \xrightarrow{\text{pr}_2^*} H^1_{\text{dR}}(pr_2^* A/\tilde{M}_{\mathcal{H},1}) \xrightarrow{s^* - \text{Id}^*} H^1_{\text{dR}}(pr_1^* A/\tilde{M}_{\mathcal{H},1}) \cong pr_1^* H^1_{\text{dR}}(A/M_{\mathcal{H},1}) \otimes \Omega^1_{M_{\mathcal{H},1}/S_1}.
\]

This connection coincides with the usual Gauss–Manin connection on the relative de Rham cohomology (cf. [29]).

1.3. Automorphic bundles and de Rham complexes.

**Definition 1.12.** The principal \( G_1 \)-bundle over \( M_{\mathcal{H},1} \) is the \( G_1 \)-torsor

\[
\mathcal{E}_{G_1} := \text{Isom}_{\mathcal{O}}(\mathcal{O}_{M_{\mathcal{H},1}}) \otimes (H^1_{\text{dR}}(A/M_{\mathcal{H},1}), \langle \cdot, \cdot \rangle _{\lambda}, \mathcal{O}_{M_{\mathcal{H},1}}(1)),
\]

\[
((L_{0,1} \oplus L_{0,1}'(1)) \otimes \mathcal{O}_{M_{\mathcal{H},1}}, \langle \cdot, \cdot \rangle _{\text{can.}}, \mathcal{O}_{M_{\mathcal{H},1}}(1))),
\]

the sheaf of isomorphisms of \( \mathcal{O}_{M_{\mathcal{H},1}} \)-sheaves of symplectic \( \mathcal{O} \)-modules.

The group \( G_1 \) acts as automorphisms on \( (L \otimes \mathcal{O}_{M_{\mathcal{H},1}}, \langle \cdot, \cdot \rangle _{\text{can.}}, \mathcal{O}_{M_{\mathcal{H},1}}(1)) \) by definition. The third entries in the tuples represent the values of the pairings. We allow isomorphisms of symplectic modules to modify the pairings up to units.
Definition 1.13. The principal $P_1$-bundle over $M_{H,1}$ is the $P_1$-torsor

$$
\mathcal{E}_{P_1} := \text{Isom}_{\mathcal{O}_Z \otimes \mathcal{O}_{M_{H,1}}} \left( (\mathcal{H}^d_{\text{dR}}(A/M_{H,1}), \langle \cdot, \cdot \rangle_\lambda, \mathcal{O}_{M_{H,1}}(1), \mathcal{L}(\mathcal{O}_{\mathcal{M}_{H,1}})), (\langle L_0, 1 + L_0(1) \rangle \otimes \mathcal{O}_{M_{H,1}}, \langle \cdot, \cdot \rangle_{\text{can}}, \mathcal{O}_{M_{H,1}}(1), L_0(1) \otimes \mathcal{O}_{M_{H,1}}) \right),
$$

the sheaf of isomorphisms of $\mathcal{O}_{M_{H,1}}$-sheaves of symplectic $\mathcal{O}$-modules with maximal totally isotropic $\mathcal{O} \otimes R_1$-submodules.

Similarly to the previous definition, the group $P_1$ acts as automorphisms on $(L \otimes \mathcal{O}_{M_{H,1}}, \langle \cdot, \cdot \rangle_{\text{can}}, \mathcal{O}_{M_{H,1}}(1), L_0(1) \otimes \mathcal{O}_{M_{H,1}})$ by definition.

The principal bundles $\mathcal{E}_{G_1}$ and $\mathcal{E}_{P_1}$ are (étale) torsors (of the respective group schemes $G_1$ and $P_1$) because $(\mathcal{H}^d_{\text{dR}}(A/M_{H,1}), \langle \cdot, \cdot \rangle_\lambda, \mathcal{O}_{M_{H,1}}(1), \mathcal{L}(\mathcal{O}_{\mathcal{M}_{H,1}}))$ and

$$
(\langle L_0, 1 + L_0(1) \rangle \otimes \mathcal{O}_{M_{H,1}}, \langle \cdot, \cdot \rangle_{\text{can}}, \mathcal{O}_{M_{H,1}}(1), L_0(1) \otimes \mathcal{O}_{M_{H,1}})
$$

are étale locally isomorphic by the theory of infinitesimal deformations (cf. for example [29, Ch. 2]) and the theory of Artin’s approximations (cf. [11, Thm. 1.10 and Cor. 2.5]).

Definition 1.14. The principal $M_1$-bundle over $M_{H,1}$ is the $M_1$-torsor

$$
\mathcal{E}_{M_1} := \text{Isom}_{\mathcal{O}_Z \otimes \mathcal{O}_{M_{H,1}}} \left( \left( \mathcal{L}(\mathcal{O}_{\mathcal{M}_{H,1}}), \mathcal{O}_{M_{H,1}}(1), (L_0(1) \otimes \mathcal{O}_{M_{H,1}}) \right), \right),
$$

the sheaf of isomorphisms of $\mathcal{O}_{M_{H,1}}$-sheaves of $\mathcal{O} \otimes R_1$-modules.

We view the second entries in the pairs as an additional structure, inherited from the corresponding objects for $P_1$. The group $M_1$ acts as automorphisms on $(L_0(1) \otimes \mathcal{O}_{M_{H,1}}, \mathcal{O}_{M_{H,1}}(1))$ by definition.

Definition 1.15. For any $R_1$-algebra $R$, we denote by $\text{Rep}_R(G_1)$ (resp. $\text{Rep}_R(P_1)$, resp. $\text{Rep}_R(M_1)$) the category of $R$-modules of finite presentation with algebraic actions of $G_1 \otimes R$ (resp. $P_1 \otimes R$, resp. $M_1 \otimes R$).

Definition 1.16. Let $R$ be any $R_1$-algebra. For any $W \in \text{Rep}_R(G_1)$, we define

$$
\mathcal{E}_{G_1,R}(W) := (\mathcal{E}_{G_1} \otimes_R R_1) \times W,
$$

and call it the automorphic sheaf over $M_{H,1} \otimes R$ associated with $W$. It is called an automorphic bundle if $W$ is locally free as an $R$-module. We define similarly $\mathcal{E}_{P_1,R}(W)$ (resp. $\mathcal{E}_{M_1,R}(W)$) for $W \in \text{Rep}_R(P_1)$ (resp. $W \in \text{Rep}_R(M_1)$) by replacing $G_1$ with $P_1$ (resp. with $M_1$) in the above expression (1.17).

Lemma 1.18. Let $R$ be any $R_1$-algebra. The assignment $\mathcal{E}_{G_1,R}(\cdot)$ (resp. $\mathcal{E}_{P_1,R}(\cdot)$, resp. $\mathcal{E}_{M_1,R}(\cdot)$) defines an exact functor from $\text{Rep}_R(G_1)$ (resp. $\text{Rep}_R(G_1)$, resp. $\text{Rep}_R(G_1)$) to the category of coherent sheaves on $M_{H,1}$.

Proof. Étale locally over $M_{H,1}$, the principal bundle $\mathcal{E}_{G_1,R}$ (resp. $\mathcal{E}_{P_1,R}$, resp. $\mathcal{E}_{M_1,R}$) is isomorphic to the pullback of $G_1$ (resp. $P_1$, resp. $M_1$) from $S_1 = \text{Spec}(R_1)$ to $M_{H,1}$. Therefore, $\mathcal{E}_{G_1,R}(W)$ (resp. $\mathcal{E}_{P_1,R}(W)$, resp. $\mathcal{E}_{M_1,R}(W)$) is locally isomorphic to the pullback of $W$ from $S_1$ to $M_{H,1}$, and the assignment is functorial and exact because $M_{H,1} \to S_1$ is flat. □
Lemma 1.19. Let $R$ be any $R_1$-algebra. If we consider an object $W \in \Rep_R(G_1)$ as an object in $\Rep_R(P_1)$ by restriction to $P_1$, then we have a canonical isomorphism $\mathcal{E}_{G_1,R}(W) \cong \mathcal{E}_{P_1,R}(W)$.

Proof. By definition, we have a natural morphism $\mathcal{E}_{P_1,R} \times W \to \mathcal{E}_{G_1,R} \times W$ inducing a natural morphism $\mathcal{E}_{P_1,R}(W) \to \mathcal{E}_{G_1,R}(W)$. Reasoning as in the proof of Lemma 1.18, we see that this morphism is an isomorphism, because it is étale locally identified with the identity morphism $W \to W$.

Lemma 1.20. Let $R$ be any $R_1$-algebra. If we view an object $W \in \Rep_R(M_1)$ as an object in $\Rep_R(P_1)$ in the canonical way (under the canonical surjection $P_1 \to M_1$), then we have a canonical isomorphism $\mathcal{E}_{P_1,R}(W) \cong \mathcal{E}_{M_1,R}(W)$.

Proof. This follows from the very definitions of $\mathcal{E}_{P_1}$ and $\mathcal{E}_{M_1}$.

Corollary 1.21. Let $R$ be any $R_1$-algebra. Suppose $W \in \Rep_R(P_1)$ has a decreasing filtration by subobjects $F^s(W) \subset W$ in $\Rep_R(P_1)$ such that each graded piece $\Gr^s_1(W) := F^s(W)/F^{s+1}(W)$ can be identified with an object of $\Rep_R(M_1)$. Then $\mathcal{E}_{P_1,R}(W)$ has a filtration $\mathcal{E}_{P_1,R}(F^s(W))$ with graded pieces $\mathcal{E}_{M_1,R}(\Gr^s_1(W))$.

Proof. This follows from the exactness of the functor $\mathcal{E}_{P_1,R}$ in Lemma 1.18.

Example 1.22. We have $\mathcal{E}_{G_1,R_1}(L_1) \cong \mathcal{E}_{P_1,R_1}(L_1) \cong H^1_{\text{dR}}(A/M_{H_1}, \omega^1)$, with Hodge filtration defined by the submodule $\mathcal{E}_{P_1,R_1}((L_1)^+)^1(1)) \cong \mathcal{E}_{M_1,R_1}(L_0^+(1)) \cong \text{Lie}_{A^s/M_{H_1}}$, and with top graded piece $\mathcal{E}_{P_1,R_1}(L_0,1) \cong \mathcal{E}_{M_1,R_1}(L_0,1) \cong \text{Lie}_{A/M_{H_1}}$.

In Definition 1.11 the Gauss–Manin connection is defined by the difference between the two isomorphisms $\text{Id}^*, s^*: \Pr_1^* H^1_{\text{dR}}(A/M_{H_1}) \to \Pr_1 H^1_{\text{dR}}(A/M_{H_1})$ lifting the identity morphism on $H^1_{\text{dR}}(A/M_{H_1})$. Since $s^*$ has a simple definition, we can interpret $\text{Id}^*$ (whose definition as in [29, Prop. 2.1.6.4] is far from simple) as induced by the Gauss–Manin connection (and $s^*$). The same is true if we base change (horizontally) from $R_1$ to any $R_1$-algebra $R$. By construction of $\mathcal{E}_{G_1,R}(\cdot)$ (cf. 1.17), for any $W \in \Rep_R(G_1)$, the two isomorphisms above induce two isomorphisms $\text{Id}^*, s^*: \Pr_1^* \mathcal{E}_{G_1,R}(W) \to \Pr_1(\mathcal{E}_{G_1,R}(W))$ lifting the identity morphism on $\mathcal{E}_{G_1,R}(W)$. Hence the difference $s^* - \text{Id}^*$ induces an integrable connection

$$\nabla: \mathcal{E}_{G_1,R}(W) \to \mathcal{E}_{G_1,R}(W) \otimes \Omega^1_{\mathcal{M}_{H_1}/S^1_R}.$$ 

**Definition 1.24.** The integrable connection $\nabla$ in (1.23) above is called the Gauss–Manin connection for $\mathcal{E}_{G_1,R}(W)$. The complex $(\mathcal{E}_{G_1,R}(W) \otimes \Omega^*_{\mathcal{M}_{H_1}/S^1_R}, \nabla)$ it induces is called the de Rham complex for $\mathcal{E}_{G_1,R}(W)$.

2. Representation theory

2.1. Decomposition of reductive groups. Using the decomposition of $\mathcal{O}_{F,1}$ in (1.5), we obtain a corresponding decomposition

$$\mathcal{O}_1 \cong \prod_{\tau \in \Upsilon} \mathcal{O}_\tau,$$

where $\mathcal{O}_F$ acts on the factor $\mathcal{O}_\tau$ via the homomorphism $\mathcal{O}_F \to \mathcal{O}_{F,\tau}$ defined by $\tau$.

By [29, Lem. 1.1.3.4], there is a unique (up to isomorphism) indecomposable projective $\mathcal{O}_\tau$-module for each $\tau \in \Upsilon$, which we shall denote by $V_\tau$. When $\mathcal{O}_\tau \cong$
Let \( (p_\tau)_\tau \in \mathcal{Y} \) be the multi-rank of \( L_{0,1} \) (resp. \( L_{0,1}(1) \)). Then \( q_\tau = p_\tau + q_\tau \), and \( c : \mathcal{O}_F \to \mathcal{O}_F \) is the restriction of \( \mathcal{O} \to \mathcal{O} \). Then the multi-rank of \( L_1 \) is \( (p_\tau + q_\tau)_\tau \in \mathcal{Y} \), because we have the isomorphism \([16]\) over \( R_1 \).

Choose and fix an isomorphism \( L_{0,1} \cong \bigoplus_{\tau \in \mathcal{Y}} V^{\oplus p_\tau}_\tau \), as well as the isomorphisms \( V^{\vee}_{r_\tau}(1) := \operatorname{Hom}_{R_1}(V_{r_\tau}, R_1(1)) \cong V_{r_\tau} \) (for \( \tau \in \mathcal{Y} \)). These chosen isomorphisms canonically induce an isomorphism

\[
L_1 \cong \left( \bigoplus_{\tau \in \mathcal{Y}} V^{\oplus p_\tau}_\tau \right) \oplus \left( \bigoplus_{\tau \in \mathcal{Y}} (V^{\vee}_{r_\tau}(1))^{\oplus q_\tau} \right) \cong \bigoplus_{\tau \in \mathcal{Y}} V^{\oplus (p_\tau + q_\tau)}_\tau
\]

by \([16]\), matching the pairing \( \langle \cdot, \cdot \rangle \) with the pairing

\[
(\langle x_1, \tau \rangle, \langle x_2, \tau \rangle) \in \mathcal{Y} \Rightarrow \sum_{\tau \in \mathcal{Y}} (f_2, \tau(x_1, \tau) - f_1, \tau(x_2, \tau)).
\]

**Lemma 2.4.** There exists a cocharacter \( G_{m, \otimes R_1} \to G_1 \) splitting the similitude character \( \psi : G_1 \to G_{m, \otimes R_1} \) which acts trivially on the submodule \( L_{0,1}(1) \) of \( L_1 \) (under the identification \([16]\)).

**Proof.** Let \( R \) be any \( R_1 \)-algebra. Let \( t_0 \) be any element in \((G_{m, \otimes R_1})(R) = R^x\).

In the decomposition \([2.2]\), if we let \( t_0 \) act as \( t_0 \) on \( V^{\oplus p_\tau}_\tau \), and act trivially on \( (V^{\vee}_{r_\tau}(1))^{\oplus q_\tau} \), for any \( \tau \in \mathcal{Y} \), then the pairing \([2.3]\) is multiplied by \( t_0 \). This gives an element in \( G_1(R) \) with similitude \( t_0 \) and with trivial action on \( L_{0,1}(1) \), as desired. \( \square \)

For each \( \tau \in \mathcal{Y} \), set \( L_\tau := V^{\oplus p_\tau}_\tau \oplus (V^{\vee}_{r_\tau}(1))^{\oplus q_\tau} \), and define the canonical pairing \( \langle \cdot, \cdot \rangle_\tau : L_\tau \times L_\tau \to R_1(1) \) by \( \langle (x_1, \tau), (x_2, \tau) \rangle \mapsto f_2, \tau(x_1, \tau) - f_1, \tau(x_2, \tau) \). Then the pairing \([2.3]\) is simply the sum of \( \langle \cdot, \cdot \rangle_\tau \) over \( \tau \in \mathcal{Y} \). Note that \( \operatorname{GL}_{\mathcal{O}, \otimes R}(L_\tau \otimes R_1) \cong \operatorname{GL}_{\mathcal{O}, \otimes R}(L^{\vee}_{r_\tau}(1) \otimes R) \) for any \( R_1 \)-algebra \( R \). If we define

\[
G_\tau(R) := \left\{ g \in \operatorname{GL}_{\mathcal{O}, \otimes R}(L_\tau \otimes R) : \langle gx, gy \rangle_\tau = \langle x, y \rangle_\tau, \forall x \in L_\tau, \forall y \in L^{\vee}_{r_\tau}(1) \right\}
\]

for each \( R_1 \)-algebra \( R_1 \), then we obtain a group functor \( G_\tau \) over \( \operatorname{Spec}(R_1) \), which falls into only three possible cases:

(1) \( G_\tau \cong \operatorname{Sp}_{2r_\tau} \otimes R_1 \), where \( r_\tau = q_\tau \) and \( \operatorname{Sp}_{2r_\tau} \) is the (split) symplectic group of rank \( r_\tau \) over \( \operatorname{Spec}(\mathbb{Z}) \). (This is a factor of type C.)

(2) \( G_\tau \cong \operatorname{O}_{2r_\tau} \otimes R_1 \), where \( r_\tau = q_\tau \) and \( \operatorname{O}_{2r_\tau} \) is the (split) even orthogonal group of rank \( r_\tau \) over \( \operatorname{Spec}(\mathbb{Z}) \). (This is a factor of type D.)

(3) \( G_\tau \cong \operatorname{GL}_{r_\tau} \otimes R_1 \), where \( r_\tau = p_\tau + q_\tau \) and \( \operatorname{GL}_{r_\tau} \) is the general linear group of rank \( r_\tau \) over \( \operatorname{Spec}(\mathbb{Z}) \). (This is a factor of type A.)

Thus we obtain a decomposition

\[
G_1 \cong \left( \prod_{\tau \in \mathcal{Y}/c} G_\tau \right) \times (G_{m, \otimes R_1}),
\]
where \( \tau \in \mathcal{Y}/c \) means (by abuse of language) we pick exactly one representative \( \tau \) in its \( c \)-orbit in \( \mathcal{Y} \), and where the last factor \( (\mathbf{G}_m \otimes R_1) \) is given by the cocharacter given by Lemma 2.3, splitting the similitude character.

### 2.2. Decomposition of parabolic subgroups

Under the identification (1.6), the submodule \( L_{0,1}^\vee(1) \) of \( L_1 \) can be identified with the submodule
\[
0 \oplus \bigoplus_{\tau \in \mathcal{Y}} (V_{\tau}^{\vee}(1))^{\oplus q_{\tau}}
\]
of the second member in (2.2). For each \( \tau \in \mathcal{Y} \), define group functors \( P_{\tau} \) and \( M_{\tau} \) over Spec(\( R_1 \)) by setting for each \( R_1 \)-algebra \( R \)
\[
P_{\tau}(R) := \begin{cases}
g \in G_{\tau}(R) : g(0 \oplus (V_{\tau}^{\vee}(1))^{\oplus q_{\tau}} \otimes R)) = (0 \oplus (V_{\tau}^{\vee}(1))^{\oplus q_{\tau}} \otimes R)) \\
\text{in } L_1 \otimes R = (V_{\tau}^{\oplus p_{\tau}} \otimes R) \oplus ((V_{\tau}^{\vee}(1))^{\oplus q_{\tau}} \otimes R)
\end{cases}
\]
and
\[
M_{\tau}(R) := \begin{cases}
g \in P_{\tau}(R) : g((V_{\tau}^{\oplus p_{\tau}} \otimes R) \oplus 0) = ((V_{\tau}^{\oplus p_{\tau}} \otimes R) \oplus 0) \\
\text{in } L_1 \otimes R = (V_{\tau}^{\oplus p_{\tau}} \otimes R) \oplus ((V_{\tau}^{\vee}(1))^{\oplus q_{\tau}} \otimes R)
\end{cases}.
\]

Then the subgroup \( P_1 \) of \( G_1 \) can be identified with the subgroup
\[
\left( \prod_{\tau \in \mathcal{Y}/c} P_{\tau} \right) \times (\mathbf{G}_m \otimes R_1) \subset \left( \prod_{\tau \in \mathcal{Y}/c} G_{\tau} \right) \times (\mathbf{G}_m \otimes R_1),
\]
and the canonical surjection \( P_1 \to M_1 \) has a splitting \( M_1 \subset P_1 \) given by
\[
\left( \prod_{\tau \in \mathcal{Y}/c} M_{\tau} \right) \times (\mathbf{G}_m \otimes R_1) \subset \left( \prod_{\tau \in \mathcal{Y}/c} P_{\tau} \right) \times (\mathbf{G}_m \otimes R_1).
\]

For each \( \tau \in \mathcal{Y} \), we have \( \text{Hom}_{\mathcal{O}}(V_\tau, V_\tau) \cong \text{Hom}_{\mathcal{O}_L}(V_{\tau}^{\vee}(1), V_{\tau}^{\vee}(1)) \cong \mathcal{O}_{F,\tau} \cong R_1 \). Therefore, we have diagonal actions of \( G_m^{p_{\tau}}(R) \) on \( V_{\tau}^{\oplus p_{\tau}} \otimes R \) and of \( G_m^{q_{\tau}}(R) \) on \( (V_{\tau}^{\vee}(1))^{\oplus q_{\tau}} \otimes R \), which are functorial in \( R \) and hence define a homomorphism \( (G_m^{p_{\tau}} \times G_m^{q_{\tau}}) \otimes R_1 \to M_\tau \).

### 2.3. Hodge filtration

Let \( R \) be any \( R_1 \)-algebra. Fix any choice of a cocharacter as in Lemma 2.3 and consider its reciprocal \( H : \mathbf{G}_m \otimes R_1 \to G_1 \). Note that by definition \( H \) factors through \( P_1 \).

**Definition 2.9.** Given any object \( W \in \text{Rep}_R(P_1) \), the induced action of \( \mathbf{G}_m \otimes R_1 \) decomposes \( W \) into weight spaces \( W^{(a)} \) for \( \mathbf{G}_m \otimes R_1 \), indexed by integers. Then the Hodge filtration \( F \) on \( W \) is the decreasing filtration \( F(W) = \{ F^a(W) \}_{a \in \mathbb{Z}} \) defined by \( F^a(W) := \bigoplus_{b \geq a} W^{(b)} \).

**Example 2.10.** Since the cocharacter \( H \) acts with weight 0 on \( L_{0,1}^\vee(1) \) (as a submodule of \( L_1 \)) and with weight \(-1\) on \( L_{0,1} \) (as a quotient module of \( L_1 \)), the Hodge filtration \( F \) on \( L_1 \) is given by \( F^{-1}(L_1) = L_1, F^0(L_1) = L_{0,1}^\vee(1), \) and \( F^1(L_1) = \{0\} \). Thus the only possibly nonzero graded pieces are \( Gr_{F}^{-1}(L_1) = L_{0,1} \) and \( Gr_{F}^0(L_1) = L_{0,1}^\vee(1) \). Note that the choice of \( H \) is not unique, but the resulting filtration is independent of this choice.
Lemma 2.11. Let $W \in \text{Rep}_R(P_1)$ and let $\{F^a(W)\}_{a \in \mathbb{Z}}$ denote the Hodge filtration defined in Definition 2.12. Then the unipotent radical $U_1$ of $P_1$ acts trivially on $\text{Gr}^a_F(W)$ for any $a \in \mathbb{Z}$. In other words, each graded piece $\text{Gr}^a_F(W)$ can be identified with an object in $\text{Rep}_R(M_1)$.

Proof. Since the adjoint action of $H$ on $\text{Lie}(U_1)$ has weight $-1$, the action of $\text{Lie}(U_1)$ decreases the $H$-weights by $1$, as desired.

By Corollary 2.12, the Hodge filtration on $W$ defines submodules of $E_{P_1,R}(W)$, which we denote by $F^a(E_{P_1,R}(W))$ for $a \in \mathbb{Z}$.

Definition 2.12. The filtration $F(E_{P_1,R}(W)) = \{F^a(E_{P_1,R}(W))\}_{a \in \mathbb{Z}}$ is called the Hodge filtration on $E_{P_1,R}(W)$.

By Corollary 2.12, we have $\text{Gr}^a_F(E_{P_1,R}(W)) \cong E_{M_1,R}(\text{Gr}^a_F(W))$.

Definition 2.13. Let $W \in \text{Rep}_R(G_1)$. By considering $W$ as an object of $\text{Rep}_R(P_1)$ by restriction from $G_1$ to $P_1$, we can define the Hodge filtration on $E_{G_1,R}(W)$ (see Lemma 1.19) as in Definition 2.12. The Hodge filtration on the de Rham complex $E_{G_1,R}(W) \otimes_{\Omega^*_M,R} \Omega^*_M,R/S_R$ is defined by

$$F^a(E_{G_1,R}(W) \otimes_{\Omega^*_M,R} \Omega^*_M,R/S_R) := F^a(E_{P_1,R}(W) \otimes_{\Omega^*_M,R} \Omega^*_M,R/S_R)$$

It is a subcomplex of $E_{G_1,R}(W) \otimes_{\Omega^*_M,R} \Omega^*_M,R/S_R$ for the Gauss–Manin connection thanks to the Griffiths transversality. (The only de Rham complexes we will need for our main results are those realized by geometric plethysm as in Lemma 4.7 below, for which the Griffiths transversality is clear. For de Rham complexes attached to an arbitrary $W \in \text{Rep}_R(G_1)$, see [30].)

Lemma 2.14. Suppose $W_1$ and $W_2$ are two objects in $\text{Rep}_R(G_1)$ such that the induced actions of $P_1$ and $\text{Lie}(G_1)$ on them satisfy $W_1|\text{Lie}(P_1) \cong W_2|\text{Lie}(P_1)$ and $W_1|\text{Lie}(G_1) \cong W_2|\text{Lie}(G_1)$. Then we have a canonical isomorphism $(E_{G_1,R}(W_1) \otimes_{\Omega^*_M,R,S_R} \nabla) \cong (E_{G_1,R}(W_2) \otimes_{\Omega^*_M,R,S_R} \nabla)$ respecting the Hodge filtrations on both sides.

Proof. By Lemma 1.19, we have isomorphisms $F^a(E_{G_1,R}(W_1) \otimes_{\Omega^*_M,R} \Omega^*_M,R/S_R) \cong F^a(E_{G_1,R}(W_2) \otimes_{\Omega^*_M,R} \Omega^*_M,R/S_R)$ between the individual terms because they are defined by $P_1$-modules. Then the lemma is true because the definition of the connections only involves differentials on $M_{H,R}$ and $G_1 \otimes R$ (relative to $R$).

Remark 2.15. Lemma 2.14 will be needed only when $G_1$ is not connected, i.e. when $\mathcal{O}_Z \otimes \mathbb{Q}$ involves simple factors of type $D$ (as in [29, Def. 1.2.1.15]).

While we claim that the two automorphic bundles in Lemma 2.14 are isomorphic as abstract vector bundles with integrable connections, we do not claim that the Hecke operators on their cohomology are identical. This is harmless for our purpose, but the reader should not make similar identifications for questions about the Galois or Hecke actions.
2.4. Roots and weights. We shall choose a maximal torus $T_\tau$ of $M_\tau$ by choosing a subgroup of $(G_m^r \times G_m^{q_r}) \otimes \mathbb{Z} \mathbb{R}_1$ that embeds into $M_\tau$ under the natural homomorphism $(G_m^r \times G_m^{q_r}) \otimes \mathbb{Z} \mathbb{R}_1 \to M_\tau$ defined at the end of Section 2.2. There are two cases:

1) If $\tau = \tau \circ c$, then $p_{\tau} = q_{\tau}$ and we take $T_\tau = \{ t_\tau = (t_{\tau,i})_{1 \leq i \leq \tau} \}$, embedded in $(G_m^r \times G_m^{q_r}) \otimes \mathbb{Z} \mathbb{R}_1$ by $t_\tau \mapsto (t_{\tau,-1}, t_\tau)$.

2) If $\tau \neq \tau \circ c$, then we take $T_\tau = \{ t_\tau = (t_{\tau,i})_{1 \leq i \leq \tau} \}$ and identify it with $(G_m^r \times G_m^{q_r}) \otimes \mathbb{Z} \mathbb{R}_1$ by sending $(t_{\tau,i})_{1 \leq i \leq \tau}$ to $((t_{\tau,q_{r+i}}, t_{\tau,i}))_{1 \leq i \leq \tau}$, irreducible algebraic group scheme $G_1$ over $\text{Spec}(\mathbb{R}_1)$. For any root $\alpha \in \Phi_{G_1}$, we shall denote by $\alpha^\vee \in \Phi_{G_1}^+$ the associated coroot.

The choice of the positive roots $\Phi_{G_1}^+$ in $\Phi_{G_1}$ corresponds to the choice of a Borel subgroup $B_1$ in $G_1$. By choosing $B_1$ to contain the unipotent radical $U_1$ of $P_1$ (using the explicit identifications in (2.5), (2.7), (2.8), and (2.16)), we can choose $\Phi_{G_1}^+$ such that the set $X_{G_1}^+$ of dominant weights of $G_1$ consists of those $\mu \in X$ as above with $\mu_{\tau,i} \geq \mu_{\tau,i+1}$ for any $\tau \in \mathbb{Y}/c$ and for any $1 \leq i_r < r_\tau$, satisfying in addition:

1) If $G_\tau \cong Sp_{2r_\tau} \otimes \mathbb{Z} \mathbb{R}_1$, then $\mu_{\tau,r_\tau} \geq 0$.

2) If $G_\tau \cong O_{2r_\tau} \otimes \mathbb{Z} \mathbb{R}_1$, then $\mu_{\tau,r_\tau-1} \geq |\mu_{\tau,r_\tau}|$.

(If $G_\tau \cong GL_{r_\tau} \otimes \mathbb{Z} \mathbb{R}_1$, then there is no other condition on $\mu_{\tau,r_\tau}$.)

Remark 2.17. When $G_\tau \cong O_{2r_\tau} \otimes \mathbb{Z} \mathbb{R}_1$ for some $\tau \in \mathbb{Y}$, irreducible algebraic representations of $G_\tau$ are not exactly parameterized by dominant weights, due to the presence of an element in $O_{2r_\tau} \otimes \mathbb{Z} \mathbb{R}_1$ flipping the two weights $\mu_{\tau} = (\mu_{\tau,1}, \ldots, \mu_{\tau,r_\tau-1}, \mu_{\tau,r_\tau})$ and $(\mu_{\tau,1}, \ldots, \mu_{\tau,r_\tau-1}, -\mu_{\tau,r_\tau})$. (A concise discussion on this matter can be found in [17 §5.5.5].) By Lemma 2.14 two representations of $O_{2r_\tau} \otimes \mathbb{Z} \mathbb{R}_1$ will serve the same purpose for us if their restrictions to $SO_{2r_\tau} \otimes \mathbb{Z} \mathbb{R}_1$ are isomorphic. Therefore, in what follows, we will denote by $[\mu]$ the set of highest
dominant weights that appear in the irreducible representation of $G_1$ containing the dominant weight $\mu$. This does not, for example, distinguish the determinant representation of $O_{2r} \otimes R_1$ from the trivial representation, but will be sufficient for our purpose. Then there is always a unique $\mu'$ in $[\mu]$ satisfying the additional condition that $\mu'_{\tau,r} \geq 0$ for any $\tau \in \Upsilon$ such that $G_{\tau} \cong O_{2r} \otimes R_1$.

Let $\Phi_{M_1}$ be the roots of the split reductive group scheme $M$ over $\text{Spec}(R_1)$. Then intersection of $M_1$ (realized as a subgroup in $P_1$ as above) with the $B_1$ chosen above determines a set of positive roots $\Phi_{M_1}^+$ in $\Phi_{M_1}$, so that $\Phi_{M_1}^+ = \Phi_{M_1} \cap \Phi_{G_1}^+$. The set $X_{M_1}^+$ of dominant weights of $M_1$ consists of those $\mu \in X$ as above with $\mu_{\tau,r} \geq \mu_{\tau,i_r+1}$ for any $\tau \in \Upsilon/c$ and for any $1 \leq i_r < q_r$ or $q_r < i_r < r_\tau$.

It is conventional to say that a root $\alpha \in \Phi_G$ is compact if it is an element of $\Phi_{M_1}$, and that $\alpha$ is non-compact otherwise. We denote the non-compact roots of $\Phi_G$, by $\Phi^{M_1}$, and denote the collection of positive non-compact roots by $\Phi^{M_1,+}$. For negative roots, we replace $+$ with $-$ in the above notation.

Let $W_G$, (resp. $W_{M_1}$) be the Weyl group of $G_1$ (resp. of $M_1$). The realization of $M_1$ as a subgroup of $G_1$ containing $T_1$ identifies $W_{M_1}$ as a subgroup of $W_G$. We define

$$W_{M_1} := \{ w \in W_G : w(X_{M_1}^+) \subseteq X_{M_1}^+ \}.$$ 

Then any element $w$ in $W_G$ has a unique expression as $w = w_1 w_2$ with $w_1 \in W_{M_1}$ and $w_2 \in W_{M_1}$. Let $\rho := \frac{1}{2} \sum_{\alpha \in \Phi_{G_1}^+} \alpha$. The dot action of $W_G$ (and hence the subset $W_{M_1}$ of it) is defined by $w \cdot \mu := w(\mu + \rho) - \rho$ for any $w \in W_G$.

2.5. Plethysm for representations. In this subsection, we denote by $\text{GL}_r$, $\text{Sp}_{2r}$, $O_{2r}$, etc, the split forms of the groups over $\mathbb{Z}$, and we denote the base change to other rings by subscripts. We shall explain in our context the construction of representations of classical groups using Weyl’s invariant theory. (It may be helpful to consult [15], [17], [20], and [47] for more information.)

Let $r \geq 0$ be any integer, and let $\nu = (\nu_1, \nu_2, \ldots, \nu_r)$ be any tuple of integers satisfying $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_r$. We know that $\nu$ is the weight of an algebraic irreducible $\mathbb{Q}$-representation of $\text{GL}_r$. Let us define $|\nu| := \sum_{1 \leq i \leq r} \nu_i$. If $\nu_r \geq 0$, we say the tuple $\nu$ and the corresponding $\mathbb{Q}$-representation are polynomial, and write $\nu \geq 0$.

For any polynomial weight $\nu$, we plot the so-called Young diagram by putting $\nu_1$ blocks in the first row, $\nu_2$ in the second rows, and so on. By filling in numbers (in arbitrary order) from 1 to $|\nu|$, we obtain a so-called Young tableau for $\nu$. (See, e.g., [15], pp. 45.) We shall denote a particular choice of Young tableau of $\nu$ by $D_\nu$. Let $\mathfrak{S}_{|\nu|}$ denote the symmetric group of permutations on $\{1, 2, \ldots, |\nu|\}$. Based on the choice of $D_\nu$, we define $\mathfrak{D}_{|\nu|}$ (resp. $\mathfrak{Q}_{|\nu|}$) to be the subgroup of $\mathfrak{S}_{|\nu|}$ consisting of elements permuting numbers in each row (resp. column) of $D_\nu$. Let $\mathbb{Z}[\mathfrak{S}_{|\nu|}]$ be the group algebra with generators $e_h$ for each $h \in \mathfrak{S}_{|\nu|}$. Let us define $a_{D_\nu} := \sum_{h \in \mathfrak{D}_{|\nu|}} e_h$ and $b_{D_\nu} := \sum_{h \in \mathfrak{Q}_{|\nu|}} \text{sgn}(h) e_h$. Then the Young symmetrizer is $c_{D_\nu} := a_{D_\nu} b_{D_\nu}$.

**Lemma 2.18.** Let $n = |\nu|$. Then we have the following facts in $\mathbb{Z}[\mathfrak{S}_n]$:

1. $c_{D_\nu} \mathbb{Z}[\mathfrak{S}_n] c_{D_\nu} \subseteq \mathbb{Z} c_{D_\nu}$.
(2) \(c_D, c_p = d_D, c_D, p\) for some integer \(d_D\), dividing \(n!\) (i.e. factorial).

(3) Let \(D, v\) be the Young tableau for some \(v \geq 0\) with \(|v| = n\). Then \(c_D, c_D, v = 0\) if \(v \neq v'\).

(4) Let \(V_{D, v} := \mathbb{Z}[\mathfrak{S}_n]c_p\). Then \(V_{D, v}\) is an irreducible \(Q\)-representation of \(\mathfrak{S}_n\), and \(V_{D, v} \cong V_{D, v'}\) (for some \(D, v\) with \(|v| = n\)) if and only if \(v = v'\).

Proof. In [47, Ch. IV, §3] or [13], §4.23, 4.25, and 4.26], variants of these are stated over \(\mathbb{C}\), but the proofs are valid for our statements above over \(\mathbb{Z}\) or \(Q\).

Let \(V_{\text{std}, v} := \mathbb{Z}^{\oplus r}\) be the standard representation of \(\text{GL}_r\). Let \(n \geq 0\) be any integer. Then \((g, h) \in \text{GL}_r \times \mathfrak{S}_n\) acts on \(V_{\text{std}, r} \otimes V_{D, v}\) by
\[
g(v_1 \otimes v_2 \otimes \ldots \otimes v_n) := g(v_1) \otimes g(v_2) \otimes \ldots \otimes g(v_n)
\]
\[
h(v_1 \otimes v_2 \otimes \ldots \otimes v_n) := v_{h^{-1}(1)} \otimes v_{h^{-1}(2)} \otimes \ldots \otimes v_{h^{-1}(n)}
\]
for any \(v_1, v_2, \ldots, v_n \in V_{\text{std}, r}\). (These relations are interpreted functorially.)

**Proposition 2.19** (cf. [20, 2.4.3]). There is an isomorphism
\[
V_{\text{std}, r} \otimes \mathbb{Q} \cong \bigoplus_{\nu \geq 0, |\nu| = n} (V_{\nu, \mathbb{Q}} \otimes V_{D, v})
\]
between \(Q\)-representations of \(\text{GL}_r \times \mathfrak{S}_n\), called **Schur duality**, where \(V_{\nu, \mathbb{Q}}\) is the algebraic \(Q\)-representation of \(\text{GL}_r, Q\) of highest weight \(\nu\), and where \(D, v\) is any Young tableau for \(\nu\). As a result, we obtain **Weyl’s construction**, an isomorphism
\[
V_{\nu, \mathbb{Q}} \cong c_D, v \otimes V_{\text{std}, r} \otimes Q
\]
between \(Q\)-representations of \(\text{GL}_r, Q\) for any polynomial weight \(\nu\) of \(\text{GL}_r, Q\).

Proof. The proof of (2.20) in [20, 2.4.3] is carried out over \(\mathbb{C}\). Once (2.20) is known over \(\mathbb{C}\), we know (2.21) over \(\mathbb{C}\) by Lemma 2.18. Then (2.21) is true over \(Q\) because both sides of (2.21) are absolutely irreducible and defined over \(Q\), and hence (2.20) is also true over \(Q\).

**Definition 2.22.** Let \(r \geq 0\) be any integer. Let \(V_{\text{std}, r} = \mathbb{Z}^{\oplus r} \cong \mathbb{Z}^{\oplus r} \oplus \mathbb{Z}^{\oplus r}\) be equipped with the standard symplectic pairing \((\cdot, \cdot)_{\text{std}}\) with matrix \((0_{1r}, 1_r)\), and with the standard symmetric pairing \([\cdot, \cdot]_{\text{std}}\) with matrix \((1_r, 0_r)\). Then we have a canonical action of \(\text{Sp}_{2r}\) on \(V_{\text{std}, 2r}\) preserving \((\cdot, \cdot)_{\text{std}}\), and a canonical action of \(O_{2r}\) on \(V_{\text{std}, 2r}\) preserving \([\cdot, \cdot]_{\text{std}}\). For any integer \(n \geq 0\), and for any \(1 \leq i < j \leq n\), we define \(\phi_{i,j}^{(\cdot, \cdot)} : V_{\text{std}, n} \otimes V_{\text{std}, 2r} \rightarrow V_{\text{std}, 2r} \otimes V_{\text{std}, n}^{(n-2)}\) by
\[
\phi_{i,j}^{(\cdot, \cdot)}(v_1 \otimes v_2 \otimes \ldots \otimes v_n) := (v_i, j, \tilde{v}_i \otimes \ldots \otimes \tilde{v}_j \otimes \ldots \otimes v_n),
\]
and define similarly \(\phi_{i,j}^{([\cdot, \cdot])} : V_{\text{std}, n} \otimes V_{\text{std}, 2r} \rightarrow V_{\text{std}, 2r} \otimes V_{\text{std}, n}^{(n-2)}\) by replacing \((v_i, j)\) with \([v_i, j]\) in the above expression. (Here \(\tilde{v}_i\) and \(\tilde{v}_j\) denote omissions of entries as usual. When \(n < 2\), we declare \(V_{\text{std}, 2r} = 0\) and hence \(\phi_{i,j}^{(\cdot, \cdot)} = 0\) or \(\phi_{i,j}^{([\cdot, \cdot])}\).) Then we define
\[
\psi_{i,j}^{(n)} := \bigcap_{1 \leq i < j \leq n} \ker(\phi_{i,j}^{(\cdot, \cdot)}) \text{ and } \psi_{i,j}^{(n)} := \bigcap_{1 \leq i < j \leq n} \ker(\phi_{i,j}^{([\cdot, \cdot])}).
\]

Note that \(V_{\text{std}, 2r}\) is its own dual under either \((\cdot, \cdot)\) or \([\cdot, \cdot]\). Therefore, the maps \(\phi_{i,j}^{(\cdot, \cdot)}\) and \(\phi_{i,j}^{([\cdot, \cdot])}\) define, by duality, the maps \(\psi_{i,j}^{(\cdot, \cdot)} : V_{\text{std}, n} \otimes V_{\text{std}, 2r} \rightarrow V_{\text{std}, 2r} \otimes V_{\text{std}, n}^{(n-2)}\) and \(\psi_{i,j}^{([\cdot, \cdot])} : V_{\text{std}, n} \otimes V_{\text{std}, 2r} \rightarrow V_{\text{std}, 2r} \otimes V_{\text{std}, n}^{(n-2)}\), respectively, by inserting the pairings into the \(i\)-th and \(j\)-th factors. (See [15, §17.3 and §19.5].) By taking a standard symplectic basis as in the proof of [15, (17.12)], we see that \(\phi_{i,j}^{(\cdot, \cdot)} \psi_{i,j}^{(\cdot, \cdot)} = 2r\), and
hence \((2r)\ker(\phi_{i,j}^{\nu}) \subseteq ((2r)\Id - \psi_{i,j}^{\nu})(V_{\text{std},2r}^\nu) \subseteq \ker(\phi_{i,j}^{\nu})\). Similarly, \(\phi_{i,j}^{\nu} = 2r\) and hence \((2r)\ker(\phi_{i,j}^{\nu}) \subseteq ((2r)\Id - \psi_{i,j}^{\nu})(V_{\text{std},2r}^\nu) \subseteq \ker(\phi_{i,j}^{\nu})\). These relations will be especially useful when \(2r\) is invertible in the rings we consider. (See Section 3.4 below.)

**Proposition 2.23.** Let \(\nu = (\nu_1, \nu_2, \ldots, \nu_r)\) be the weight of an irreducible algebraic \(Q\)-representation \(V_{\nu,Q}^{\nu} \subseteq \text{Sp}_{2r,Q}\) satisfying \(\nu_1 \geq \nu_2 \geq \ldots \geq \nu_r \geq 0\). We view \(\nu\) as a polynomial weight of \(\text{GL}_r\) by supplying zeros in the end. Then we have an isomorphism \(V_{\nu,Q}^{\nu} \cong V_{\text{std},2r,Q}(c_{D_{\nu}}V_{\text{std},2r,Q}^\nu)\) between \(Q\)-representations of \(\text{Sp}_{2r,Q}\) for any choice of Young tableau \(D_{\nu}\) for \(\nu\).

**Proof.** This is stated (without proof) in [17, Ch. VI, §3] and proved in [15, Thm. 17.11] over \(C\). It is then valid over \(Q\) because both sides of the isomorphism are absolutely irreducible and defined over \(Q\).

**Proposition 2.24.** Let \(\gamma_r\) be the element of \(O_{2r,Q}\) flipping the two weights \((\mu_1, \mu_2, \ldots, \mu_r)\) and \((\mu_1, \mu_2, \ldots, -\mu_r)\) of \(O_{2r,Q}\) for any integers \(\mu_1 \geq \mu_2 \geq \ldots \geq \mu_r \geq 0\) (cf. Remark 2.17). Let \(\nu = (\nu_1, \nu_2, \ldots, \nu_r)\) be the weight of an irreducible algebraic \(Q\)-representation \(V_{\nu,Q}^{\nu} \subseteq \text{Sp}_{2r,Q}\) satisfying \(\nu_1 \geq \nu_2 \geq \ldots \geq \nu_{r-1} \geq \nu_r \geq 0\). When \(\nu_r = 0\), we require moreover that the action of \(\gamma_r\) is trivial on \(V_{\nu,Q}^{\nu} \setminus \nu_r\). We view \(\nu\) as a polynomial weight of \(\text{GL}_r\) by supplying zeros in the end. Then we have an isomorphism \(V_{\nu,Q}^{\nu} \cong V_{\text{std},2r,Q}(c_{D_{\nu}}V_{\text{std},2r,Q}^\nu)\) between \(Q\)-representations of \(O_{2r,Q}\) for any choice of Young tableau \(D_{\nu}\) for \(\nu\).

**Proof.** This is proved in [17, Ch. V, §7] and stated (without proof) in [15, Thm. 19.19] over \(C\). A modern treatment can be found in [17, §10.2.5]. It is then valid over \(Q\) because both sides of the isomorphism are absolutely irreducible and defined over \(Q\).

**Remark 2.25.** When \(\nu_r = 0\), there is another irreducible representation of \(O_{2r,Q}\) containing the weight \(\nu\) on which \(\gamma_r\) acts nontrivially. According to [17, §10.2.5], it is isomorphic to \(V_{\text{std},2r,Q}(c_{D_{\nu}}V_{\text{std},2r,Q}^\nu)\), where \(\nu_k = (\nu_{k1}, \ldots, \nu_{kr})\) is the polynomial weight of \(\text{GL}_r\) such that, for \(1 \leq i \leq r, \nu_{ki} := \nu_i\) and \(\nu_{kr} := 0\) when \(\nu_i > 0\), while \(\nu_{ki} := \nu_{ki}^{r+1} := 1\) when \(\nu_i = 0\). In other words, it can be constructed by a variant of the isomorphism in Proposition 2.24. However, for simplicity, we shall ignore these representations. (As in Remark 2.17 this is justified by Lemma 2.14.)

As in [12, 1.5], a \(Z\)-lattice in a \(Q\)-representation of a group scheme over \(Z\) is called admissible if it is stable under the action of the group scheme.

**Definition 2.26.** Let \(\nu = (\nu_1, \nu_2, \ldots, \nu_r)\) be a weight satisfying \(\nu_1 \geq \nu_2 \geq \ldots \geq \nu_r\).

1. Let \(\nu_{r+1}\) be any integer such that \(\nu_r \geq \nu_{r+1}\), put \(\nu' := (\nu_1 - \nu_{r+1}, \nu_2 - \nu_{r+1}, \ldots, \nu_r - \nu_{r+1})\), and choose any Young tableau \(D_{\nu'}\) for \(\nu'\). Then we define \(V_{\nu',\nu_{r+1}}\) to be the admissible \(Z\)-lattice

\[V_{\nu',\nu_{r+1}} := (c_{D_{\nu'}}V_{\text{std},r}^\nu)(\wedge^r V_{\text{std},r})^{\otimes \nu_{r+1}}\]

in \(V_{\nu,Q} \cong V_{\nu,Q} \otimes V_{\nu',\nu_{r+1}} \cong (c_{D_{\nu'}}V_{\text{std},r,q}^\nu)(\wedge^r V_{\text{std},r})^{\otimes \nu_{r+1}}\). (Here \(V_{\nu',\nu_{r+1}}\) depends on the choice of \(\nu_{r+1}\), but \(V_{\nu',\nu_{r+1}} \otimes Q \cong V_{\nu,Q}\) does not.)
Definition 2.27. Suppose three cases for factors $G$ we shall assume that $X \in V$ Young symmetrizers, using $V$.

Then we set $V^\nu_r$.

Definition 2.28. Then we set $V^\nu_r$.

The admissibility of these $Z$-lattices is clear because the constructions using Young symmetrizers, using $V^\nu_r$, and using $V^\nu_r$ are all compatible with the actions of the group schemes (over $Z$).

**Definition 2.27.** Suppose $\mu = ((\mu_\tau)_{\tau \in \Gamma / c}; \mu_0) = (((\mu_\tau, r_\tau)_{1 \leq i_\tau, r_\tau})_{\tau \in \Gamma / c}; \mu_0) \in X^+_G$. By replacing $\mu$ with another element in $[\mu]$ (see Remark 2.17) if necessary, we shall assume that $\mu_\tau, r_\tau \geq 0$ for any $\tau \in \Gamma$ such that $G_\tau \cong O_{2r_\tau} \otimes R_1$. There are three cases for factors $G_\tau$ of $G_1$:

1. If $G_\tau \cong Sp_{2r_\tau} \otimes Z R_1$, then we set $V_\mu, := V^\nu_r \otimes Z R_1$.
2. If $G_\tau \cong O_{2r_\tau} \otimes Z R_1$, then we set $V_\mu, := V^\nu_r \otimes Z R_1$.
3. If $G_\tau \cong GL_{r_\tau} \otimes Z R_1$, and if $\mu_\tau, r_\tau + 1$ is the even integer such that $1 \geq \mu_\tau, r_\tau - \mu_\tau, r_\tau + 1 \geq 0$, then we set $V_\mu, := V^\nu_r \otimes Z R_1$.

Here the modules $V^\nu_r$, $V^\nu_r$, and $V^\nu_r$ are defined in Definition 2.26. Then we set

$$V^{\nu_r} := \bigotimes_{\tau \in \Gamma / c} V^{\nu_r}_{\tau} \otimes Z R_1,$$

where $v$ is the free rank one $R_1$-module on which $G_1$ acts via the similitude character.

**Definition 2.28.** Suppose $\mu = ((\mu_\tau)_{\tau \in \Gamma / c}; \mu_0) = (((\mu_\tau, r_\tau)_{1 \leq i_\tau, r_\tau})_{\tau \in \Gamma / c}; \mu_0) \in X^+_M$. There are two cases for factors $M_\tau$ of $M_1$:

1. If $\tau = \tau \circ c$, then $M_\tau \cong GL_{r_\tau} \otimes Z R_1$, and we take $W_{\mu,} := V^\nu_r \otimes Z R_1$.
2. If $\tau \neq \tau \circ c$, then $M_\tau \cong (GL_{q_r} \times GL_{p_r}) \otimes Z R_1$, and we take

$$W_{\mu,} := \bigotimes_{\tau \in \Gamma / c} V^\nu_r \otimes Z R_1.$$
2.6. $p$-small weights and Weyl modules.

**Definition 2.29.** We say $\mu \in X$ is $p$-small for $G_1$ if $(\mu + \rho, \alpha^\vee) \leq p$ for every $\alpha \in \Phi_{G_1}$. We say $\mu \in X$ is $p$-small for $M_1$ if $(\mu + \rho, \alpha^\vee) \leq p$ for every $\alpha \in \Phi_{M_1}$. We denote the subset of $X$ that are $p$-small for $G_1$ (resp. $M_1$) by $X_{G_1}^{\leq p}$ (resp. $X_{M_1}^{\leq p}$), and we set $X_{G_1}^{+,<p} := X_{G_1}^{+} \cap X_{G_1}^{\leq p}$ (resp. $X_{M_1}^{+,<p} := X_{M_1}^{+} \cap X_{M_1}^{\leq p}$).

**Remark 2.30 (cf. [42, 1.9]).** The dot action of $W_{G_1}$ sends a $p$-small weight of $G_1$ to a $p$-small weight of $M_1$. The second statement in Definition 2.29 makes sense because $\rho_{M_1} := \frac{1}{2} \sum_{\alpha \in \Phi_{M_1}} \alpha$ satisfies $(\rho - \rho_{M_1}, \alpha^\vee) = 0$ for any $\alpha \in \Phi_{M_1}$. Thus, if $\mu \in X$ is $p$-small for $G_1$, then $\omega \cdot \mu$ is $p$-small for $M_1$ for any $\omega \in W_{G_1}$.

Since $G_1$ (resp. $M_1$) is split over $R_1$, there exists a split reductive algebraic group $G_{\text{split}}$ (resp. $M_{\text{split}}$) over $\mathbb{Z}(p)$ such that $G_1 \equiv G_{\text{split}} \otimes R_1$ (resp. $M_1 \equiv M_{\text{split}} \otimes R_1$). Note that $G_{\text{split}}$ (resp. $M_{\text{split}}$) has the same roots and weights as $G_1$ (resp. $M_1$), and is a (semi-direct) product of $G_{\mathbb{G}_m}$ with groups of the three types in Propositions 2.19, 2.23 and 2.24 over $\mathbb{Z}(p)$. For $\mu \in X_G^{\leq p}$ (resp. $\mu \in X_{M_1}^{\leq p}$), let $V_{[\mu],\mathbb{Q}}$ (resp. $W_{\mu,\mathbb{Q}}$) be the irreducible representation of $G_{\text{split}} \otimes \mathbb{Q}$ (resp. $M_{\text{split}} \otimes \mathbb{Q}$) containing the dominant weight $\mu$ (see Remark 2.17 for the meaning of $[\mu]$) with simple factors (modulo the similitude character) of the forms given in Propositions 2.19, 2.23 and 2.24. (See also Remark 2.25.) Let $V_{[\mu],\mathbb{Z}(p)} \subset V_{[\mu],\mathbb{Q}}$ (resp. $W_{\mu,\mathbb{Z}(p)} \subset W_{\mu,\mathbb{Q}}$) be the Weyl module over $\mathbb{Z}(p)$ defined as in [42, 1.3], (namely the span of a highest weight vector under the action of the group scheme over $\mathbb{Z}(p)$) which is minimal among admissible $\mathbb{Z}(p)$-lattices in $V_{[\mu],\mathbb{Q}}$ (resp. $W_{\mu,\mathbb{Q}}$) containing the same highest weight vector. (See [42, 1.5].)

According to [42, Cor. 1.9], if $\mu \in X_{G_1}^{+,<p}$ (resp. $\mu \in X_{M_1}^{+,<p}$), then all admissible $\mathbb{Z}(p)$-lattices in $V_{[\mu],\mathbb{Q}}$ (resp. $W_{\mu,\mathbb{Q}}$) are isomorphic to each other. Therefore, it necessarily follows (cf. [42, Cor. 5]) that $V_{[\mu]} \equiv V_{[\mu],\mathbb{Z}(p)} \otimes R_1$ (resp. $W_{\mu} \equiv W_{\mu,\mathbb{Z}(p)} \otimes R_1$), regardless of the artificial choices made in Definitions 2.27 and 2.28. We set $V_{[\mu],R} := V_{[\mu]} \otimes R$ (resp. $W_{\mu,R} := W_{\mu} \otimes R$) for any $R_1$-algebra $R$.

3. Geometric realizations of automorphic bundles

The aim of this and the next sections is to explain how automorphic bundles and their cohomology can be realized geometrically using the cohomology of fiber products of $A \rightarrow S_1$ (with trivial coefficients).

3.1. Standard representations. Consider the decomposition (2.1) induced by (1.5). By [29, Prop. 1.1.1.17], we have $O_\tau \cong M_{1_\tau}(O_{F,\tau})$ for some integer $t_\tau \geq 1$. There are three possibilities, depending on the classification of the group $G_\tau$, or rather the restriction of $*$ to $O_\tau$. (See [29, Lem. 1.2.3.2] and its proof, with several misleading typos corrected in the revision.)

Suppose $G_\tau \cong \text{Sp}_{2t_\tau} \otimes_{\mathbb{Z}} R_1$. This happens exactly when $\tau = \tau \circ c$ and the restriction of $*$ to $O_\tau$ is of the form $x \mapsto c^t x c^{-1}$ for some element $c \in O_\tau$ satisfying $t c = c$. Let us take $\varepsilon_\tau \in O_\tau \cong M_{1_\tau}(O_{F,\tau})$ to be the elementary idempotent matrix $E_{11}$ with unique nonzero entry 1 at the most upper-left corner.
Then we have \( t\varepsilon = \varepsilon, O_\tau \varepsilon_\tau O_\tau = O_\tau, \) and \( L_{\text{std},\tau} := \varepsilon_\tau(L_1) \subset L_1 \) is a free \( R_1 \)-module of rank 2\( \tau \), whose \( O_\tau \)-span in \( L_1 \) is \( L_\tau \) (under the identification \( \tau \)). For any \( R_1 \)-algebra \( R \), to check if \( g \in \text{GL}_{Q_\tau}(L_\tau) \) lies in \( G_\tau \), we need to verify if \( \langle gx, gy \rangle = \langle x, y \rangle \) for \( x, y \in L_\tau \otimes R \). We may assume that \( x \in \varepsilon_\tau(L_1 \otimes R) \).

Let us write \( x = \varepsilon_\tau x_0 \) and \( y = cy_0 \) for some \( x_0, y_0 \in L_\tau \). Then \( x = \varepsilon_\tau x, \) and \( \langle x, y \rangle = \langle \varepsilon_\tau x, \varepsilon_\tau y \rangle = \langle x, \varepsilon_\tau^2 \varepsilon_\tau^{-1} y \rangle = \langle x, \varepsilon_\tau \varepsilon_\tau^{-1} y \rangle = \langle x, \varepsilon_\tau y \rangle \) shows that it suffices to check if the action induced by \( g \) on \( L_{\text{std},\tau} \) preserves the pullback to \( \tau \) of the pairing \( \langle \cdot, \cdot \rangle_{\text{std},\tau} : L_{\text{std},\tau} \times L_{\text{std},\tau} \to R_1(1) \) defined by \( \langle x, z \rangle_{\text{std},\tau} := \langle x, cz \rangle \) for any \( x, z \in L_{\text{std},\tau} \). (This pairing is alternating because \( c^* = c \cdot c^{-1} = c \).) Then we view \( (L_{\text{std},\tau}, \langle \cdot, \cdot \rangle_{\text{std},\tau}) \) as the standard representation of \( G_\tau \cong \text{Sp}_{2\tau} \otimes R_1 \).

Suppose \( G_\tau \cong O_{2\tau} \otimes R_1 \). This happens exactly when \( \tau = \tau \circ c \) and the restriction of \( * \) to \( O_\tau \) is of the form \( x \mapsto d^t x d^{-1} \) for some \( d \in O_\tau \) satisfying \( t d = -d \).

Let us take \( \varepsilon_\tau \in O_\tau \cong M_{1\tau}(O_{F,\tau}) \) to be the elementary idempotent matrix \( E_{11} \) with unique nonzero entry 1 at the most upper-left corner. Then we have \( t\varepsilon = \varepsilon, \quad O_\tau \varepsilon_\tau O_\tau = O_\tau, \quad \) and \( L_{\text{std},\tau} := \varepsilon_\tau(L_1) \subset L_1 \) is a free \( R_1 \)-module of rank 2\( \tau \), whose \( O_\tau \)-span in \( L_1 \) is \( L_\tau \) (under the identification \( \tau \)). By an analogous procedure as in the symplectic case, we define the pairing \( \langle \cdot, \cdot \rangle_{\text{std},\tau} : L_{\text{std},\tau} \times L_{\text{std},\tau} \to R_1(1) \) by \( \langle x, z \rangle_{\text{std},\tau} := \langle x, dz \rangle \) for any \( x, z \in L_{\text{std},\tau} \). (This pairing is symmetric because \( d^* = d^t d^{-1} = -d \).) Then we view \( (L_{\text{std},\tau}, \langle \cdot, \cdot \rangle_{\text{std},\tau}) \) as the standard representation of \( G_\tau \cong O_{2\tau} \otimes R_1 \).

Suppose \( G_\tau \cong GL_{2\tau} \otimes R_1 \). This happens exactly when \( \tau \neq \tau \circ c \). Then \( * \) switches the two factors \( O_\tau \) and \( O_{\tau \circ c} \) in \( \tau \). Let us take \( \varepsilon_\tau \in O_\tau \cong M_{1\tau}(O_{F,\tau}) \) to be the elementary idempotent matrix \( E_{11} \) with unique nonzero entry 1 at the most upper-left corner. Then we have \( O_\tau \varepsilon_\tau O_\tau = O_\tau, \quad O_{\tau \circ c} \varepsilon_\tau O_{\tau \circ c} = O_{\tau \circ c}, \quad \) and \( L_{\text{std},\tau} := \varepsilon_\tau(L_1) \subset L_1 \) and \( L_{\text{std},\tau}^* := \varepsilon_\tau^*(L_1) \subset L_1 \) are free \( R_1 \)-modules of rank \( \tau \) whose \( O_\tau \)-spans in \( L_1 \) are respectively \( L_\tau \) and \( L_{\tau \circ c} \) (under the identification \( \tau \)). Then the restriction of \( \langle \cdot, \cdot \rangle \) to \( L_\tau \times L_\tau \) is determined by its restriction to \( L_{\text{std},\tau} \times L_{\text{std},\tau}^* \), so that the action of \( G_\tau \) on \( L_{\text{std},\tau} \) is determined by its action on \( L_{\text{std},\tau}^* \); and we view \( L_{\text{std},\tau} \) as the standard representation of \( G_\tau \cong O_{2\tau} \otimes R_1 \).

Any element \( b \otimes r \in O_\tau \otimes R_1 \) acts on \( H^1_{\text{dR}}(A/M_{H,1}) \) by
\[
(b \otimes r)_* := r i(b)_* : H^1_{\text{dR}}(A/M_{H,1}) \to H^1_{\text{dR}}(A/M_{H,1}),
\]
where \( i : O \to \text{End}_{M_{H,1}}(A) \) is the \( O \)-endomorphism structure inducing \( i(b) \), by functoriality, and where \( r \) acts via the \( R_1 \)-module structure. (Similar actions work for any reasonable homology or cohomology of \( A \) with coefficients in \( R_1 \)-modules.) Since \( \varepsilon_\tau \) is an idempotent, we obtain an \( R_1 \)-module summand
\[
L_{\text{std},\tau} := (\varepsilon_\tau)_*(H^1_{\text{dR}}(A/M_{H,1})),
\]
of \( H^1_{\text{dR}}(A/M_{H,1}) \). By functoriality and exactness of \( \mathcal{E}_{G_1}(\cdot) \), we have
\[
\mathcal{E}_{G_1}(L_{\text{std},\tau}) \cong L_{\text{std},\tau}.
\]

### 3.2. Lieberman’s trick

Let \( m, n \geq 0 \) be two integers. Let \( Z \) denote the multiplicative semi-group of integers, and let \( Z^n \) denote its \( n \)-fold product. Then \( Z^n \) has
a natural componentwise action on $L^\oplus_1$, inducing canonically an action on
\[ (3.1) \quad L^\oplus_1 \cong \bigoplus_{i_1, i_2, \ldots, i_n \geq 0} \left( (\wedge i_1 L_1) \otimes (\wedge i_2 L_1) \otimes \cdots \otimes (\wedge i_n L_1) \right), \]
with $(l_1, l_2, \ldots, l_n)$ acting as $l_1^i l_2^j \cdots l_n^i$ on $(\wedge i_1 L_1) \otimes (\wedge i_2 L_1) \otimes \cdots \otimes (\wedge i_n L_1)$.

When $m = n$, the summand with $i_1 = i_2 = \ldots = i_n = 1$ is just $L^\oplus_1$.

Suppose $m < p$. For each $0 \leq i \leq m$ except $i = 1$, choose an integer $1 \leq l(i) < l$ such that $l(i) - l(i)$ is a unit in $\mathbb{Z}(p)$. This is always possible because $m < p$. Let $\varepsilon_{n,i,j}^{L}$ denote the element $l(i)^i(1, 1, 1, \ldots, 1) - (1, l(i), \ldots, 1)$ in $\mathbb{Z}[\mathbb{Z}]^n$ with $l(i)$ appearing in the $j$-th entry in the second term (with all the other entries 1). Then $\varepsilon_{n,m}^{L,i,j}$ acts as zero on all summands in \[ (3.1) \] except for $L^\oplus_1$ when $m = n$, on which it acts as 1 instead. This shows that $\varepsilon_{n,m}^{L,i,j}$ acts as an idempotent on $\wedge^m(L^\oplus_1)$, defining a projection to $L^\oplus_1$ when $m = n$. We shall denote $\varepsilon_{n,m}^{L,i,j}$ by $\varepsilon_{n,m}^{L}$ for simplicity.

Now suppose we have a tuple $\underline{n} = (n_\tau)_{\tau \in \mathcal{Y}/c}$ such that $n = |\underline{n}| := \sum_{\tau \in \mathcal{Y}/c} n_\tau$ satisfies $n < p$. Consider the componentwise action of $\mathcal{O}_1^n$ on $L^\oplus_1$. To be precise, we shall denote elements in $\mathcal{O}_1^n$ by $b = (b_{\tau,i})_{1 \leq i, \tau \leq n_\tau}$ in $\mathcal{O}_1^n$. Consider the idempotent $\varepsilon_{n} = (\varepsilon_{n,i,\tau})_{1 \leq i, \tau \leq n_\tau}$ in $\mathcal{O}_1^n$ with $\varepsilon_{\tau,i,\tau} = \varepsilon_{\tau}$ for any $\tau \in \mathcal{Y}/c$ and any $1 \leq i, \tau \leq n_\tau$. Then we have

\[ \bigotimes_{\tau \in \mathcal{Y}/c} L^\oplus_{n_\tau} \cong \varepsilon_{n} L^\oplus_1. \]

Geometrically, we can realize $\wedge^m(L^\oplus_1)$ by taking the $n$-fold fiber product $A^n$ of $A$ over $M_{H,1}$ and then taking the $m$-th relative de Rham homology

\[ H^dR_m(A^n/M_{H,1}) \cong \wedge^m(H^dR(A/M_{H,1})^\oplus_n). \]

Then we obtain natural isomorphisms

\[ E_{\mathcal{G}_1}(L^\oplus_{n_\tau}) \cong \bigotimes_{\tau \in \mathcal{Y}/c} (\varepsilon_{n_\tau})_*(\varepsilon_{n})_*(H^dR_m(A^n/M_{H,1})). \]

### 3.3. Young symmetrizers

Now suppose we have an element $\mu \in X^+_1$ such that $\mu = ((\mu_{\tau})_{\tau \in \mathcal{Y}/c}; \mu_0) = ((\mu_{\tau,i})_{1 \leq i, \tau \leq n_\tau})_{\tau \in \mathcal{Y}/c}$, where $\mu_{\tau,i,\tau} = \varepsilon_{\tau}$ and $\mu_{\tau,i,\tau} = 0$ for any $\tau \in \mathcal{Y}/c$ such that $\mu_{\tau,i,\tau} \geq 0$ for any $\tau \in \mathcal{Y}$ such that $G_{\tau} \cong \mathrm{O}_{2r_{\tau}, \mathbb{Z}} \otimes R_1$. For each $\tau \in \mathcal{Y}/c$, we have two possibilities:

1. $G_{\tau} \cong \mathrm{Sp}_{2r_\tau, \mathbb{Z}} \otimes R_1$ or $G_{\tau} \cong \mathrm{O}_{2r_\tau, \mathbb{Z}} \otimes R_1$, we view $\mu_{\tau}$ as a polynomial weight $\mu'_{\tau}$ of $\mathrm{GL}_{2r_{\tau}}$ by supplying zeros in the end. We set $t_{\mu_{\tau}} := 0$ in this case.

2. $G_{\tau} \cong \mathrm{GL}_{r_{\tau}, \mathbb{Z}} \otimes R_1$, we take $\mu_{\tau,r_{\tau}+1}$ to be the unique integer such that $1 \geq \mu_{\tau,r_{\tau}+1} \geq 0$, and take the polynomial weight $\mu'_{\tau} = (\mu_{\tau,1} - \mu_{\tau,r_{\tau}+1}, \mu_{\tau,2} - \mu_{\tau,r_{\tau}+1}, \ldots, \mu_{\tau,r_{\tau}+1} - \mu_{\tau,r_{\tau}+1})$ of $\mathrm{GL}_{r_{\tau}, \mathbb{Z}} \otimes R_1$. We set $t_{\mu_{\tau}} := (1/2)\tau \mu_{r_{\tau}+1}$ in this case.
In either case, we take a Young tableau $D_{\mu}^r$ for $\mu^r$, and define the Young symmetrizer $c_{D_{\mu}^r}$ in $\mathbb{Z}[\mathcal{S}[\mu^r]]$. By Lemma 2.18, $c_{D_{\mu}^r}c_{D_{\mu}^r} = d_{D_{\mu}^r}c_{D_{\mu}^r}$ for some integer $d_{D_{\mu}^r}$ dividing $|\mu^r|!$ (i.e. factorial).

**Definition 3.2.** Set $|\mu|_Y := \max_{\tau \in \Upsilon/c} |\mu^r|_\tau$ and $|\mu|_L := \sum_{\tau \in \Upsilon/c} |\mu^r|_\tau$. (Here $\mu^r$ is defined after replacing $\mu$ with the element in $[\mu]$ (see Remark 2.17) satisfying $\mu_{r,r} \geq 0$ for any $\tau \in \Upsilon$ such that $G_\tau \cong O_{2r} \otimes \mathbb{Z}$.) By abuse of notation, we shall also write $|\mu^r|_L = |\mu^r|$. We say a weight $\mu$ in $X^+_{G_1}$ is $p$-small for Young symmetrizers (resp. for Lieberman’s trick) if $|\mu|_Y < p$ (resp. $|\mu|_L < p$). Obviously, $|\mu|_L < p$ implies $|\mu|_Y < p$, and they coincide when $\Upsilon/c$ is a singleton. If $|\mu| < p$ and $\mu \in X^+_{G_1}$, we say $\mu$ is $p$-small for the geometric realization of Weyl’s construction. We denote by $X^+_{G_1}$ (resp. $X^{+,<}\mathbb{Z}$, resp. $X^{+,<}\mathbb{Z}$) the set of weights $p$-small for Young symmetrizers (resp. for Lieberman’s trick, resp. the geometric realization of Weyl’s construction).

Now suppose $\mu \in X^+_{G_1}$ (and hence $\mu \in X^{+,<}\mathbb{Z}$). Then $d_{D_{\mu}^r}^{-1}c_{D_{\mu}^r} \in \mathbb{Z}(p)[\mathcal{S}[\mu^r]]$ for each $\tau \in \Upsilon/c$, and we define

$$
\varepsilon^Y_\mu := \bigotimes_{\tau \in \Upsilon/c} (d_{D_{\mu}^r}^{-1}c_{D_{\mu}^r}) \in \bigotimes_{\tau \in \Upsilon/c} \mathbb{Z}(p)[\mathcal{S}[\mu^r]] \xrightarrow{\text{can}} \mathbb{Z}(p)[\mathcal{S}[\mu^r]],
$$

which acts on $\bigotimes_{\tau \in \Upsilon/c} L^{\otimes |\mu^r|_\tau}$ as an idempotent. Since $\mathcal{S}[\mu^r]$ acts naturally on $A^{\mu}$ by permutations, we can realize the geometric action $(\varepsilon^Y_\mu)_*$ on $H^{\text{DR}}_m(A^{\mu}|_{L}/\mathcal{M}_{H,1})$ by functionality.

We shall denote by $\varepsilon^S$ the $\varepsilon_\mu$ in Section 3.2 with $n = (|\mu^r|)_{\tau \in \Upsilon/c}$.

### 3.4. Poincaré bundles. We retain the setting in the previous section.

Suppose $\tau \in \Upsilon/c$ satisfies $\tau = \tau \circ c$. Suppose $(x,y)_{\text{std},\tau} = (x,c_\tau y)$ for some $c_\tau \in \mathcal{O}_\tau$ (which was either $c$ or $d$ in Section 3.1, depending on whether we were in the symplectic or orthogonal case) such that $\varepsilon_\tau = c_\tau \varepsilon_\tau c_\tau^{-1}$, for any $x,y \in L_{\text{std},\tau} = \varepsilon_\tau(L_1)$.

For any $1 \leq i < j \leq |\mu^r|$, we define $\phi_{i,j}^{(c_\tau \varepsilon_\tau^{-1})_{\text{std},\tau}} : L^{\otimes |\mu^r|_{i-1}} \to L^{\otimes |\mu^r|_{i-1}}(1)$ by

$$
\phi_{i,j}^{(c_\tau \varepsilon_\tau^{-1})_{\text{std},\tau}}(v_1 \otimes v_2 \otimes \ldots \otimes v_{|\mu^r|}) := (v_i,v_j)_{\text{std},\tau}(v_1 \otimes \ldots \otimes \hat{v}_i \otimes \ldots \otimes \hat{v}_j \otimes \ldots \otimes v_{|\mu^r|})
$$

for $v_1,\ldots,v_{|\mu^r|} \in L_{\text{std},\tau}$, and define $\phi_{i,j}^{(c_\tau \varepsilon_\tau^{-1})} : L^{\otimes |\mu^r|_{i-1}} \to L^{\otimes |\mu^r|_{i-1}}(1)$ by

$$
\phi_{i,j}^{(c_\tau \varepsilon_\tau^{-1})}(v_1 \otimes v_2 \otimes \ldots \otimes v_{|\mu^r|}) := (v_i,c_\tau \varepsilon_\tau v_j)(v_1 \otimes \ldots \otimes \hat{v}_i \otimes \ldots \otimes \hat{v}_j \otimes \ldots \otimes v_{|\mu^r|})
$$

for $v_1,\ldots,v_{|\mu^r|} \in L_1$. (Here $\hat{v}_i$ and $\hat{v}_j$ denote omissions of entries as usual.)

**Lemma 3.3.** We have $\ker(\phi_{i,j}^{(c_\tau \varepsilon_\tau^{-1})}) = \varepsilon_{\tau,|\mu^r|} \ker(\phi_{i,j}^{(c_\tau \varepsilon_\tau^{-1})})$ in $L^{\otimes |\mu^r|}$, where $\varepsilon_{\tau,|\mu^r|} \in \mathcal{O}^{\mu^r}$ has all its entries equal to $\varepsilon_\tau$.

**Proof.** This is because $(x,c_\tau \varepsilon_\tau y) = \langle x,c_\tau \varepsilon_\tau^{-1} y \rangle = \langle x,c_\tau \varepsilon_\tau^{-1} c_\tau y \rangle = \langle \varepsilon_\tau x,c_\tau \varepsilon_\tau y \rangle$ for any $x,y \in L_1$. (See Section 3.3. □)

Now let us turn to geometric realizations. The first Chern class $c_1((\text{Id}_A \times X)^*P_A) \in H^{\text{DR}}_m(A^2/\mathcal{M}_{H,1})(1)$ induces, by Künneth decomposition, the pairing $\langle \cdot , \cdot \rangle_\lambda : H^{\text{DR}}_1(A/\mathcal{M}_{H,1}) \times H^{\text{DR}}_1(A/\mathcal{M}_{H,1}) \to \mathcal{O}_{\mathcal{M}_{H,1}}(1)$,
which is the geometric realization of \( \langle \cdot, \cdot \rangle : L_1 \times L_1 \to R_1(1) \). Thus, if 
\[ c_{\tau} = \sum_{\alpha \in I} b_{\alpha} \circ \tau_{\alpha} \in O_{\tau} = O \otimes R_1, \] 
then \( \langle \cdot, c_{\tau} \rangle \) is realized geometrically by
\[ c_{\tau}^\lambda := \sum_{\alpha \in I} r_{\alpha} (I_d A \times i(b_{\alpha}))^*(c_1((I_d A \times \lambda)^*P_A)) \in H^2_{\text{dR}}(A^2/M_{H,1})(1) \]
For any \( 1 \leq i < j \leq \vert \mu_r' \vert \), consider the Künneth morphisms
\[ K_{\tau}^{i,j} : H^{|\mu_r'|-2}_{\text{dR}}(A|\mu_r'|^{-2}/M_{H,1}) \otimes H^2_{\text{dR}}(A^2/M_{H,1}) \to H^{|\mu_r'|}_{\text{dR}}(A|\mu_r'|/M_{H,1}) \]
corresponding to the \( i \)-th and \( j \)-th factors in \( A|\mu_r'| \). (Note that the image of \( K_{\tau}^{i,j} \) can also be cut out by a variant of Lieberman’s trick.) Then the composition
\[ H^{|\mu_r'|-2}_{\text{dR}}(A|\mu_r'|^{-2}/M_{H,1}) \cong H^{|\mu_r'|-2}_{\text{dR}}(A|\mu_r'|^{-2}/M_{H,1}) \otimes H^0_{\text{dR}}(A^2/M_{H,1}) \]
\[ \to H^{|\mu_r'|-2}_{\text{dR}}(A|\mu_r'|^{-2}/M_{H,1}) \otimes H^2_{\text{dR}}(A^2/M_{H,1})(1) \xrightarrow{K_{\tau}^{i,j}} H^{|\mu_r'|}_{\text{dR}}(A|\mu_r'|/M_{H,1})(1) \]
is dual to the morphism \( H^{|\mu_r'|}_{\text{dR}}(A|\mu_r'|/M_{H,1}) \to H^{|\mu_r'|-2}_{\text{dR}}(A|\mu_r'|^{-2}/M_{H,1})(1) \) inducing the geometric realization
\[ (4.4) \quad \phi_{\tau}^{i,j} : H^{|\mu_r'|}_{\text{dR}}(A/M_{H,1}) \otimes |\mu_r'| \to H^{|\mu_r'|}_{\text{dR}}(A/M_{H,1}) \otimes |\mu_r'|^{-2}(1) \]
of \( \phi_{\tau}^{i,j} \). That is, we take the cup product of the image of \( H^{|\mu_r'|-2}_{\text{dR}}(A|\mu_r'|^{-2}/M_{H,1}) \) under \( K_{\tau}^{i,j} \) in \( H^{|\mu_r'|-2}_{\text{dR}}(A|\mu_r'|^{-2}/M_{H,1}) \) with the pullback of \( \chi_{\tau}^\lambda \) to \( A|\mu_r'| \).

On the other hand, the pairing \( \langle \cdot, \cdot \rangle \) identifies \( H^{|\mu_r'|}_{\text{dR}}(A/M_{H,1}) \) with its own dual, with values in \( O_{M_{H,1}}(1) \). Therefore we obtain a morphism
\[ (3.5) \quad \psi_{\tau}^{i,j} : H^{|\mu_r'|}_{\text{dR}}(A/M_{H,1}) \otimes |\mu_r'|^{-2}(1) \to H^{|\mu_r'|}_{\text{dR}}(A/M_{H,1}) \otimes |\mu_r'| \]
geometrically realizing the map \( \psi_{\tau}^{i,j} : L^1 \otimes \langle |\mu_r'|^{-2}(1) \to L^1 \otimes |\mu_r'| \) inserting \( \langle \cdot, c_{\tau} \rangle \) into the \( i \)-th and \( j \)-th component. Since \( \varepsilon_{\tau}^\lambda = c_{\tau} \varepsilon_{\tau} c_{\tau}^{-1} \), the geometric action of \( \varepsilon_{\tau}^\lambda \) commutes with \( \phi_{\tau}^{i,j} \) and \( \psi_{\tau}^{i,j} \), and induces
\[ \phi_{\tau}^{i,j} : L_{\text{std}, \tau} \otimes |\mu_r'| \to L_{\text{std}, \tau} \otimes \langle |\mu_r'|^{-2}(1) \quad \text{and} \quad \psi_{\tau}^{i,j} : L_{\text{std}, \tau} \otimes \langle |\mu_r'|^{-2}(1) \to L_{\text{std}, \tau}. \]
Now assume that either \( r_{\tau} = 0 \) or \( p \uparrow 2r_{\tau} \). This is true, for example, if \( \max(2, r_{\tau}) < p \). As explained in the paragraph following Definition 2.22 we have \( \mathcal{E}_G_{1}(\text{ker}(\phi_{\tau}^{i,j})_{\text{std}, \tau}) \) isomorphic to \( \text{Id} - (2r_{\tau})^{-1} \psi_{\tau}^{i,j} \phi_{\tau}^{i,j} \). (We cannot define \( (2r_{\tau})^{-1} \)) when \( r_{\tau} = 0 \), but at the same time \( L_{\text{std}, \tau} \) is trivial. In this case, we shall maintain the abuse of language that \( \text{Id} - (2r_{\tau})^{-1} \psi_{\tau}^{i,j} \phi_{\tau}^{i,j} \) and similar operators below are defined symbolically and act trivially.) Combining all possible \( 1 \leq i < j \leq |\mu_r'| \), we define \( \varepsilon_{\tau, i, j} \) to be the \( R_1 \)-linear combination of algebraic correspondences on \( A|\mu_r'| \) acting as the idempotent
\[ (3.6) \quad (\varepsilon_{\tau, i, j} |\mu_r'|)^* = \prod_{1 \leq i < j \leq |\mu_r'|} (\text{Id} - (2r_{\tau})^{-1} \psi_{\tau, i, j} \phi_{\tau, i, j}) \]
on \( H^{|\mu_r'|}_{\text{dR}}(A/M_{H,1}) \otimes |\mu_r'| \). Then \( (\varepsilon_{\tau, i, j} |\mu_r'|)(L_{\text{std}, \tau}) \) is isomorphic to \( \mathcal{E}_G_{1}(L_{\text{std}, \tau}(|\mu_r'|)) \) (resp. \( \mathcal{E}_G_{1}(L_{\text{std}, \tau}) \)) when \( G_r \cong \text{Sp}_{2r_{\tau}} \otimes R_1 \) (resp. \( G_r \cong O_{2r_{\tau}} \otimes R_1 \)).
Finally, in the case $\tau \neq \tau \circ c$ we set $\varepsilon_{\tau, [\mu]_L}^\lambda$ to be trivial, so that $(\varepsilon_{\tau, [\mu]_L}^\lambda)_* = \text{Id}$. Using the K"unneth morphisms, we define $\varepsilon_{\mu}^\lambda$ to be the product of pullbacks of $\varepsilon_{\tau, [\mu]_L}^\lambda$, so that $(\varepsilon_{\mu}^\lambda)_*$ acts on $\bigotimes_{\tau \in \mathcal{Y}/c} L^\otimes_{\mathcal{Y}, n\tau}$ as the idempotent

$$(\varepsilon_{\mu}^\lambda)_* = \bigotimes_{\tau \in \mathcal{Y}/c} (\varepsilon_{\tau, [\mu]_L}^\lambda)_*.$$ 

### 3.5. Geometric plethysm

We can summarize our constructions as follows:

**Proposition 3.7** Suppose $\mu \in X^+_{\text{G}_1},$ with $0 \leq n := |\mu|_{\text{L}} < p$, as in Definition 3.2. Then $\mu \in X^+_{\text{G}_1}$ as well, so that the Weyl module $V_{[\mu]}$ is defined. (See Section 2.6.) Suppose moreover that $\max(2, r_\tau) < p$ whenever $\tau = \tau \circ c$. Consider the $n$-fold fiber product $A^n$ of $A$ over $M_{H,1}$. Consider the coherent sheaf $\mathcal{H}^R_n(A^n/M_{H,1}) \cong \wedge^n(\mathcal{H}^R_1(A/M_{H,1} \otimes \mathcal{E}))$ equipped with the canonical action of $R_1[O^\text{reg}_1 \times \text{S}_n]$ induced functorially by the $O$-endomorphism structure $i: O \to \text{End}_{M_{H,1}}(A)$ and by permuting factors. Let $\varepsilon_n^\lambda, \varepsilon^\mu_n,$ and $\varepsilon^\lambda_n$ be the elements in $R_1[O^\text{reg}_1 \times \text{S}_n]$ defined in Sections 3.2–3.3, and let $\varepsilon^\lambda_n$ be the one defined in Section 3.4, all acting as idempotents on $\mathcal{H}^R_n(A^n/M_{H,1})$. Put $\varepsilon_{\mu} := \varepsilon^\lambda_n \varepsilon^\mu_n \varepsilon^S_n$, so that

$$(\varepsilon_{\mu})_* = (\varepsilon^\lambda_n)_* (\varepsilon^\mu_n)_* (\varepsilon^S_n)_*(\varepsilon^L_n)_*,$$

and let

$$t_{\mu} := \mu_0 + \sum_{\tau \in \mathcal{Y}/c} (\varepsilon_{\mu}^\lambda)_*$$

be the total number of Tate twists. (The order of $\varepsilon^\lambda_n, \varepsilon^\mu_n, \varepsilon^S_n,$ and $\varepsilon^L_n$ in the definition of $\varepsilon_{\mu}$ does not matter, and their product $\varepsilon_{\mu}$ acts as an idempotent, because they commute with one another by definition.) Then we have canonical isomorphisms

$$V_{[\mu]} := \mathcal{E}_{G_1}(V_{[\mu]}) \cong (\varepsilon_{\mu})_* \mathcal{H}^R_n(A^n/M_{H,1})(t_{\mu}).$$

and (by duality)

$$V_{[\mu]}^\vee := \mathcal{E}_{G_1}(V_{[\mu]}^\vee) \cong (\varepsilon_{\mu})^* \mathcal{H}^R_n(A^n/M_{H,1})(-t_{\mu}).$$

Moreover, the $F$-filtration on $\mathcal{E}_{G_1}(V_{[\mu]})$ coincides with the Hodge filtration on $\mathcal{H}^R_n(A^n/M_{H,1})(t_{\mu})$. The duality between $\mathcal{E}_{G_1}(V_{[\mu]})$ and $\mathcal{E}_{G_1}(V_{[\mu]}^\vee)$ is obvious.

**Proof.** Since $\mu \in X^+_{\text{G}_1}$, the construction in Section 2.5 shows that $V_{[\mu]}$ can be constructed using the same collection of idempotents. Hence the result follows from the identifications in Example 1.22 and from the matching between powers of the similitude character $\nu$ and Tate twists. \qed

**Remark 3.8.** Since $\varepsilon_{\mu}$ acts as an idempotent, the vector bundle $V_{[\mu]}$ (resp. $V_{[\mu]}^\vee$) is a direct summand of $\mathcal{H}^R_n(A^n/M_{H,1})(t_{\mu})$ (resp. $\mathcal{H}^R_n(A^n/M_{H,1})(-t_{\mu})$).

**Definition 3.9.** We set $d := \dim_{\text{S}_1}(M_{H,1}),$ $|\mu|_{\text{re}} := d + |\mu|_{\text{L}},$ and $|\mu|_{\text{tot}} := \dim_{\text{S}_1}(A^n) = d + \dim_{M_{H,1}}(A) |\mu|_{\text{L}}$. We call $|\mu|_{\text{re}}$ (resp. $|\mu|_{\text{tot}}$) the realization size (resp. total size) of $\mu$.

**Remark 3.10.** According to Remark 1.10, we have the simple formula $d = \dim_{R_1}(G_1) - \dim_{R_1}(P_1)$.
Remark 3.11. Note that $|\mu|_{re}$ and $|\mu|_{tot}$ are always non-negative and are insensitive to the entry $\mu_0$ in $\mu$. In particular, they are different from the so-called motivic weight of the local system $V^\vee_{[\mu]}$. (Nowhere in the various bounds in our results on torsion coefficients will appear the motivic weight.)

3.6. Construction without Poincaré duality. We retain the assumptions of Proposition 3.7 in this subsection.

The definition of the idempotent $\varepsilon_\mu$, which we have employed to realize $V^\vee_{[\mu]}$ as a direct summand of $H^n_{\text{dR}}(A^n/M_{H,1})(-t_\mu)$ (see Remark 3.8), relies on Poincaré duality when $\tau = \tau \circ c$ (i.e., for types C and D). For technical reasons (that will be clarified in Section 5.2), it is preferable to avoid this dependence, and here is how it can be done.

So suppose $\tau \in \mathfrak{Y}$ satisfies $\tau = \tau \circ c$. For simplicity, let us assume that $G_\tau \cong \text{Sp}_{2r_\tau} \otimes \mathbb{Z}$. (The case when $G_\tau \cong \mathbb{O}_{2r_\tau} \otimes \mathbb{Z}$ is similar.) Then we know that $$(e_{r_\tau,|\mu_{\tau}|})_*(L^\otimes_{\text{std},\tau}|\mu_{\tau}|) \cong \mathcal{E}_{G_1}(L^{(|\mu_{\tau}|)}_{\text{std},\tau})$$ is the kernel of

\begin{equation}
(3.12)
\phi_{r_\tau,i,j}^\lambda : L^\otimes_{\text{std},\tau}|\mu_{\tau}| \to \bigoplus_{1 \leq i < j \leq n} L^\otimes_{\text{std},\tau}|\mu_{\tau}|^{-2}(-1).
\end{equation}

(See the paragraph containing (3.4).) Here the notation $\phi_{r_\tau,i,j}^\lambda$ makes sense (as a restriction) because the geometric action of $\varepsilon_{\tau,|\mu_{\tau}|}$ commutes with $\phi_{r_\tau,i,j}^\lambda$. Equivalently, $(e_{r_\tau,|\mu_{\tau}|})^*(L^\otimes_{\text{std},\tau}|\mu_{\tau}| \otimes|\mu_{\tau}|)$ is the cokernel of

\begin{equation}
(3.13)
\sum_{1 \leq i < j \leq n} (\phi_{r_\tau,i,j}^\lambda)^\vee : \bigoplus_{1 \leq i < j \leq n} (L^\otimes_{\text{std},\tau}|\mu_{\tau}| \otimes|\mu_{\tau}|)^{-2}(-1) \to (L^\otimes_{\text{std},\tau}|\mu_{\tau}|)^{-1}.
\end{equation}

As explained in Section 3.4, the definition of each $(\phi_{r_\tau,i,j}^\lambda)^\vee$ involves only functoriality and cup product with the pullback of $e_{r_\tau}^\lambda$.

Lemma 3.14. The image of the morphism (3.13) is globally a direct summand (as a coherent module with connection).

Proof. This is because it is the kernel of the idempotent $(e_{r_\tau,|\mu_{\tau}|})^*$.

Remark 3.15. The point is that, while we use $(e_{r_\tau,|\mu_{\tau}|})^*$ in the proof, we do not need it in the definition using the morphism (3.13).

Lemma 3.16. If $|\mu|_{re} < p$, then the kernel of the morphism (3.13) is globally a direct summand.

Proof. Equivalently, we can show that the image of the dual morphism (3.12) is a direct summand. Without using a convenient idempotent like $(e_{r_\tau,|\mu_{\tau}|})^*$, it suffices to notice that (3.12) is the functorial image under $\mathcal{E}_{G_1,R_1}(\cdot)$ of a similar morphism in $\text{Rep}_R(G_1)$, the question is whether the surjection from the source to the image of this morphism (in $\text{Rep}_R(G_1)$) splits (non-canonically). By [42, 1.10, Cor.] (or rather by the same proof there), it suffices to show that all the objects in (3.13) lie in the image under $\mathcal{E}_{G_1,R_1}(\cdot)$ of representations with $p$-small weights, between which there cannot be any nontrivial extension classes. Since $|\mu|_{re} = d + |\mu|_L < p$, it suffices to check that, for any integer $m$ such that $0 \leq m < p - d$, all the weights of the representation $L^\otimes_{\text{std},\tau}^m$ of $G_1$ (or rather of $G_\tau$) are $p$-small. Any weight $\nu$ of $L^\otimes_{\text{std},\tau}^m$ satisfies $|\nu_\tau| := \sum_{1 \leq i_\tau < j_\tau \leq r_\tau} |\nu_{r_\tau,i_\tau}| \leq m < p - d$. Then, for any $1 \leq i_\tau < j_\tau \leq r_\tau$,
we have \(|\nu_{r,i} + i_r| + |\nu_{j,r} + j_r| \leq m + d < p\). This implies that \((\nu + \rho, \alpha') \leq p\), i.e. \(\nu\) is \(p\)-small, as desired. \(\square\)

**Remark 3.17.** Lemma 3.16 is needed only in Section 5.2.

4. Cohomology of automorphic bundles

In this section, we fix a choice of \(\mu \in \mathbb{C}^{1+,<w} \) and take \(n = |\mu|_L\). We shall maintain the running assumption that \(\max(2, r_{\tau}) < p\) whenever \(\tau = \tau(\sigma, \eta)\), so that the element \(\varepsilon_\mu = \varepsilon_\mu \otimes \varepsilon_\eta \otimes \varepsilon_\mu \) in Proposition 3.7 is defined. Let \(f_n : A^n \to M_{H,1}\) be the structural morphism.

4.1. Koszul and Hodge filtrations. By smoothness of \(f_n\), we have the exact sequence \(0 \to f_n^*(\Omega_{\mathcal{M}_{H,1}/S_1}^1) \to \Omega_{A^n/S_1}^1 \to \Omega_{A^n/M_{H,1}}^1 \to 0\), which induces the Koszul filtration [24, 1.2, 1.3] \(K^q(\Omega_{A^n/S_1}^*) := \text{image}(\Omega_{A^n/S_1}^{* - a} \otimes f_n^*(\Omega_{\mathcal{M}_{H,1}/S_1}^{a}) \to \Omega_{A^n/S_1}^*)\) on \(\Omega_{A^n/S_1}^1\), with graded pieces \(\text{Gr}_F^p(\Omega_{A^n/S_1}^*) \cong \Omega_{A^n/S_1}^{* - a} \otimes f_n^*(\Omega_{\mathcal{M}_{H,1}/S_1}^{a})\).

On the other hand, we have the Hodge filtration \(F^p(\Omega_{A^n/S_1}^*) := \Omega_{A^n/S_1}^{* - a} \otimes f_n^*(\Omega_{\mathcal{M}_{H,1}/S_1}^{a})\), giving the Hodge filtration \(F^p(H^i_{\text{dR}}(A^n/M_{H,1})) := \text{image}(R^i(f_n)_*(\Omega_{A^n/S_1}^*)) \to R^i(f_n)_*(\Omega_{A^n/S_1}^1)\) on \(H^i_{\text{dR}}(A^n/M_{H,1})\). By applying \(R^i(f_n)_*\) to the short exact sequence

\begin{equation}
0 \to \Omega_{A^n/M_{H,1}}^1 \otimes f_n^*(\Omega_{\mathcal{M}_{H,1}/S_1}^1) \to \mathcal{K}/\mathcal{K}^2 \to \Omega_{A^n/M_{H,1}}^* \to 0,
\end{equation}

we obtain in the long exact sequence the connecting homomorphisms \(H^i_{\text{dR}}(A^n/M_{H,1}) = R^i(f_n)_*(\Omega_{A^n/M_{H,1}}^1) \cong \bigoplus R^{i+1}(f_n)_*(\Omega_{A^n/M_{H,1}}^1 \otimes \Omega_{\mathcal{M}_{H,1}/S_1}^1) \cong H^i_{\text{dR}}(A^n/M_{H,1}) \otimes \Omega_{\mathcal{M}_{H,1}/S_1}\), which is the Gauss–Manin connection. If we take the F-filtration on \((4.1),\) we obtain \(0 \to F^{a-1}(\Omega_{A^n/M_{H,1}}^*) \otimes f_n^*(\Omega_{\mathcal{M}_{H,1}/S_1}^1) [-1] \to F^a(\mathcal{K}/\mathcal{K}^2) \to F^a(\Omega_{A^n/S_1}^1) \to 0\) and hence the Griffiths transversality (as in [24, Prop. 1.4.16]).

Since \(A^n \to M_{H,1}\) is an abelian scheme, the Hodge to de Rham spectral sequence \(E^a_{i,j} := R^{i-a}(f_n)_*(\Omega_{A^n/M_{H,1}}^a) \Rightarrow H^i_{\text{dR}}(A^n/M_{H,1})\) degenerates at \(E_1\). (See for example [4, Prop. 2.5.2]). Then \(\text{Gr}_F^a(H^i_{\text{dR}}(A^n/M_{H,1})) \cong R^{i-a}(f_n)_*(\Omega_{A^n/M_{H,1}}^a)\), and we can conclude (as in [24, Prop. 1.4.1.7]) that the induced morphism \(\nabla : \text{Gr}_F^a H^i_{\text{dR}}(A^n/M_{H,1}) \to \text{Gr}_F^{a-1} H^i_{\text{dR}}(A^n/M_{H,1}) \otimes \Omega_{\mathcal{M}_{H,1}/S_1}\) agrees with the morphism \(R^{i-a}(f_n)_*(\Omega_{A^n/M_{H,1}}^a) \to R^{i-a+1}(f_n)_*(\Omega_{A^n/M_{H,1}}^a) \otimes \Omega_{\mathcal{M}_{H,1}/S_1}\) defined by cup product with the Kodaira–Spencer class.

The Koszul filtration gives a spectral sequence

\begin{equation}
E^{a,b}_{1} := R^{a+b}(f_n)_*(\Omega_{A^n/M_{H,1}}^{a+b}) \Rightarrow R^{a+b}(f_n)_*(\Omega_{A^n/M_{H,1}}^a),
\end{equation}

where each \(E^{a,b}_{1}\) can be canonically identified with \(H^a_{\text{dR}}(A^n/M_{H,1}) \otimes \Omega_{\mathcal{M}_{H,1}/S_1}\).

As in [24, (3.2.5)] (with the notation \(K\) here being \(F\) there), the de Rham complex \((H^a_{\text{dR}}(A^n/M_{H,1}) \otimes \Omega_{\mathcal{M}_{H,1}/S_1})\) is the complex \((E^{a,b}_{1}, d^{a,b}_{1})\) in the \(b\)-th row of the
By taking cohomology over $M_{H,1}$, we obtain the Leray spectral sequence (cf. [23 Rem. 3.3])

$$E_2^{a,b} := H^a_{dR}(M_{H,1}/S_1, H^b_{dR}(A^n/M_{H,1})) \Rightarrow H^{a+b}_{dR}(A^n/S_1).$$

(The left-hand side of (4.3) stands for $H^a(M_{H,1}, H^b_{dR}(A^n/M_{H,1}) \otimes \Omega^b_{M_{H,1}/S_1}$, for simplicity.)

For any integer $l$, we denote by $[l]$ the multiplication by $l$ morphism on the abelian scheme $A^n$ over $M_{H,1}$. This lets the algebra $\mathbb{Z}(p)/\mathbb{Z}$ (spanned by the symbols $[l]$) act on the (relative and absolute) cohomology groups of $A^n$. Essential in the Lieberman’s trick is the observation that $[l]$ acts as multiplication by $l^i$ on the relative cohomology $H^i$ of any abelian scheme.

**Proposition 4.4.** Suppose $2d < p$. Then the Leray spectral sequence (4.3) degenerates at $E_2$.

**Proof.** The algebra $\mathbb{Z}(p)/\mathbb{Z}$ acts on the spectral sequence by functoriality. Let $l_0$ be an integer reducing modulo $p$ to a generator of $\mathbb{F}_p$. Then for any pair of integers $i$ and $j$, the integer $l_0^i - l_0^j$ is invertible in $\mathbb{Z}(p)$ unless $i \equiv j \mod p - 1$. For an integer $b_0$ such that $0 \leq b_0 \leq N := 2n \dim(A/M_{H,1})$, put

$$\varepsilon^\deg_{b_0} := \prod_{0 \leq i \leq N, i \neq b_0 \mod p - 1} (l_0^i - l_0^{-1})^{-1} \in \mathbb{Z}(p)/\mathbb{Z}.$$  

It annihilates $E_2^{a,b}$ unless $b \equiv b_0 \mod p - 1$, acts as a unit on $E_2^{a,b}$ when $b \equiv b_0 \mod p - 1$, and acts as 1 on $E_2^{a,b_0}$. Already from the terms on the $E_2$ page of (4.3), we have $E_2^{a,b} = 0$ for all $r \geq 2$, unless $a \in [0, 2d]$ and $b \in [0, N]$. Any differential between terms in two rows of $E_r$ with the vertical distance at least $p - 1$ is zero, since $p - 1 \geq 2d$. With varying $b_0$, we obtain the degeneration of (4.3).

**Remark 4.6.** The degeneration itself is not strictly necessary in the main line of proofs of our results. However, we will make use of the element (4.5).

### 4.2. De Rham cohomology.

**Lemma 4.7.** With the assumptions as in the beginning of Section 4, the application of $(\varepsilon_\mu)^*$ and the Tate twist in Proposition 3.7 gives

$$\nabla \otimes \Omega^*_{M_{H,1}/S_1} \cong (\varepsilon_\mu)^* (H^0_{dR}(A^n/M_{H,1}) \otimes \Omega^*_{M_{H,1}/S_1} \nabla)(-t_\mu)$$

and respects the Hodge filtrations on both sides.

**Proof.** The operator $\varepsilon_\mu$ was defined using the product of certain $R_1$-linear combinations of pullbacks via morphisms between $M_{H,1}$-schemes, the first Chern class of the Poincaré line bundle, the cup product, and the Künneth decomposition. As such, $(\varepsilon_\mu)^*$ is horizontal with respect to the Gauss–Manin connection. The Hodge filtrations are respected because they are so when $V_{[\mu]} \cong L_1$ as in Example 2.10.

**Proposition 4.8.** With the assumptions as in the beginning of Section 4, suppose moreover that $2d < p$. Let $\varepsilon_n^{\deg} \in \mathbb{Z}(p)/\mathbb{Z}$ be defined by (4.5) (with some choice of $l_0$ and with $b_0 = n$). Then we have a canonical isomorphism

$$H^i_{dR}(M_{H,1}/S_1, \nabla_{[\mu]}) \cong (\varepsilon_\mu)^* (\varepsilon_n^{\deg})^* H^{i+n}_{dR}(A^n/S_1)(-t_\mu)$$

for every integer $i$. 

Proof. According to the proof of Proposition 4.4 under the application of $(\epsilon^{\deg})^*$, only the term $F^{i,n}_2$ survives among the terms $F^{a,b}_2$ with $a + b = i + n$ in (4.3). Therefore the result follows from Lemma 4.7. \qed

Remark 4.10. Everything in Sections 4.1, 4.2 remains valid if we base change (horizontally) from $R_1$ to an $R_1$-algebra $R$.  

4.3. Étale and Betti cohomology. Let $F^{ac}_0$ be the algebraic closure of $F_0$ in $\mathbb{C}$. By abuse of notation, we shall write $M_{H,F}^{ac} := M_{H,0} \otimes F^{ac}_0$ and denote by $A^{ac}_F$ the pullback (to $M_{H,F}^{ac}$) of the universal family from $M_{H,0}$, rather than from $M_{H,1}$. Let $f_{n,F} : A^{ac}_F \rightarrow M_{H,F}^{ac}$ denote the structural morphism. We shall use similar notation for pullbacks to $\mathbb{C}$.

Let $\Lambda$ be an integral domain, finite flat over the notation for pullbacks to $\mathbb{C}$. Moreover that $\Lambda$ is a direct summand of $R_1$. Let $\Lambda)$ be any algebraically closed subfield of $\Lambda$.

Remark 4.11. For the same reason as in Remark 3.8, the sheaves $\epsilon_n V^\gamma_{[\mu]}$ (resp. $\eta_n V^\gamma_{[\mu]}$) is a direct summand of $R^n(f_{n,F}^{ac})_*(\Lambda)(-t_{\mu})$ (resp. $R^n(f_{n,F})_*(\Lambda)(-t_{\mu})$).

Proposition 4.12. With the assumptions as in the beginning of Section 4, suppose moreover that $2d < p$. Let $\mathbb{Z}_p$ be as in Proposition 4.8. Then, for any $i$, we have canonical isomorphisms

(4.13) $H^n_{\text{ét}}(M_{H,F}^{ac}, \epsilon_n V^\gamma_{[\mu]}) \cong (\epsilon_n^*)^* H^n_{\text{ét}}(A^{ac}_F, \Lambda)(-t_{\mu})$ and

$H^n_{\text{B}}(M_{H,F}^{ac}, \eta_n V^\gamma_{[\mu]}) \cong (\epsilon_n^*)^* H^n_{\text{B}}(A^{ac}_F, \Lambda)(-t_{\mu})$.

Proof. The same argument as in the proof of Proposition 4.8 using a Leray spectral sequence analogous to (4.3) works here. \qed

Proposition 4.14. Let $K^{ac}$ be any algebraically closed subfield of $\mathbb{C}$ containing $F^{ac}_0$. The embeddings $F^{ac}_0 \hookrightarrow K^{ac} \hookrightarrow \mathbb{C}$ determine canonical isomorphisms $H^n_{\text{ét}}(M_{H,F}^{ac}, \epsilon_n V^\gamma_{[\mu]}) \cong H^n_{\text{ét}}(M_{H,K^{ac}}, \epsilon_n V^\gamma_{[\mu]}) \cong H^n_{\text{B}}(M_{H,K^{ac}}, V^\gamma_{[\mu]})$ for all $i$.

Proof. By [8] Arcata, V. Cor. 3.3, the embeddings between separably closed fields determine canonical isomorphisms $H^n_{\text{ét}}(A^{ac}_F, \Lambda) \cong H^n_{\text{ét}}(A^{ac}_{K^{ac}}, \Lambda) \cong H^n_{\text{B}}(A^{ac}_F, \Lambda)$. By [2] XI, Thm. 4.4, there is a canonical isomorphism $H^n_{\text{ét}}(A^{ac}_F, \Lambda) \cong H^n_{\text{B}}(A^{ac}_F, \Lambda)$. Thus the result follows from Proposition 4.12 by applying $(\epsilon_n^*)^*$ and Tate twists. \qed

Thus Proposition 4.14 relates the Betti cohomology in the Question of the Introduction with the étale cohomology, which might be more interesting because it realizes Galois representations. Moreover, for our purpose, the main technical advantage of the (torsion) étale cohomology is that (with the reduction steps to be introduced in later sections) it can be studied using techniques only available in positive characteristics via $p$-adic comparison theorems.
5. Crystalline comparison isomorphisms

To prove the vanishing and the torsion-freeness of the Betti (or étale) cohomology in the Introduction, we will first prove the corresponding statements for the de Rham (or crystalline) cohomology, and apply the crystalline comparison isomorphism. We will only use the basic case of a projective smooth scheme over an absolutely unramified $p$-adic base ring.

First, let us fix the notation. The structural homomorphism $O_{F_0} \to R_1$ determines a $p$-adic place of $F_0$, and we will denote the completion of $O_{F_0}$ at this place by $W$; recall that $p$ is unramified in $O_{F_0}$, and we will identify $W$ with the ring of Witt vectors of its residue field. By passing to the completions, $W$ embeds canonically into the $p$-adic completion of $R_1$. Let $K := \text{Frac}(W)$, and fix an algebraic closure $K^{ac}$ of $K$. We also fix an isomorphism $\iota : K^{ac} \to \mathbb{C}$ of $F_0$-algebras, and identify $F_0^{ac}$ (under $\iota$) with the algebraic closure of $F_0$ in $K^{ac}$.

We let $M_{H,W} := M_{H,0} \otimes W$ and denote by $A_W$ the pullback (to $M_{H,W}$) of the universal family from $M_{H,0}$ (rather than from $M_{H,1}$). We shall use similar notations for pullbacks to $K$ and $K^{ac}$.

5.1. Constant coefficients. For an integer $s \geq 1$, we write $W_s = W/p^sW$ and use the abelian category $\underline{MF}_\text{tor}^{fr}$ defined in [6, 3.1.1]. For the sake of brevity, we shall refer to an object $(M, (\text{Fil}^a(M))_{0 \leq a \leq r}, (\varphi_a)_{0 \leq a \leq r})$ of $\underline{MF}_\text{tor}^{fr}$ simply by the underlying $W$-module $M$ when there is no ambiguity about additional data.

Let $Z$ be a proper smooth scheme over $W$. For any integer $s \geq 1$, put $Z_s := Z \otimes W_s$. Then [14, II, Cor. 2.7] shows that for $0 \leq j \leq r \leq p - 1$, the de Rham cohomology $H^j(Z_s, \Omega^{*}_{Z_s})$ (with its Hodge filtration and its crystalline Frobenius, which we omit from the notation) defines an object of the category $\underline{MF}_\text{tor}^{fr}$.

Recall $A_{cr} := \lim_{s \to \infty} H^0_{cr}((O_{K^{ac}}/(pO_{K^{ac}}))/W_s)$. (See [6, 3.1.2] or [13, p. 242].)

**Definition 5.1.** (see [14, II, Cor. 2.7].) For an object $M$ of $\underline{MF}_\text{tor}^{fr}$ and an integer $s \geq 1$ such that $p^sM = 0$, we put $T_{cr}^s(M) := \text{Hom}_{W,s_{cr}}^W(M, A_{cr}/p^sA_{cr})$. (We suppress $s$ from the notation since the result is independent of the choice of $s$.) It defines a contravariant functor from $\underline{MF}_\text{tor}^{fr}$ to the category of continuous $\text{Gal}(K^{ac}/K)$-modules. We also define a covariant functor by putting $T_{cr}(M) := T_{cr}^s(M)^{\vee} = \text{Fil}^r(A_{cr} \otimes M)^{\varphi,r=1} = (-r)$.

By [6] Thm. 3.1.3.1, for $0 \leq r \leq p - 2$, the functor $T_{cr}^s$ is fully faithful.

**Theorem 5.2.** (see [6] Thm. 3.2.3, [14, III, 6.3], and [12, Thm. 5.3].) Let $Z$ be a proper smooth scheme over $W$, and let $s$ be an integer $\geq 1$. For $0 \leq j \leq r \leq p - 2$, we have a natural isomorphism $T_{cr}(H^j_{cr}(Z_s/W_s)) \cong H^j_{cr}(Z \otimes W^{ac}/Z/p^sZ)$, compatible with the action of $\text{Gal}(K^{ac}/K)$. The isomorphism is functorial in the proper smooth $W$-scheme $Z$ and is compatible with the cup product structures and with the formation of the Chern classes of line bundles over $Z$.

5.2. Automorphic coefficients. Let $\Lambda$ be an integral domain, finite flat over the $p$-adic completion of $R_1$ (and hence finite flat over $\mathbb{Z}_p$). (See the second paragraph of Section 4.3.) Assume moreover that the set $\Omega := \text{Hom}_{\mathbb{Z}_p\text{-alg}}(W, \Lambda)$ has cardinality
\[ [F_0 : \mathbb{Q}] \], so that there is a natural decomposition

\[(5.3) \quad W \otimes \Lambda \cong \prod_{\sigma \in \Omega} W_{\sigma}, \]

where each \( W_{\sigma} \) is a copy of \( \Lambda \) on which \( W \) acts via \( \sigma : W \to \Lambda \).

Let \( \mu \in X^{<\mathbb{Q},<W}_0 \) with \( n := |\mu|_L \). According to Theorem 5.2 for any integer \( s \geq 1 \) and any \( 0 \leq j \leq p - 2 \), we have a natural isomorphism

\[(5.4) \quad T_{cr}(H^j_{dR}(A^n_{W,s}/W_s)) \cong H^1_{\acute{e}t}(A^K_{ac}, \mathbb{Z}/p^s\mathbb{Z}). \]

Let \( \Lambda_s := \Lambda/p^s\Lambda \), and apply \( \otimes_{\mathbb{Z}/p^s\mathbb{Z}} \) to both sides of (5.4). Then we obtain

\[(5.5) \quad T_{cr}(\bigoplus_{\sigma \in \Omega} H^j_{dR}(A^n_{W,s}/\text{Spec}(W_s)) \otimes \Lambda_s) \cong H^1_{\acute{e}t}(A^K_{ac}, \Lambda_s). \]

By taking reduction modulo \( p^s \) of (5.3), we obtain a similar decomposition

\[(5.6) \quad T_{cr} \left( \bigoplus_{\sigma \in \Omega} H^j_{dR}(A^n_{W,s}/W_{s,s}) \right) \cong H^1_{\acute{e}t}(A^K_{ac}, \Lambda_s). \]

Suppose \( 2d < p \), and \( \max(2, r_\tau) < p \) whenever \( \tau = \tau \circ c \). Let \( \varepsilon_{\mu} = \varepsilon_{\mu^h}^Y \varepsilon_{\mu}^S \varepsilon_{\mu}^L \varepsilon_n^{\text{deg}} \) be defined in Proposition 3.7 and let \( \varepsilon_n^{\text{deg}} \) be defined as in 4.5, with \( b_0 = n \). Then the sheaves \( \varepsilon_{\mu} V^i_{[\mu]} \) and \( H^i_{\acute{e}t}(\varepsilon_{\mu} V^i_{[\mu]}) \) are defined as in Section 4.3 and Propositions 4.8 and 4.12 relate the cohomology of automorphic sheaves to those of the fiber products of \( \Lambda \).

Suppose moreover that \( |\mu|_{\text{re}} < p \). Then Lemmas 3.14 and 3.16 imply that the action of the idempotent \( (\varepsilon_{\mu}^n)^* \) can be achieved by taking cokernels of morphisms from cohomology groups of lower degrees, defined by functoriality and by cup products with Chern classes of line bundles. (We use Lemma 3.16 to ensure that the cokernel of the cohomology of the cokernel of (3.13) is the cokernel of the induced morphism between cohomology groups.) On the other hand, all the actions of \( \varepsilon_{\mu}^Y, \varepsilon_{\mu}^S, \varepsilon_{\mu}^L, \) and \( \varepsilon_n^{\text{deg}} \) involve only functoriality. Therefore, by (4.9) and (4.13), the natural properties satisfied by the comparison isomorphism in Theorem 5.2 imply that

\[(5.7) \quad T_{cr} \left( \bigoplus_{\sigma \in \Omega} H^i_{dR}(M_{\mathcal{H}, W_{s,s}}/S_{W_{s,s}}, \varepsilon_{\mu} V^i_{[\mu]}) \right) \cong H^i_{\acute{e}t}(M_{\mathcal{H}, K^{\text{ac}}}, \varepsilon_{\mu} V^i_{[\mu], \Lambda_s}) \]

for any \( 0 \leq i \leq 2d \) such that \( j = i + n \leq p - 2 \).

**Proposition 5.8.** With the assumptions on \( \mu \) and \( p \) above, if \( H^i_{dR}(\mathcal{M}_{\mathcal{H}_0, \varepsilon_{\mu} V^i_{[\mu], \Lambda_s}}) = 0 \) for some integer \( i \) such that \( i + n \leq p - 2 \), then \( H^i_{\acute{e}t}(M_{\mathcal{H}, K^{\text{ac}}}, \varepsilon_{\mu} V^i_{[\mu], \Lambda_s}) = 0 \) for the same \( i \).

**Proof.** This follows from (5.7) and Proposition 4.14. \( \square \)

**Definition 5.9.** We set \( |\mu|_{\text{comp}} := 2d + n \), called the comparison size of \( \mu \).

**Remark 5.10.** The definition of \( |\mu|_{\text{comp}} \) depends on the comparison theorem we use. Using the crystalline comparison that allows non-constant coefficients, \( |\mu|_{\text{comp}} \) can be made smaller.
6. Illusie’s vanishing theorem

6.1. Statement. We use Illusie’s notation in this subsection, which is somewhat different from ours. As we will rely on the vanishing theorem only in the form of Corollary 6.2 in the next subsection, this should not create any confusion.

Let $k$ be a perfect field of characteristic $p > 0$, and let $(X, D)$ and $(Y, E)$ be pairs of smooth schemes over $k$ endowed with simple normal crossings divisors. Suppose $f : (X, D) \to (Y, E)$ is a proper semistable morphism (see [22, §1]), and consider the relative logarithmic de Rham cohomology sheaves $H^m(f) = R^m f_* (ω_{X/Y}^m)$ for integers $m \geq 0$, equipped with the Hodge filtration and the Gauss–Manin connection (the two satisfying the Griffiths transversality).

**Theorem 6.1** (Illusie; cf. [22, Cor. 4.16]). Assume that $f$ lifts to $\tilde{f}$ over $W_2(k)$ in the obvious sense (see [22, §2]), that $Y$ is proper over $k$ of pure dimension $e$, and that $L$ is an ample line bundle over $Y$. Then, for every integer $m < p - e$, we have

1. $H^{i+j}(Y, L \otimes \text{gr}^i ω_Y(\text{H}^m(f))) = 0$ for $i + j > e$; and
2. $H^{i+j}(Y, L^{-1} \otimes \text{gr}^i ω_Y(\text{H}^m(f))) = 0$ for $i + j < e$.

**Proof.** The assumptions imply that the conclusion of [22, Thm. 4.7] is true, namely

$\oplus_j \text{gr}^j ω_Y(\text{H}_1) \isom F_{Y/k, ω_Y}(H),$

where we abbreviated $H = H^m(f)$, and where the subscript 1 denotes the base change by the absolute Frobenius on $k$. The condition (*) in [22, Thm. 4.7] is verified for $i + j < p$ by [22, Cor. 2.4] in view of our assumptions, and this suffices for the calculations and constructions in [22, §§3–4]. Moreover, the condition $m + e < p$ implies that the subcomplex $G_{p-1}$ is the whole complex.

From this decomposition, we get our first vanishing statement just as Illusie got [22 (4.16.1)], using Serre vanishing.

The second statement is different from (4.16.2) in loc. cit., when $E$ is nonempty. Instead of applying duality, we directly apply the inequality (4.16.3) in loc. cit. to $M = L^{-1}$ repeatedly, and use Serre vanishing for high tensor powers of anti-ample line bundles.

6.2. Application to automorphic bundles. Applying Theorem 6.1 to the Shimura variety and automorphic bundles, we immediately deduce:

**Corollary 6.2.** Suppose $μ ∈ X_{G_1}^{+,<WP}$ with $n := |μ|_L$, and $\max(2, r_τ) < p$ whenever $τ = τ \circ c$. Recall that $d = \dim S(M_{H,1})$. (See Definition 3.9). Suppose moreover that $|μ|_c = d + n < p$. Let $ℒ$ be an ample line bundle over $M_{H,1}$. Let $ℒ_{κ_1} := ℒ \otimes κ_1$.

Then we have:

1. $H^i(M_{H,κ_1}, ℒ_{κ_1} \otimes \text{Gr}_{\ell} (V_{μ|_κ,κ_1} \otimes Ω_{M_{H,κ_1}/S_{κ_1}}^\bullet)) = 0$ for every $i > d$.
2. $H^i(M_{H,κ_1}, ℒ_{κ_1} \otimes \text{Gr}_{\ell} (V_{μ|_κ,κ_1} \otimes Ω_{M_{H,κ_1}/S_{κ_1}}^\bullet)) = 0$ for every $i < d$.

**Definition 6.3.** We say $μ ∈ X_{G_1}^{+,<WP}$ is $p$-small for Illusie’s theorem if $|μ|_c = d + |μ|_L < p$. (See Definition 3.9). We write in this case that $μ ∈ X_{G_1}^{+,<re-p}$. 

6.3. Reformulations using dual BGG complexes. For any $\nu \in X_+^{+,<p}$ (as in Definition 2.29), and for any $R_1$-algebra $R$, we define $W_{\omega,R} := \mathcal{E}_{M_1,R}(W_{\omega,R})$ (see Lemma 2.20). For any $\mu \in X_+^{+,<p}$ and any $w \in M_1^\nu$, we define $W_{\nu,[\mu],R} := \bigoplus_{t \leq w[\mu]} W_{\nu,R}$, and define $W_{\nu,[\mu],R}$ and $W_{\nu,[\mu],R}$ in the similar, obvious way.

For any integer $a \geq 0$, we denote by $W_{M_1}(a)$ the elements $w$ in $W_{M_1}$ with length $l(w) = a$.

**Theorem 6.4 (Faltings; cf. 111 §3, 13 Ch. VI, §5, and 37 §5).** Let $R$ be any $R_1$-algebra. For any $\mu \in X_+^{+,<p}$, there is an $F$-filtered complex $BGG^\bullet(V_{[\nu],R}^\vee)$, with trivial differentials on $F$-graded pieces, such that

$$Gr^F(\text{BGG}^\bullet(V_{[\nu],R}^\vee)) \cong \bigoplus_{w \in W_{M_1}(a)} W_{[\nu],R}$$

as $\mathcal{O}_{M_{H,R}}$-modules, together with a canonical quasi-isomorphic embedding

$$Gr^F(\text{BGG}^\bullet(V_{[\nu],R}^\vee)) \hookrightarrow Gr^F(V_{[\nu],R}^\vee \otimes \Omega_{M_{H,R}}^{\bullet}(M_{H,R}/S_{R}))$$

(of complexes of $\mathcal{O}_{M_{H,R}}$-modules) between $F$-graded pieces.

If $G_1$ has no type D factors, then this is well known. The same method in 13 Ch. VI, §5 and 37 §5, using 42 Thm. D as the main representation-theoretic input, carries over with little modification. However, after consulting Patrick Polo and after checking the details more carefully, we realize that the method involves only the (compatible) actions of $P_1$ and Lie($G_1$) (cf. Lemma 2.14), and that, if one use a simple variant of 42 Thm. A instead of 42 Thm. D, the method also works when $G_1$ has type D factors. For more detailed explanations, see 30.

**Corollary 6.5.** For any $\mu \in X_+^{+,<p}$ and any $R_1$-algebra $R$,

$$H^i(M_{H,R},Gr^F(V_{[\nu],R}^\vee \otimes \Omega_{M_{H,R}}^{\bullet}(M_{H,R}/S_{R})) \cong \bigoplus_{w \in W_{M_1}} H^{i-l(w)}(M_{H,R},W_{[\nu],R}^\vee).$$

Combining Corollary 6.2 and Theorem 6.4, we obtain:

**Corollary 6.7.** Suppose $\mu \in X_+^{+,<p}$ (see Definition 6.3), and $\max(2,\tau_r) < p$ whenever $\tau = \tau \circ c$. Let $L$ be an ample line bundle over $M_{H,1}$. Let $L_{\kappa_1} := L \otimes \kappa_1$.

Then, for any $w \in W_{M_1}$, we have:

1. $H^{i-l(w)}(M_{H,\kappa_1},L_{\kappa_1} \otimes W_{[\nu],R}^\vee) = 0$ for $i > d$.

2. $H^{i-l(w)}(M_{H,\kappa_1},L_{\kappa_1} \otimes W_{[\nu],R}^\vee) = 0$ for $i < d$.

Clearly, Corollary 6.7 will be more useful if $L$ is an automorphic bundle (in the sense of Definition 1.16). We shall investigate this possibility in Section 7.

7. Ample Automorphic Line Bundles

**Definition 7.1.** Any weight $\nu \in X_+^{+,<p}$ such that $W_{\nu}$ is a rank one free $R_1$-module is called a generalized parallel weight. We say in this case that $W_{\nu}$ is an automorphic line bundle. For simplicity, we say $\nu$ is positive if the associated automorphic line bundle $W_{\nu}$ is ample over $M_{H,1}$.
According to [2.8], we have $M_1 \cong \left( \prod_{\tau \in \mathcal{Y}/c} M_\tau \right) \times (G_m \otimes R_1)$, with two possibilities for the factors $M_\tau$:

1. If $\tau = \tau \circ c$, then $M_\tau \cong \text{GL}_{p_\tau} \otimes R_1 = \text{GL}_{r_\tau} \otimes R_1$.

2. If $\tau \neq \tau \circ c$, then $M_\tau \cong (\text{GL}_{p_\tau} \times \text{GL}_{q_\tau}) \otimes R_1$.

This shows that:

**Lemma 7.2.** The generalized parallel weights $\nu$ in $X_{M_1}^{+,<p}$ are exactly those $\nu = \left( (\nu_\tau)_{\tau \in \mathcal{Y}/c}; \nu_0 \right) = \left( ((\nu_{\tau,i})_{1 \leq i \leq r_\tau})_{\tau \in \mathcal{Y}/c}; \nu_0 \right)$ satisfying the following conditions:

1. If $\tau = \tau \circ c$, then $\nu_\tau = k_\tau(1,1,\ldots,1)$, where $k_\tau \in \mathbb{Z}$.

2. If $\tau \neq \tau \circ c$, then $\nu_\tau = k_\tau(1,1,\ldots,1,0,0,\ldots,0) - k_{\tau \circ c}(0,0,\ldots,0,1,1,\ldots,1)$, where $k_\tau, k_{\tau \circ c} \in \mathbb{Z}$, so that the value of $\nu_\tau$ and $\nu_{\tau \circ c}$ are independent of the choice of representatives in $\mathcal{Y}/c$.

(There are no restrictions on the sizes of $k_\tau$ or $k_{\tau \circ c}$. Weights $\nu$ of the above form are all $p$-small.)

**Definition 7.3.** The integers $(k_\tau)_{\tau \in \mathcal{Y}}$ in Lemma 7.2 are called the coefficients of the generalized parallel weight $\nu$.

An important feature of a generalized parallel weight is that $W_\nu \otimes W'_\mu \cong W_{\mu - \nu}$ and $W'_\nu \otimes W_\mu \cong W'_{\mu + \nu}$ for any $\mu \in X_{M_1}^{+,<p}$. (Adding or subtracting a generalized parallel weight does not affect $p$-smallness of a weight in $X_{M_1}^+$.) Therefore, tensoring with an automorphic line bundle simply shifts the weight of an automorphic vector bundle.

Corollary 6.7 implies in particular that:

**Corollary 7.4.** Suppose $\mu \in X_{G_1}^{+,<r_\tau p}$, and $\text{max}(2,r_\tau) < p$ whenever $\tau = \tau \circ c$. Suppose $w \in W_{M_1}$, and $\nu \in X_{M_1}^{+,<p}$ is a positive generalized parallel weight. Then we have:

1. $H^{i-\langle w \rangle}(M_{\mathcal{H},k_\tau}, W'_{w-[\mu]-\nu,k_\tau}) = 0$ for every $i > d$.

2. $H^{i-\langle w \rangle}(M_{\mathcal{H},k_\tau}, W'_{w+[\mu]+\nu,k_\tau}) = 0$ for every $i < d$.

Changing our perspective a little bit:

**Corollary 7.5.** Suppose $\mu \in X_{G_1}^{+,<r_\tau p}$, $w \in W_{M_1}$, and $\text{max}(2,r_\tau) < p$ whenever $\tau = \tau \circ c$. Suppose that, for each $\mu' \in [\mu]$, there exist positive generalized parallel weights $\nu_+, \nu_- \in X_{M_1}^{+,<p}$ such that the condition $\mu' \pm w^{-1}(\nu_{\pm}) \in X_{G_1}^{+,<r_\tau p}$ is satisfied. (The choices of $\nu_{\pm}$ may depend on $\mu'$.) Then $H^{i-\langle w \rangle}(M_{\mathcal{H},k_\tau}, W'_{w-[\mu]-\nu,k_\tau}) = 0$ for every $i \neq d$.

Combining Corollaries 6.5 and 7.3, we obtain:

**Theorem 7.6.** Suppose $\mu \in X_{G_1}^{+,<r_\tau p}$, and $\text{max}(2,r_\tau) < p$ whenever $\tau = \tau \circ c$. Suppose that, for each $w \in W_{M_1}$ and each $\mu' \in [\mu]$, there exist positive generalized parallel weights $\nu_+, \nu_- \in X_{M_1}^{+,<p}$ such that the condition $\mu' \pm w^{-1}(\nu_{\pm}) \in X_{G_1}^{+,<r_\tau p}$ is satisfied. Then we have $H^i(M_{\mathcal{H},k_\tau}, \text{Gr}((V_{[\mu],k_\tau}^\vee \otimes \Omega^{\bullet}_{M_{\mathcal{H},k_\tau}/S_{k_\tau}})) = 0$ and $H^i_{\text{dR}}(M_{\mathcal{H},k_\tau}/S_{k_\tau}, V_{[\mu],k_\tau}^\vee) = 0$ for every $i \neq d$. 


Proof: The first statement follows from Corollaries 6.5 and 7.5. The second statement then follows from the Hodge to de Rham spectral sequence
\[
E_1^{a,b} := H^{a+b}(\mathcal{M}_{\mathcal{H},1}, \text{Gr}_F^a(\mathcal{L}_{\nu}^\vee \otimes \Omega^*_{\mathcal{H},1/S_1})) \Rightarrow H^{a+b}_{\text{dR}}(\mathcal{M}_{\mathcal{H},1}/S_1, \mathcal{L}_{\nu}^\vee)
\]
associated with the hypercohomology of filtered complexes. 

7.2. Ampleness. The most well-known (and perhaps the only known) way to produce ample automorphic line bundles is to use variants of the Hodge line bundle:

**Proposition 7.8.** The line bundle \( \omega := \wedge^{\text{top}} \text{Lie}_{\mathcal{A}/\mathcal{H}}^\vee \) is ample over \( \mathcal{M}_{\mathcal{H}} \).

Proof. For the case of Siegel moduli schemes with principal levels at least 3, this is recorded in [38 IX, Thm. 3.1; cf. VII, Def. 4.3.3]. The case for \( \mathcal{M}_{\mathcal{H}} \) bundles, we may assume that there exists some finite forgetful morphism (defined by the universal polarized abelian scheme) from \( \mathcal{M}_{\mathcal{H}} \) to a Siegel moduli scheme with principal level at least 3. The second way is to refer to [29, Thm. 7.2.4.1] (following and generalizing [13 Thm. 2.5]).

**Lemma 7.9.** The line bundle \( \omega \) is isomorphic to \( W_\nu \) with coefficients \( (k_\nu)_{\nu \in \Upsilon} \) of \( \nu \) satisfying \( k_\nu = \text{rk}_{R_1}(V_\nu) \). (See Section 2.1 for the definition of \( V_\nu \).)

Proof. This is because \( \text{Lie}_{\mathcal{A}/\mathcal{H}}^\vee \cong \text{Lie}_{\mathcal{L}/\mathcal{H}}^\vee \cong \mathcal{E}_{\mathcal{M}_1,R}(\mathcal{L}_{0,1}^\vee) \) as vector bundles over \( \mathcal{M}_{\mathcal{H},1} \) (ignoring Tate twists). (See Definition 1.13 and Example 1.22).

**Proposition 7.10** (Correction of the originally published version). An automorphic line bundle \( W_\nu \) defines a torsion element in the Picard group of \( \mathcal{M}_{\mathcal{H},1} \) if its coefficients \( (k_\nu)_{\nu \in \Upsilon} \) of \( \nu \) satisfy the condition that \( k_\nu + k_{\nu \in \Upsilon} = 0 \) for all \( \nu \in \Upsilon \).

Proof. Suppose that the condition in the proposition holds. Then the representation \( W_\nu \) is trivial after pullback to the complexification of the maximal compact subgroup of \( \text{G}(\mathbb{R}) \), and hence the pullback \( W_{\nu,C} \) of \( W_\nu \) under any ring homomorphism \( R_1 \to \mathbb{C} \) is trivial, by the comparison in [27 §5.2]. Suppose \( R \) is any discrete valuation ring finite flat over \( R_1 \) such that \( K := \text{Frac}(R) \) is Galois over \( K_1 := \text{Frac}(R_1) \), and such that the connected components of \( \mathcal{M}_{\mathcal{H},K} = \mathcal{M}_{\mathcal{H},1} \otimes K \) are geometrically connected. Let \( k \) and \( \varpi \) denote the residue field and uniformizer of \( R \), respectively. Let \( M \) to be any connected component of \( \mathcal{M}_{\mathcal{H},1} \otimes R \), and let \( W \) denote the pullback of \( W_{\nu,C} \) to \( M \). By taking norms with respect to the action of \( \text{Gal}(K/K_1) \), it suffices to show that \( W \) is trivial. Since the structural morphism \( \mathcal{M}_{\mathcal{H}} \to S_0 = \text{Spec}(\mathcal{O}_{\mathcal{H},0}(p)) \) is proper and smooth, all fibers of \( M \to \text{Spec}(R) \) are geometrically integral, so that \( H^0(M, \mathcal{O}_M) \cong R \). Since \( W_{\nu,C} \) is trivial, both \( H^0(M, W) \) and \( H^0(M, W_{\nu}) \) are nonzero. Suppose \( s \) and \( t \) are nonzero elements of these two groups, respectively, whose product \( st \) defines an element of \( H^0(M, \mathcal{O}_M) \cong R \). Let \( V(s) \) (resp. \( V(t) \)) denote the closed subsets of \( M \) where the morphism \( \mathcal{O}_M \to W \) (resp. \( W \to \mathcal{O}_M \)) defined by \( s \) (resp. \( t \)) fails to be an isomorphism. Suppose \( st = \varpi t \) for some \( \nu \in R \), so that \( \varpi k \subseteq V(s) \cup V(t) \). Since \( \varpi k \) is integral, either \( M \otimes k \subseteq V(s) \) and \( s = \varpi s' \) for some \( s' \in H^0(M, W) \), or \( M \otimes k \subseteq V(t) \) and \( t = \varpi t' \) for some \( t' \in H^0(M, W_{\nu}) \). Up to replacing \( s \) with \( s' \) or \( t \) with \( t' \), and by repeating this process, we may assume that \( st \in R^X \), in which case \( V(s) = 0 = V(t) \), and so \( W \) is trivial, as desired.
**Definition 7.11.** We say our linear algebraic data \((\mathcal{O}, *, L, \langle \cdot, \cdot \rangle, h_0)\) is \(\mathbb{Q}\)-simple if \(F\) is a field, or equivalently if \(\mathcal{O} \otimes \mathbb{Q}\) is a simple algebra.

**Definition 7.12.** We say that two elements \(\tau, \tau' : \mathcal{O}_F \to R_1\) in \(\Upsilon = \text{Hom}_{\mathbb{Z}_{\text{alg}}}(\mathcal{O}_F, R_1)\) are equivalent over \(\mathbb{Q}\), and write \(\tau \sim_\mathbb{Q} \tau'\), if they factor through the same simple factor of \(\mathcal{O}_F\). The equivalence class containing \(\tau\) is denoted by \([\tau]_\mathbb{Q}\).

Then our linear algebraic data is simple if and only if \(\Upsilon\) has a single equivalence class under \(\sim_\mathbb{Q}\).

**Lemma 7.13.** If our linear algebraic data is simple, then \(\text{rk}_{R_1}(V_{\tau})\) is a constant independent of \(\tau \in \Upsilon\).

**Proof.** Since we assumed that \(\mathcal{O}_F\) is split over \(R_1\), if our linear algebraic data is simple, then \(\mathcal{O}_*\) is abstractly the same algebra over \(R_1\) for all \(\tau \in \Upsilon\). Hence \(\text{rk}_{R_1}(V_{\tau})\) is a constant independent of \(\tau\), as desired. \(\square\)

**Definition 7.14.** We say that a generalized parallel weight \(\nu\) with coefficients \((k_\tau)_\tau \in \Upsilon\) is parallel if \([k]_\tau := k_\tau + k_{\tau^0}\) satisfies \([k]_\tau = [k]_{\tau'}\) whenever \(\tau \sim_\mathbb{Q} \tau'\).

**Proposition 7.15.** Let \(\nu\) be a generalized parallel weight with coefficients \((k_\tau)_\tau \in \Upsilon\). Then the automorphic line bundle \(W_\nu\) over \(\mathcal{M}_{\mathcal{H}, 1}\) is ample if it is parallel (as in Definition 7.14), and if all the numbers \([k]_\tau\) are positive.

**Proof.** By decomposing \(F\) into simple factors over \(\mathbb{Q}\), by decomposing our linear algebraic data accordingly, and by replacing \(\mathcal{H}\) with a finite index subgroup (which is harmless as in the proof of Proposition 7.8), we may assume that there exists a finite morphism from \(\mathcal{M}_{\mathcal{H}, 0}\) to a product of (base changes from possibly smaller rings of) analogous moduli problems defined by simple linear algebraic data. Since the conditions we listed respect this decomposition, we may assume that our moduli problem is defined by a simple linear algebraic data. By Proposition 7.8 Lemma 7.9 and Lemma 7.13 we know that an automorphic line bundle with coefficients \((k_\tau)_\tau \in \Upsilon\) is ample when \(k_\tau\) is positive and independent of \(\tau \in \Upsilon\). Then the result follows from Proposition 7.10. \(\square\)

### 7.3. Positive parallel weights of minimal size.

For each \(\tau \in \Upsilon\), let \(d_\tau := \dim_{R_1}(G_\tau) - \dim_{R_1}(P_\tau)\), and let \(d_{[\tau]_\mathbb{Q}} := \max_{\tau' \in [\tau]_\mathbb{Q}} (d_{\tau'})\). Note that \(d_{[\tau]_\mathbb{Q}} = d_{\tau}\) whenever \(\tau = \tau \circ c\).

**Definition 7.16.** We say that a parallel weight \(\nu \in X_{\Phi_0^{+, \leq p}}\) (as in Definition 7.14) is positive of minimal size if its coefficients \((k_\tau)_\tau \in \Upsilon\) satisfy the following conditions:

1. If \(d_{[\tau]_\mathbb{Q}} = 0\), then \(k_\tau = 0\).
2. If \(d_{[\tau]_\mathbb{Q}} > 0\) and \(\tau = \tau \circ c\), then \(k_\tau = 1\).
3. If \(d_{[\tau]_\mathbb{Q}} > 0\) and \(\tau \neq \tau \circ c\), then \((k_\tau, k_{\tau^0})\) is either \((1, 0)\) or \((0, 1)\).

Using (2.5), we can say if a root \(\alpha \in \Phi_0^{+, \leq p}\) comes from \(G_\tau\) for some \(\tau \in \Upsilon/c\).

**Proposition 7.17.** Suppose \(\mu \in X_{G_\tau}^{+, \leq p}\), and suppose \(\nu \in X_{\Phi_0^{+, \leq p}}\) is parallel and positive of minimal size as in Definition 7.16. Then the condition \(\mu' \pm w^{-1}(\nu) \in X_{\Phi_0^{+, \leq p}}\) is satisfied for every \(\mu' \in [\mu]\) and \(w \in W^{-1}\), if the following conditions are satisfied for all \(\alpha \in \Phi_0^{+, \leq p}\).
(1) If \( \alpha \) comes from \( G_\tau \) such that \( \tau = \tau \circ c \), then \((\mu', \alpha') \geq \min(|\alpha'|, d_{[\tau]_0})\).

(Here the norm \(|\alpha'|\) defined by the Killing form is at most 2.)

(2) If \( \alpha \) comes from \( G_\tau \) such that \( \tau \neq \tau \circ c \), then \((\mu', \alpha') \geq \min(1, d_{[\tau]_0})\).

Proof. If \( \alpha \) comes from \( G_\tau \), then \((\mu', \alpha') = (\mu'_\tau, \alpha')\). If \( \tau = \tau \circ c \), then \((w^{-1}(\nu), \alpha')\) has entries either \( \pm 1 \) or 0. Hence \(|(w^{-1}(\nu), \alpha')|\) is at most \(\min(|\alpha'|, d_{[\tau]_0})\) for \( \alpha \in \Phi^+_G \). If \( \tau \neq \tau \circ c \), then \((w^{-1}(\nu), \alpha')\) has entries either 0 or 1. Hence \(|(w^{-1}(\nu), \alpha')|\) is at most \(\min(1, d_{[\tau]_0})\) for \( \alpha \in \Phi^+_G \).

Definition 7.17. We say that \( \mu \in X^+_{G_1} \) is sufficiently regular if it satisfies the conditions (1) and (2) in Proposition 7.14. We shall denote the set of sufficiently regular elements in \( X^+_{G_1} \) (resp. \( X^{+,<p}_{G_1} \)) by \( X_{G_1}^\circ \) (resp. \( X_{G_1}^{+,<p} \)).

Remark 7.19. If \( \tau \neq \tau \circ c \) for all \( \tau \in \Upsilon \), which implies that \( G_1 \) has only A factors, then being regular implies being sufficiently regular.

Lemma 7.20. Suppose that a weight \( \mu \in X^{+,<p}_{G_1} \) satisfies \(|\mu|_{re} \leq p - \min(2, d)\), and that \( \nu \in X^{+,<p}_{M_1} \) is a positive parallel weight of minimal size. Then \((\mu' \pm w^{-1}(\nu) + \rho, \alpha') \leq p \) for any \( \nu \in W_{M_1} \), \( \mu' \in [\mu] \), and any \( \alpha \in \Phi^+_G \).

Proof. By Definition 3.3, \(|\mu|_{re} = d + |\mu|_L \). By Definition 3.2, \(|\mu|_L = \sum_{\tau \in \Upsilon/c} |\mu'_\tau|\), where \( \mu'_\tau \) means \( \mu'_\tau \) in Section 3.3 (see in particular the explanation in Definition 3.2); we modified the notation here simply to avoid a conflict with the \( \mu' \in [\mu] \) in the statement of this lemma. Since \( d = \sum_{\tau \in \Upsilon/c} d_{r_\tau} \) with \( d_{r_\tau} = \dim R_\tau(G_\tau) - \dim R_\tau(P_\tau) \), it suffices to prove the inequalities for each individual \( \tau \)-factor.

If \( \tau = \tau \circ c \), then \( \mu'_{\tau,r} = \mu'_{\tau,r} \geq \mu'_{\tau,\tau+1} = |\mu'_{\tau,\tau+1}| \geq 0 \) for every \( 1 \leq \tau < r_\tau \). The condition \(|\mu|_{re} \leq p - \min(2, d)\) implies that \( d_{r_\tau} + |\mu|_L = d_{r_\tau} + \sum_{1 \leq \tau \leq r_\tau} |\mu'_{\tau}| \leq d_{r_\tau} + \sum_{1 \leq \tau \leq r_\tau} |\mu'_{\tau}| \leq p - \min(2, d) - d_{r_\tau} \), and hence \((\mu' + \rho, \alpha') \leq p - \min(2, d)\) for any \( \alpha \in \Phi^+_G \), because \((\rho, \alpha') \leq d_{r_\tau}\). Then the result is true because \(|(w^{-1}(\nu), \alpha)| \leq \min(|\alpha'|, d_{[\tau]_0}) \leq \min(2, d)\). (We use sufficient regularity of \( \mu \) when \( \tau = \tau \circ c \) only to make sure that the condition \((\mu' \pm w^{-1}(\nu) \in X^+_{G_1}\) is satisfied for every \( \nu \in W_{M_1}\).)

If \( \tau \neq \tau \circ c \), then the sufficient regularity of \( \mu' \) implies \( |\mu'_{\tau}| \geq 0 \) is a strictly decreasing sequence of integers for \( 1 \leq \tau \leq r_\tau \) except when \( d_{[\tau]_0} = 0 \). We may assume that \( d_{[\tau]_0} \neq 0 \), because otherwise \( \nu = 0 \) by Definition 7.16 in which case there is nothing to prove. Since \(|\mu|_{re} = d + |\mu|_L < p \) and \( \mu'_{\tau} \) is strictly decreasing (because \( d_{[\tau]_0} \neq 0 \)), we have, for any \( 1 \leq a < b \leq r_\tau \), \((\mu'_{\tau} - \mu'_{\tau} + \frac{1}{2} (b-a)(b-a-1) \leq \sum_{a \leq \tau \leq b} |\mu|_{L} \leq p - 1 - d_{r_\tau} \), and hence \((\mu'_{\tau} - \mu'_{\tau} + (b-a) \leq p - d \). This implies that \((\mu' + \rho, \alpha') \leq p - d \) for any \( \alpha \in \Phi^+_G \). Then the result follows from \(|(w^{-1}(\nu), \alpha)| \leq \min(1, d_{[\tau]_0}) \leq \min(1, d)| \).
Proposition 7.21. Suppose that a weight \( \mu \in X_{G_1}^{+,<p} \) satisfies the condition
\[
|\mu|_{re,+} := |\mu|_{re} + \sum_{\tau \in \mathcal{T}/c} \min(1, d_\tau) \max(p_\tau, q_\tau) < p.
\]
Then \( \mu \) belongs to \( X_{G_1}^{+,<wp} \) and satisfies the condition in Theorem 7.6: that is, for each \( w \in W^{M_1} \) and each \( \mu' \in [\mu] \), there exist positive parallel weights \( \nu_+, \nu_- \in X_{M_1}^{+,<p} \) such that the condition \( \mu' \pm w^{-1}(\nu_\pm) \in X_{G_1}^{+,<wp} \) is satisfied.

Proof. Under the condition (7.22), we claim that, for each \( w \in W^{M_1} \) and each \( \mu' \in [\mu] \), there exist positive parallel weights \( \nu_+ \) and \( \nu_- \) of minimal size such that:

1. \( |\mu' \pm w^{-1}(\nu_\pm) + \rho, \alpha^\vee| \leq p \) for all \( \alpha \in X_{G_1}^+ \).
2. \( |\mu'|_{re} < p \) and \( |\mu' \pm w^{-1}(\nu_\pm) - \nu_\pm w^{-1}(\nu_\pm)|_{re} < p \).
3. \( |\mu'|_{L} < p \) and \( |\mu' \pm w^{-1}(\nu_\pm)\nu_\pm w^{-1}(\nu_\pm)|_{L} < p \).

Since \( p_\tau \) cannot be both zero when \( d_\tau \geq 1 \), the condition \( |\mu'|_{re,+} < p \) implies \( |\mu'|_{re} < p - \min(2, d) \). Hence (1) follows from Lemma 7.20. Moreover, (3) follows from (2) because \( |\mu'|_{re} = d + |\mu'|_{L} \).

Let us verify (2) by bounding \( |\mu' \pm w^{-1}(\nu_\pm)|_{L} - |\mu'|_{L} \). As always, it suffices to prove the inequalities for each individual \( \tau \)-factor. We may assume that \( d_\tau|_{\hat{S}} > 0 \), because otherwise \( \nu_{\pm,\tau} = 0 \). If \( \tau = \tau \circ c \), then \( p_\tau = q_\tau = r_\tau \) and \( |\mu' \pm w^{-1}(\nu_\pm,\tau)|_{L} \leq |\mu'_{\tau,L} + r_\tau| \), because \( r_\tau \) entries in \( \mu'_{\tau} \) are added or subtracted by 1. If \( \tau \neq \tau \circ c \), then the definition of \( |\cdot|_{L} \) (in Section 3.3) depends on the parity of the last entry. Since the two choices of positive parallel weights of minimal size have disjoint nonzero entries, we can choose \( \nu_{\pm,\tau} \) such that \( \mu'_{\tau} \pm w^{-1}(\nu_{\pm,\tau}) \) have the same last entry as \( \mu'_{\tau} \). Therefore, in the calculation of \( |\mu'_{\tau} \pm w^{-1}(\nu_{\pm,\tau})|_{L} \) and \( |\mu'_{\tau,L} |_{L} \), at most \( \max(p_\tau, q_\tau) \) entries in \( \mu'_{\tau} \) are added or subtracted by 1. Hence \( |\mu' \pm w^{-1}(\nu_{\pm,\tau})|_{L} \leq |\mu'_{\tau,L} |_{L} + \max(p_\tau, q_\tau) \), as desired. \( \square \)

Remark 7.23. Although the number \( \sum_{\tau \in \mathcal{T}/c} \min(1, d_\tau) \max(p_\tau, q_\tau) \) in (7.22) can be large, it depends only on the real group \( G \otimes \mathbb{R} \).

Lemma 7.24. Suppose that \( \mu \in X_{G_1}^{+,<p} \) satisfies the condition (7.22). Then \( \max(2, r_\tau) < p \) whenever \( \tau = \tau \circ c \).

Proof. If \( G_\tau \cong \text{Sp}_{2r_\tau} \otimes \mathbb{Z} \), then \( \max(2, r_\tau) \leq \frac{1}{2} r_\tau(r_\tau + 1) + r_\tau = d_\tau + r_\tau \leq |\mu|_{re,+} \).

If \( G_\tau \cong \text{O}_{2r_\tau} \otimes \mathbb{Z} \), then \( r_\tau \leq \frac{1}{2} r_\tau(r_\tau + 1) = d_\tau + r_\tau \leq |\mu|_{re,+} \) unless \( d_\tau = 0 \), in which case \( r_\tau < 2 \). On the other hand, \( 2 < p \) because we assume (see Section 1.1) that \( p \neq 2 \) if \( G \otimes \mathbb{Q} \) involves simple factors of type D (as in [29] Def. 1.2.1.15]). Hence \( \max(2, r_\tau) < p \) in all cases. \( \square \)

8. Main results and consequences

8.1. De Rham and Hodge cohomology

Theorem 8.1. Suppose \( \mu \in X_{G_1}^{+,<p} \) satisfies \( |\mu|_{re,+} < p \). (See Definition 7.18 and (7.22).) Then, for every \( i \neq d \), \( H^i(M_{H,\kappa,1} \otimes \Omega^*_{M\kappa,\kappa_1} / S_{\kappa_1}) \cong \bigoplus_{w \in W^{M_1}} H^{i-\ell(w)}(M_{H,\kappa_1}, \mathcal{W}^w_{\mu,\kappa_1}) = 0 \) and \( H^i_{dR}(M_{H,\kappa_1} / S_{\kappa_1}, \mathcal{V}^w_{\mu,\kappa_1}) = 0 \).
Proof. This follows from Theorem \ref{thm:7.6} (and its proof using Corollaries \ref{cor:6.5} and \ref{cor:7.5}), Proposition \ref{prop:7.21} and Lemma \ref{lem:7.24}.

Theorem 8.2. With the assumptions of Theorem 8.1, the Hodge to de Rham spectral sequence (7.7) degenerates at $E_1$ and defines the Hodge decomposition
\[
\text{Gr}_F(H^i_{\text{dR}}(M_{H,R}/S_R, V^\vee_{[\mu],R})) \cong \bigoplus_{w \in W_{\mathfrak{M}}} H^{i-l(w)}(M_{H,R}, W^w_{\mathfrak{O}[\mu],R})
\]
for any $R_1$-algebra $R$. The two sides are zero unless $i = d$, and each summand $H^{i-l(w)}(M_{H,R}, W^w_{\mathfrak{O}[\mu],R})$ on the right-hand side is a free $R$-module of finite rank that surjects onto $H^{i-l(w)}(M_{H,R}, W^w_{\mathfrak{O}[\mu],\kappa_R})$, where $\kappa_R := R \otimes \kappa_1$, under the canonical homomorphism $R_1 \to \kappa_1$ given by reduction modulo $p$.

Proof. Let us begin with the case $R = R_1$. Then (6.6) gives a decomposition
\[
H^i(M_{H,1}, \text{Gr}_F(V^\vee_{[\mu]} \otimes \mathcal{O}_{M_{H,1}/S_1})) \cong \bigoplus_{w \in W_{\mathfrak{M}}} H^{i-l(w)}(M_{H,1}, W^w_{\mathfrak{O}[\mu]}).
\]
Because $M_{H,1} \to S_1$ is proper and flat, and because the sheaves $W^w_{\mathfrak{O}[\mu]}$ are locally free, the upper semi-continuity of dimensions of cohomology (cf. [39, §5, Cor. (a)]) and Theorem 8.1 show that the summands $H^{i-l(w)}(M_{H,1}, W^w_{\mathfrak{O}[\mu]})$ on the right-hand side are zero unless $i = d$. A similar semi-continuity argument (cf. [39, §5, Cor. 2]) proves that these summands are free and that they surject onto the similar cohomology groups over $\kappa_1$ when $i = d$. All the cohomology groups being free over $R_1$, these statements remain true after base change from $R_1$ to any $R_1$-algebra $R$. Finally, the degeneration of (7.7) is trivial because $E_{1,a,b} = 0$ whenever $a + b \neq d$.

Corollary 8.3. With the assumptions of Theorem 8.1, the following are true for any $R_1$-algebra $R$:

1. $H^i_{\text{dR}}(M_{H,R}/S_R, V^\vee_{[\mu],R}) = 0$ for every $i \neq d$.
2. $H^i_{\text{dR}}(M_{H,R}/S_R, V^\vee_{[\mu],R})$ is a free $R$-module of finite rank.
3. The tensor product of the de Rham complex of $V^\vee_{[\mu]}$ with the canonical short exact sequence $0 \to pR \to R \to \kappa_R = R \otimes \kappa_1 \to 0$ induces an exact sequence
\[
0 \to H^d_{\text{dR}}(M_{H,R}/S_R, p(V^\vee_{[\mu],R})) \to H^d_{\text{dR}}(M_{H,R}/S_R, V^\vee_{[\mu],R}) \to H^d_{\text{dR}}(M_{H,\kappa_R}/S_{\kappa_1}, V^\vee_{[\mu],\kappa_R}) \to 0.
\]

Proof. By [23, Thm. 8.0], it suffices to treat the case $R = R_1$. We have already seen (1) in Theorem 8.2, but here is another argument to prove it and the other two statements. Since all terms in the long exact sequence associated with the short exact sequence $0 \to pR \to R \to \kappa_R = R \otimes \kappa_1 \to 0$ are finitely generated $R_1$-modules, and since $H^i_{\text{dR}}(M_{H,\kappa_1}/S_{\kappa_1}, V^\vee_{[\mu],\kappa_1}) = 0$ for all $i \neq d$ by Theorem 8.1, we obtain (1) by Nakayama’s lemma. Then (2) and (3) follow tautologically.

8.2. Cohomological automorphic forms. Let $w_0$ be the unique Weyl element in $W_{\mathfrak{M}}$, such that $w_0 \Phi_{\mathfrak{M}_1} = \Phi_{\mathfrak{M}_1}$ and $W_\nu \cong W^\vee_{-w_0(\nu)}$ for any $\nu \in X^+_{\mathfrak{M}_1}$. We define $X^+_{\mathfrak{M}_1}$ to be the set of all $\nu$ such that $\nu = w_0(\nu)$ for some $w \in W_{\mathfrak{M}_1}$.

Definition 8.4. We say that a weight $\nu \in X^+_{\mathfrak{M}_1}$ is cohomological if there exist $\mu \in X^+_{\mathfrak{G}_1}$ and $\mu' \in [\mu]$ such that $-w_0(\nu) = w \cdot \mu'$ for some $w \in W_{\mathfrak{M}_1}$. (Here $w$, $\mu'$, and hence $[\mu]$ are unique if they exist.) We write in this case that $\mu' = \mu(\nu)$,

$[\mu] = [\mu(\nu)]$, and $w = w(\nu)$. 

Definition 8.5. Let ν ∈ X_{M_i}^{+,<p}. Let R be any R_1-algebra. An R-valued algebraic automorphic form of weight ν is an element of the graded R-module

A_ν^i(ℋ; R) := H^i(ℋ_{R,R}, \mathcal{W}_ν, R).

It is convenient to also introduce, for any R_1-module E, the E-valued forms

A_ν^i(ℋ; E) := H^i(ℋ_{R,1}, \mathcal{W}_ν ⊗ E). (This is compatible with Definition 8.5 when E = R.)

Proposition 8.6. Let R be any R_1-algebra. If ν ∈ X_{M_i}^{+,<p} is cohomological and satisfies µ(ν) ∈ X_{G_1}^{+,<p} and |µ(ν)|_{re,+} < p, then A_ν^i(ℋ; R) is concentrated in degree d − l(ν(ν)), and A_ν^{d−l(ν(ν))} is a direct summand of Gr\nu(H^d_{dg}(ℋ_{R,R}/S_R, V^\nu_{[µ(ν)], R})).

Proof. This is a special case of Theorem 8.2.

Theorem 8.7. Let R be any R_1-module, and let κ := R ⊗ R_1. Let ν ∈ X_{M_i}^{+,<p}. For simplicity, let us assume that max(2, r_τ) < p whenever τ = τ ∗ c. Then A_ν^i(ℋ; R) has the following properties:

1. If there exists a positive parallel weight ν_+ (resp. ν_-) such that ν − ν_+ (resp. ν + ν_-) is cohomological and µ(ν − ν_+) ∈ X_{G_1}^{+,<p} (resp. µ(ν + ν_-) ∈ X_{G_1}^{+,<p}), then A_ν^i(ℋ; R) = 0 for every i > d − l(ν(ν_+)) (resp. i < d − l(ν(ν_-))).
2. If R is flat over R_1, and if A_ν^{i−1}(ℋ; κ_1) = 0 for some degree i, then A_ν^{i−1}(ℋ; R) = 0 and A_ν^i(ℋ; R) is a free R-module of finite rank.
3. If A_ν^{i+1}(ℋ; R_1) = 0 = Tor\nu^R(A_ν^{i+2}(ℋ; R_1), pR) for some degree i, then A_ν^{i+1}(ℋ; pR) = 0 and the natural morphism A_ν^i(ℋ; R) → A_ν^i(ℋ; κR) induced by R_1 → κ_1 is surjective; in other words, any section of A_ν^i(ℋ; κR) is liftable, in the sense that it is the reduction modulo p of some section in A_ν^i(ℋ; R). (The condition Tor\nu^R(A_ν^{i+2}(ℋ; R_1), pR) = 0 holds, for example, when either A_ν^{i+2}(ℋ; R_1) or pR is flat over R_1. In particular, by 2, the full condition A_ν^{i+1}(ℋ; R_1) = 0 = Tor\nu^R(A_ν^{i+2}(ℋ; R_1), pR) holds when A_ν^{i+1}(ℋ; κ_1) = 0.)
4. If A_ν^{i−1}(ℋ; κ_1) = 0 and A_ν^{i+1}(ℋ; R_1) = 0 = Tor\nu^R(A_ν^{i+2}(ℋ; R_1), pR) for some degree i, then A_ν^i(ℋ; R) is a free R-module of finite rank, and we have a canonical exact sequence

0 → A_ν^i(ℋ; pR) → A_ν^i(ℋ; R) → A_ν^i(ℋ; κR) → 0.

Proof. Let us first treat the case R = R_1 (and hence κ = κ_1). The statements for A_ν^i(ℋ; κ_1) in 4 follow from a reformulation of Corollary 7.4 and the corresponding statements for A_ν^i(ℋ; R_1) follows from upper semi-continuity of dimensions of cohomology, as in the proof of Theorem 8.2. Then 2, 3, and 4 all follow from taking the long exact sequence induced by the canonical short exact sequence

0 → \mathcal{W}_ν → \mathcal{W}_ν → \mathcal{W}_ν, R → 0, as in the proof of Corollary 8.3.

For a general R_1-algebra R, essentially by [39], Thm., there exists a bounded complex \mathcal{Z} whose components are free R_1-modules of finite type (a strictly perfect complex) such that \mathcal{H}^i(\mathcal{Z} ⊗ E) = A_ν^i(ℋ; E) for any R_1-module E, where \mathcal{H}^i denotes the i-th cohomology of the complex. Consequently, since R_1 is a discrete
valuation ring, we obtain an exact sequence (the “universal coefficients theorem”)
\[(8.8) \quad 0 \to A^i_\nu(\mathcal{H}; R_1) \otimes E \to A^i_\nu(\mathcal{H}; E) \to \text{Tor}^R_1(A^{i+1}_\nu(\mathcal{H}; R_1), E) \to 0.
\]

To show (1) and (2), we use the vanishing and the freeness statements we have already proved over $R_1$. For example, if $A^i_\nu(\mathcal{H}; R_1) = 0$ for every $i \leq d - l(w(\nu))$, then $A^d_{\nu-l(w(\nu))}(\mathcal{H}; R_1)$ is free over $R_1$, and consequently $(8.8)$ with $E = R$ implies that $A^i_\nu(\mathcal{H}; R) = 0$ for every $i < d - l(w(\nu))$. To show (3), we take the cohomology long exact sequence attached to the canonical short exact sequence
\[0 \to p(W_{\nu,R}) \to W_{\nu,R} \to W_{\nu,R,R} \to 0,
\]
and deduce the vanishing $A^{i+1}_\nu(\mathcal{H}; pR) = 0$ from (8.8) with $E = pR$. Finally, to show (4), we deduce the isomorphism $A^i_\nu(\mathcal{H}; R) \cong A^i_\nu(\mathcal{H}; R_1) \otimes R$ from (8.8) with $E = R$ (and the assumption that $A^{i+1}_\nu(\mathcal{H}; R_1) = 0$), and combine this with (2) and (3).

\[\square\]

Remark 8.9. One can show using Serre vanishing that, for any positive parallel weight $\nu_+$, there exists an integer $N_0 \geq 0$ such that for all $N \geq N_0$ sections of $A^{0}_{\nu+N\nu_+}(\mathcal{H}; \kappa_R)$ (zero or not) are liftable to $A^{0}_{\nu+N\nu_+}(\mathcal{H}; R)$. However, Serre vanishing does not give an effective bound for $N_0$, and $N_0$ might have to increase with the level $\mathcal{H}$.

Remark 8.10 (cf. \[\citet{32}\] Rem. 4.5). One cannot expect the statements of Theorem 8.7 to be true for all weights, even for compact Picard modular surfaces. See \[\citet{35}\] Thm. 3.4 for counterexamples to liftability of sections of $A^0_{\nu}(\mathcal{H}; \kappa_1)$ to $A^0_{\nu}(\mathcal{H}; R_1)$ with $\mu(\nu) = 0$ and $l(w(\nu)) = d$ (so for this $\nu$ there cannot be a positive parallel weight $\nu_+$ such that $\nu - \nu_+$ is cohomological). Over such surfaces, there are global sections of the canonical bundle (the bottom Hodge piece of the de Rham cohomology with trivial coefficients) that cannot be lifted to characteristic zero. (The fact that the Hodge to de Rham spectral sequence degenerates by 2 does not help.) Similarly, there are nontrivial $p$-torsion Betti and étale cohomology classes.

8.3. Étale and Betti cohomology. Let $\Lambda$ be an integral domain, finite flat over the $p$-adic completion of $R_1$ (and hence finite flat over $\mathbb{Z}_p$). (See the second paragraph of Section 4.3.) Let $\Lambda_1 = \Lambda/p\Lambda$ (as in Section 5.2).

Lemma 8.11. Suppose there is a $\mu \in X^{++}_{G_1}$ such that $|\mu|_{\text{re}} < p$. Then $2d < p$ holds automatically. (See Proposition 4.8)

Proof. Since $|\mu|_{\text{re}} = d + |\mu|_L$, it suffices to show that $d_\tau \leq |\mu|_L$, for any $\tau \in \mathcal{Y}/c$. If $d_\tau = 0$, then this is obvious. Otherwise, since $\mu \in X^{++}_{G_1}$, we may assume that entries of $\mu_\tau'$ are strictly decreasing integers for any $\mu_\tau' \in [\mu]$. If $\tau = \tau \circ c$, then $d_\tau \leq |\mu|_L$. If $\tau \neq \tau \circ c$, then $d_\tau = p_\tau q_\tau \leq \frac{1}{2}(p_\tau + q_\tau)(p_\tau + q_\tau - 1) \leq |\mu|_L$.

$\square$

Theorem 8.12. Suppose that $\mu \in X^{++}_{G_1}$ satisfies $|\mu|_{\text{re, +}} < p$ and $|\mu|_{\text{comp}} \leq p - 2$ (see Definition 5.9). Then the following are true:

1. $H^i_{\text{ét}}(\mathcal{M}_{\mathcal{H}, F_0^{ac}}, \mathcal{O}^{\vee}_{[\mu]} \otimes_{\Lambda_1}) = 0$ for every $i \neq d$.
2. $H^d_{\text{ét}}(\mathcal{M}_{\mathcal{H}, F_0^{ac}}, \mathcal{O}^{\vee}_{[\mu]}) = 0$ for every $i \neq d$.
3. $H^d_{\text{ét}}(\mathcal{M}_{\mathcal{H}, F_0^{ac}}, \mathcal{O}^{\vee}_{[\mu]})$ is a free $\Lambda$-module of finite rank.
4. The canonical exact sequence $0 \to p(\mathcal{O}^{\vee}_{[\mu]}) \to \mathcal{O}^{\vee}_{[\mu]} \to \mathcal{O}^{\vee}_{[\mu]} \otimes \Lambda_1 \to 0$ induces an exact sequence
\[0 \to H^d_{\text{ét}}(\mathcal{M}_{\mathcal{H}, F_0^{ac}}, p(\mathcal{O}^{\vee}_{[\mu]})) \to H^d_{\text{ét}}(\mathcal{M}_{\mathcal{H}, F_0^{ac}}, \mathcal{O}^{\vee}_{[\mu]}) \to H^d_{\text{ét}}(\mathcal{M}_{\mathcal{H}, F_0^{ac}}, \mathcal{O}^{\vee}_{[\mu]} \otimes \Lambda_1) \to 0.
\]
The same are true if we base change the coefficient $\Lambda$ to any $\Lambda$-algebra.

Proof. As in the proof of Corollary 8.3 by taking the long exact sequence induced by the short exact sequence $0 \to p(\ell_{\mu, p}) \to \ell_{\mu, p} \to \ell_{\mu, p} \otimes \Lambda_1 \to 0$, the statements (2), (3), and (4) all follow from (1). (The base change statement follows from the “universal coefficient theorem” for étale cohomology; cf. the proof of Theorem 8.7.) To prove (1), we may replace $\Lambda$ with a domain finite flat over $\Lambda$ and assume that the set $\Omega := \text{Hom}_{\text{Z-alg}}(W, \Lambda)$ has cardinality $|F_0 : \mathbb{Q}|$, so that the results in Section 5.2 apply. By Lemmas 8.11 and 7.24, $|\mu|_{\text{ac}} < p$ implies that $2d < p$ and that $\max(2, r_\tau) < p$ whenever $\tau = \tau \circ c$. Since $|\mu|_{\text{comp}} < p$, Proposition 5.8 applies for any $i$ (from 0 to 2d), and (1) follows from Theorem 8.1 as desired.

Corollary 8.13. Theorem 8.12 remains true if we replace the étale cohomology with the Betti cohomology (over the complex numbers instead of $F_0^\infty$).

Proof. This follows from Proposition 4.14.

8.4. Comparison with transcendental results. Just as Deligne and Illusie deduced vanishing theorems of Kodaira type in characteristic zero from the vanishing statements in positive characteristic (see [9] and [22]), we now obtain purely algebraic proofs of (the crudest form of) certain vanishing theorems that have so far been proven only by transcendental methods.

Lemma 8.14. Suppose $\mu \in X^{+, <, \wp}_{M_1}$, and $\max(2, r_\tau) < p$ whenever $\tau = \tau \circ c$. Then $\mathfrak{B}_{\mu, 1, \zeta}$ (resp. the analytification of $\mathfrak{B}_{\mu, 1, \zeta}$) over the analytification of $M_{H, \zeta}$ can be canonically identified with the sheaf of locally constant (resp. holomorphic) sections of

$$G(\mathbb{Q}) \backslash (X \times V^\vee_{\mu, 1, \zeta}) \times G(\mathbb{A}^\infty) / H \to G(\mathbb{Q}) \backslash X \times G(\mathbb{A}^\infty) / H,$$

so that $\mathfrak{B}_{\mu, 1, \zeta}$ is canonically isomorphic to the sheaf of horizontal sections in the analytification of $(V^\vee_{\mu, 1, \zeta}, \nabla)$. A similar statement holds for $\nu \in X^{+, <, \wp}_{M_1}$ and $W^\vee_{\nu, \zeta}$, and the identifications respect the Hodge filtrations.

Proof. It suffices to verify this for $V^\vee_{\mu, 1, \zeta} = L \otimes \zeta$, together with its filtration defined by $V_0^\vee$ in (1.2), which can be canonically identified with the relative $H_1$ of the universal abelian scheme, together with its Hodge filtration. Then the result follows because this is exactly how we identify PEL-type Shimura varieties (and their universal objects) with their complex versions, as explained in, e.g., [27] §2.

Corollary 8.15. The objects $\mathfrak{B}_{\mu, 1, \zeta}$, $V^\vee_{\mu, 1, \zeta}$, and $W^\vee_{\tau, \zeta}$ in Lemma 8.14 can be defined independently of $\mu$, and we have a canonical isomorphism $H^i_B(M_{H, \zeta}, \mathfrak{B}_{\mu, 1, \zeta}) \cong H^i_{\text{dR}}(M_{H, \zeta}, V^\vee_{\mu, 1, \zeta})$ for each $i$. By abuse of language, we shall extend the definition of these objects to all dominant weights.

Note that $X^{+, <, \wp}_{G_{\zeta}} = X^{+, <, \wp}_{G_1}$ has an unambiguous meaning for any valid choice of $p$ and $R_1$. We shall write $G_{\zeta}$ in place of $G_1$ in what follows in this subsection.

Theorem 8.16. Suppose $\mu \in X^{+, +}_{G_{\zeta}}$. Then the following are true:

1. $H^i_B(M_{H, \zeta}, \mathfrak{B}_{\mu, 1, \zeta}) = 0$ for every $i \neq d$. 
The Hodge to de Rham spectral sequence for the de Rham cohomology of $V_{\nu,\mathbb{C}}^+$ degenerates at $E_1$ and defines by taking $\text{Gr}_\nu$ a Hodge decomposition
\[
\text{Gr}_\nu(H_{dR}^i(M_{H,C}/S_{\mathbb{C}}, V_{\nu,\mathbb{C}}^+)) \cong H^i(M_{H,C}, \text{Gr}_\nu V_{\nu,\mathbb{C}}^+ \otimes \Omega^\bullet_{M_{H,C}/S_{\mathbb{C}}}).
\]
Combining (1) with (2), we see that every summand on the right-hand side is zero when $i \neq d$.

Proof. By Corollary 8.15, we can choose a good prime $p$ (see Section 1.1) so large that $\mu \in X_{\text{reg}}^{+,<p}$ and $|\mu|_{\text{reg},+} < p$. Then the results follow from Theorem 8.2. \qed

Remark 8.17. To the best of our knowledge, the simplest (analytic) proof of Theorem 8.20 is given by Faltings in [11], using his construction of dual BGG complexes (based on older ideas in [3]). It is perhaps not a coincidence that our method uses this BGG idea as well. However, the proof in [11] uses $C^\infty$-resolutions of vector bundles and harmonic forms, and as such looks inadequate for dealing with torsion coefficients. In this sense, the (purely algebraic, characteristic $p > 0$) theory developed by Deligne and Illusie is as indispensable in our proof as Hodge theory is in that of Faltings.

Remark 8.18. A more general theory of vanishing theorems from the perspective of automorphic representations and group cohomology of arithmetic groups (for general reductive groups) has a good modern summary in [33, §2], with major inputs from [46, and with some updates in [34] (concerning Eisenstein cohomology classes absent in our compact case).

Remark 8.19. In works mentioned in Remarks 8.17 and 8.18 it suffices to assume that $\mu$ is regular, a weaker (and hence better) condition than ours when $G_{\mathbb{C}}$ has factors of types $C$ or $D$. (See Remark 7.19.) This is a fundamental restriction of our technique, relying on the positive parallel weights of minimal size.

Similarly (to the case of $G_1$), $X_{\text{Mc}}^+ = X_{M_1}^+$ has an unambiguous meaning, and we shall write $M_{\mathbb{C}}$ in place of $M_1$ in the remainder of this subsection.

We can extend the definition of $\text{A}^\bullet_c(H, \mathbb{C})$ to all $\nu \in X_{\text{Mc}}^+$, and deduce from Theorem 8.7 that:

**Theorem 8.20.** If there exists a positive parallel weight $\nu_+$ (resp. $\nu_-$) such that $\nu - \nu_+$ (resp. $\nu + \nu_-$) is cohomological and $\mu(\nu - \nu_+)$ is $X_{\text{Mc}}^+$ (resp. $\mu(\nu + \nu_-) \in X_{\text{Mc}}^+$), then $\text{A}^\bullet_c(H, \mathbb{C}) = 0$ for every $i > d - l(w(\nu - \nu_+))$ (resp. $i < d - l(w(\nu + \nu_-)))$.

Remark 8.21. When $\nu$ is cohomological and $\mu(\nu)$ is regular, the simplest analytic result is the same work of Faltings mentioned in Remark 8.17.

In the general non-compact case, there is a much longer story for analytic results on vanishing. We defer such discussions to [31], where we will present their algebraic (and torsion) analogues.

**References**

3. I. N. Bernstein, I. M. Gelfand, and S. I. Gelfand, *Differential operators on the base affine space and a study of g-modules*, in Gelfand [16], pp. 21–64.
33. J.-S. Li and J. Schwermer, *Automorphic representations and cohomology of arithmetic groups*, in Chen et al. [7], pp. 102–137.


37. A. Mokrane and J. Tilouine, *Cohomology of Siegel varieties with p-adic integral coefficients and applications*, in *Cohomology of Siegel varieties* [36], pp. 1–95.


42. P. Polo and J. Tilouine, *Bernstein–Gelfand–Gelfand complex and cohomology of nilpotent groups over \( \mathbb{Z}_p \) for representations with p-small weights*, in *Cohomology of Siegel varieties* [36], pp. 97–135.


