Part I. Introduction.

**Theorem 15.2 (Hörmander's Rough Theorem)**

Let $\mathcal{M}$ be a closed $n$-dim. mfd. on which there is given a smooth positive density.

Let $A$ be a self-adjoint, elliptic operator on $\mathcal{M}$ st.

$$ A_{m}(x, \xi) > 0; \quad \xi \neq 0. $$

Denote by $\lambda_j$ its eigenvalues, and $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$

Then one has the following asymptotic expansions:

- $A^{t} \to 0$, $\forall x, t \sim V_{x}(t)$ (\[= \int_{\mathcal{M}} A_{m}(x, \xi) \xi^{\ast} \right. \left. \frac{d\xi}{\lambda_{1}^{2}} \text{ for } \xi \in \mathcal{M} \) (15.6)

- $A^{t} \to 0$, $N(t) \sim V(t) \quad (\int_{\mathcal{M}} dx \int_{13}^{13} A_{m}(x, \xi) d\xi \right. \left. \frac{d\xi}{\lambda_{1}^{2}} \right. \left. \frac{d\xi}{A_{m}(x, \xi) t} \right) \quad (15.7)$

- $A^{t} \to 0$, $\lambda_{k} \sim V(1) \frac{k}{t}$ \quad (15.10)

Sketch of the proof

Set $\zeta_{A}(\xi) = \int_{\mathcal{M}} A_{2}(x, x) dx$, $Re \xi < -\frac{m}{A}$, where $A_{2}(x, y)$ is the Schwartz kernel of $A^{\xi}$.

$$ \Rightarrow \zeta_{A}(\xi) = \int_{\mathcal{M}} \xi^{\ast} \right. \left. dN(t) \right. \left. Re \xi < -\frac{m}{A} \right) \quad (By \ z_{A}(\xi) = \sum_{j=1}^{\infty} \lambda_{j}^{\xi}) $$

Then we apply the Ikehara Tauberian theorem:

If $\zeta(\xi) = \int_{0}^{\infty} \xi dN(t)$ is non-decreasing (conv. for $Re \xi < -k_{0}$) and we know the pole & residue at this pole of $\zeta(\xi)$ then we can invert $N(t) \sim \frac{A^{t}}{t^{k}}.$
Our main goal now is to give a proof of the following Hörmander

**Improved Theorem:**

**Theorem 16.1** (Hörmander's Refined Theorem).

The following estimates holds:

\[
|e(x, x, \lambda) - V(x, \lambda)| \leq C \lambda^{\frac{n+1}{m}}, \quad \lambda \geq 1, \quad x \in M.
\]

**Corollary 16.1**

The following asymptotic formula holds:

\[
N(\lambda) = V(\lambda)\left(1 + O(\lambda^{-\frac{1}{m}})\right) \quad \text{as} \quad \lambda \to +\infty.
\]

**Remark 1.**

\[
V(\lambda) = \iint_{\|x\| < \lambda} d^3x = V(1) \cdot \lambda^{\frac{n}{m}} \quad \text{(By the theory of} \; \mathbb{A}(x, \mathbb{B})) \quad \text{(By the change of variable.)}
\]

\[
N(\lambda) = V(\lambda)(1 + O(\lambda^{-\frac{1}{m}})) \iff N(\lambda) = V(\lambda) + O(\lambda^{-\frac{1}{m}})
\]

(Corollary 16.1)

Therefore, Corollary 16.1 can easily be derived from Theorem 16.1 by integrating

(16.1) both sides over \( x \).

We look at an example, first:

**Example**

Consider \( A^2 = - \frac{d^2}{dx^2} \) on the circle \( M_1 = \sqrt{2x^2 + 1} \). (= S').

The corresponding eigenfunctions are of the form:

\[ 2 \]
\( \psi_{\pm}(x) := \frac{1}{(2\pi)^{1/4}} e^{\pm ix} \), \( x = 0, \pm 1, \pm 2, \ldots \)

and the eigenvalues are \( \lambda_k = k^2 \), \( k = 0, \pm 1, \pm 2, \ldots \).

Then

- The renormalized eigenvalues are:

\[
\lambda_{(k)} \triangleq \begin{cases} 
(\frac{k}{2})^2, & k \text{ odd}, \\
(\frac{k}{2})^2, & k \text{ even}.
\end{cases}
\]

- \( m = 2, n = 1 \), \( \nu(1) = 2 \), \( \nu(k) = (2\pi) \), \( \nu(\lambda) = \frac{2\pi}{\lambda} \).

\[
\nu(\lambda) = \sum_{j < \lambda} 1 = \sum_{0 < j < \sqrt{\lambda}} 1.
\]

直覺上我們 \( \nu(\lambda) \sim 2\sqrt{\lambda} \). 實質上，由 Theorem:

- \( \nu(\lambda) = \nu(1) + O(1) = 2\sqrt{\lambda} + O(1) \), as \( \lambda \to \infty \).

- \( \lambda_{(2)} \sim \nu(1)^{-2} k^2 = (\frac{k}{2})^2 \), as \( k \to \infty \).

This example tells us:

- In general, the estimate of the remainder in (16.1) (16.2) cannot be improved!

i.e., \( \nu(\lambda) = 2\sqrt{\lambda} + O(\lambda^x) \) would not hold for any \( x < 0 \).

\[
\nu(\lambda) - 2\sqrt{\lambda} = 1 + 2 \left( \sum_{\substack{j \in \mathbb{Z} \\setminus \{0\} \ \text{ and } \ j \neq \kappa \mathbb{Z} \}} 1 - \sqrt{\lambda} \right) = 1 + 2 \left( \left[ \frac{1}{\sqrt{\lambda}} \right] - \sqrt{\lambda} \right) \overset{\text{if } \lambda \to \infty}{\longrightarrow} O(1), \quad \text{for } \lambda \to \infty.
\]

\[
\Rightarrow \left| \frac{N(\lambda) - 2\sqrt{\lambda}}{\lambda^x} \right| = \left| \left( 1 + 2 \left[ \frac{1}{\sqrt{\lambda}} \right] - \sqrt{\lambda} \right) \cdot \lambda^{-x} \right| \overset{\text{b.d.}}{\longrightarrow} C \quad \text{independent of } \lambda.
\]

\( \text{blow-up!!!} \)
In general, we have no improvement of
\[ \lambda(k) \sim \sqrt{\frac{1}{k}} \quad \text{as } k \to \infty. \]

Even though
\[ \text{(15.6)} \quad \text{改变} \quad \text{(16.1)} \]
\[ \text{(15.9)} \quad \text{(16.2)} \]

Now we state our main precise theorem:

**Main Theorem 21.2.**

Let \( A \) be a self-adjoint elliptic differential operator of order \( m \) on a closed manifold \( M \) with principal \( a_0(x,3) > 0 \),

Let us construct for nearby \( x, y \) a function \( f(x, y, 3) \) of,

\[ a_0(x, f(x, y, 3)) = a_0(y, 3) \quad \text{(21.31)} \]

\[ f|_{(x-y) \cdot 3 = 0} = 0, \quad f|_{x=y} = 3 \quad \text{(21.32)} \]

from which it follows that \( f \) is homogeneous of degree one in \( 3 \) and such that

\[ f(x, y, 3) = \langle x-y, 3 \rangle + O(1|x-y|^2) \quad \text{as } x \to y. \]

Then

\[ \text{for nearby } x, y, \quad |e(x, y, 3) - \int_{a_0(3, 3) < \lambda} e^{f(x, y, 3)} \, d\tilde{s}| \leq C(1 + \lambda) \quad \text{(21.34)} \]

\[ \Rightarrow \quad |e(x, x, 3) - \int_{a_0(3, 3) < \lambda} d\tilde{s}| \leq C(1 + \lambda) \quad \text{(21.35)} \]

\[ \Rightarrow \quad |N(\lambda) - \int_{a_0(3, 3) < \lambda} dx d\tilde{s}| \leq C(1 + \lambda) \quad \text{(21.36)} \]
2) If the pair \( x, y \) belongs to some compact set in \( M \times M \) disjoint from the diagonal, then
\[
|e^{x,y,\lambda}| \leq C(1+|x-y|)^{-n}.
\]

To prove Theorem 21.2 (operator of order \( m \in \mathbb{N} \)), it suffices to prove the \( m=1 \) case:

**Theorem 21.1**

1) Let \( \varphi(x,y,\lambda) \) be defined for nearby \( x, y \) and satisfy
\[
A_1(x, \varphi(x,y,\lambda)) = A_1(y,\lambda), \quad \varphi \big|_{(x-y,\lambda)} = 0, \quad \varphi \big|_{x=y} = \lambda \tag{21.27}
\]

Then for any nearby \( x, y \) we have.
\[
|e^{\varphi(x,y,\lambda)} - \int e^{i\varphi(x,y,\lambda)} d\beta | \leq C(1+|x-y|)^{-n} \tag{21.28}
\]

\[
\implies |e^{\varphi(x,x,\lambda)} - \sqrt{\lambda} | \leq C(1+|x-y|)^{-n} \tag{21.29}
\]

\[
\implies |N(\lambda) - \int \frac{1}{A_1(x,\lambda)} d\beta | \leq C(1+|x-y|)^{-n} \tag{21.30}
\]

2) If the pair \( x, y \) belongs to a compact set in \( M \times M \) disjoint from the diagonal, then
\[
|e^{x,x,\lambda}| \leq C(1+|x-y|)^{-n}. \tag{21.30}
\]

In Theorem 21.1, \( A \) is assumed to be a self-adjoint, classical PDO of degree 1 on \( M \) with principal symbol \( a_1(x,\lambda) > 0 \), for \( \Re \lambda \geq 0 \).
Part 2. Reduction of the order (Th. 21.2 $\rightarrow$ Thm 21.1)

Introduce the operator

$$A_1 = A^{\frac{1}{m}}.$$ 

This is an elliptic, classical, PDO of order 1 with principal symbol $a_1(x, \lambda) = am(x, \lambda)^{\frac{1}{m}}$.

Note that

- $(21.31), (21.32) \Rightarrow (21.27)$ holds for $f(x, y, \lambda)$,

$$e_1(x, y, \lambda) = \sum_{\lambda \leq \lambda} g_f(x) \overline{g_i(y)} = \sum_{\lambda \leq \lambda} g_f(x) \overline{g_i(y)} = e_1(x, y, \lambda^{\frac{1}{m}}).$$

the spectral fam of $A$

the spectral fam of $A_1$

Therefore, all statements in Theorem 21.2 follow from the corresponding statements in Theorem 21.1.

We remark first that the proof of "Rough Hörmander Theorem" can also be derived from "the asymptotic expansion of heat kernel & Karamata Tauberian Theorem", and the process are sketched below:

Problem 10.1

Let $A$ : elliptic differential operator with principal symbol $\alpha_m(x, \xi)$ on a closed mfld. $M$.
Assume that $\text{Re} \alpha_m(x, \xi) < 0$, $\xi \neq 0$.
Then the Cauchy problem

$$\frac{\partial u}{\partial t} = Au, \quad t > 0, \quad u|_{t=0} = y(x);$$

has a unique solution in $C^\infty(M)$ and $C^0(M)$.

The solution $u(t, x)$ will be of the form

$$u = e^{tA}y,$$

where the operator $e^{tA}$ is given by the contour integral,

$$e^{tA} := \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} \left( A - \lambda I \right)^{-1} d\lambda,$$

with

$$\Gamma = \Gamma_1 \cup \Gamma_2, \quad \lambda = \int_{\Gamma_1} e^{i\frac{\pi}{2} - \epsilon} + \infty > \gamma > 0, \quad \text{on } \Gamma_1,$$

$$e^{i\left( \frac{\pi}{2} + \epsilon \right)} \quad 0 < \gamma < +\infty, \quad \text{on } \Gamma_2.$$
Problem 13.4

The kernel $K(t,x,y)$ of the operator $e^{tA}$ is infinitely differentiable in $t, x, y$ for $t > 0, x, y \in M$.

As $t \to +0$, we have the following asymptotic properties of $K(t,x,y)$:

a) If $x+y = 0$, then $K(t,x,y) = O(t^N)$, $\forall N > 0$

b) $K(t,x,x) \sim \sum_{j=0}^{\infty} \sigma_j(x) t^{\frac{j-n}{m}}$, as $t \to 0$.

Furthermore, if $A = A^*$, then

$$K(t,x,y) = \sum_{j=0}^{\infty} e^{-t\lambda_j} \sigma_j(x) \overline{\sigma_j(y)}$$

and the $\Theta$-function

$$\Theta(t) = \sum_{j=0}^{\infty} e^{-t\lambda_j} \left( = \int_M K(t,x,x) \, dx = \int_0^\infty e^{-i \lambda t} \, dN(t) \right)$$

has the asymptotic expansion

$$\Theta(t) \sim \sum_{j=0}^{\infty} \sigma_j \lambda_j \frac{t^{\frac{j-n}{m}}}{j}.$$ 

Problem 14.2 (Karamata Tauberian Theorem)

Let $N(t)$ be a non-decreasing function of $t \in \mathbb{R}^+$, equal to 0 for $t < 1$ and such that the integral

$$\Theta(t) = \int_0^\infty e^{-zt} \, dN(t),$$

converges for all $z > 0$, and

$$\Theta(z) \sim A z^{-x} \text{ as } z \to +0 \text{ (here } A > 0, x > 0 \text{ are const.)}$$

Then

$$N(t) \sim \frac{A}{I(x+1)} t^x \text{ as } t \to +\infty.$$
But it needs to note that the estimate getting directly from Tauberian's Theorem is too rough!

To "invert" \( N(t) \) from these types:

\[
\begin{align*}
\theta(z) &= \int_0^\infty e^{-zt} dN(t), \\
\zeta(z) &= \int_0^\infty t^z dN(t)
\end{align*}
\]

of integrals.

Therefore, we need to "be careful" when applying the idea of Tauberian Theorem, we need some preliminaries first:

Let \( M \) be a closed \( n \)-dim. mfd with a smooth density and let \( A \) be a self-adjoint, classical PDO of degree 1 on \( M \) with principal symbol \( a_\ell(x, \xi) \) satisfying the condition:

\[
a_\ell(x, \eta) > 0 \quad \text{for } \eta \neq 0.
\]

**Definition 20.1**

The operator \( \exp(-itA) \), for \( t \in \mathbb{R} \), is defined by the formula

\[
\exp(-itA)u(x) := \sum_{\ell=1}^{\infty} \exp(-it\lambda_\ell) \, u \cdot g_\ell(x).
\]

Where \( u = \langle u, g_\ell \rangle \), \{\( g_\ell \)\}_\ell \) is complete orthonormal system of eigenvectors for \( A \) and \( \lambda_\ell \) : corresponding eigenvalues.
Remark

1. The series \((20.4)\) for \(u \in C^0(M)\) converges in the topology of \(C^0(M)\).

   In fact, we have the following Proposition:

   \[ \text{Proposition 20.1} \]
   \[ \text{If } u(x) \in C^0(M), \text{ then } u(x) = \sum_{k=1}^{\infty} u_k \Psi_k(x). \]

2. The operator \(\exp(-itA)\) is a bounded operator on \(H^s(M)\) for \(s \in \mathbb{Z}^+\).

   (Proposition 20.2).

   Note, it is also a unitary operator on \(L^2(M)\).

Let \(e(x,y,\lambda)\) be the spectral function of the 1st order operator \(A\).

\(U(t,x,y)\) be the kernel of the operator \(\exp(-itA)\).

Namely,

\[ e(x,y,\lambda) = \sum_{\lambda_j \leq \lambda} \psi_j(x) \overline{\psi_j(y)} \quad (21.1) \]

\[ U(t,x,y) = \sum_j e^{-i\lambda_j t} \psi_j(x) \overline{\psi_j(y)} \quad (21.2) \]

where the latter series is summable in the sense of distribution on \(M \times M\), depending smoothly on \(t\).

\[ \exp(-itA)u(x) = \sum_{j=1}^{\infty} e^{-i\lambda_j t} u_j \psi_j(x) = \int \left[ \sum_{j=1}^{\infty} e^{-i\lambda_j t} \psi_j(x) \overline{\psi_j(y)} \right] u(y) \, dy \]
Note that, to study the asymptotic behavior of $N(x)$, one possible way is to study the asymptotic behavior of $e(x,y,\lambda)$ as $\lambda \to \infty$. Since we have the relationship:

$$N(x) = \int_y e(x,y,\lambda) \, dx.$$ 

Observe from (21.1), (21.2) that

$$U(t,x,y) = \int_0^\infty e^{-it\lambda} \, d\lambda e(x,y,\lambda).$$  

(21.3)

--- measure of Fourier transform, Fourier-Stieltjes transform.

**Idea**

Our strategy is to "invert" the above Fourier transform in an appropriate way involving the asymptotic estimate in $\lambda$.

**Study**

For technical problem, we introduce a function $\xi(u) \in L(\mathbb{R}^1)$ s.t.

1) $\xi(\lambda) > 0$, $\forall \lambda \in \mathbb{R}^1$

2) $\hat{\xi}(0) = 1 \quad (= \int_{-\infty}^{\infty} \xi(u) \, du)$

3) $\text{supp} \hat{\xi}(t) \subseteq (-\varepsilon, \varepsilon)$, where $\varepsilon > 0$ is small enough
Let \( X \in C^0_0(\mathbb{R}^n) \) be such that \( X(x) > 0, \int X(x) = 1, \ 0 \leq \tilde{X}(\xi) \leq 1 \):

Choose the function \( X_0(x) \) s.t. \( X_0(x) e^{\hat{X}_0(\xi)}(\mathbb{R}^n) \), \( X_0(x) > 0, \int X_0(x) dx \).

Put \( X(x) := \int X_0(x+y) X_0(y) dy \) — The convolution of \( f(x) := X_0(x-y) \), \( g(x) := X_0(y) \).

Then:

- \( \text{spt} \hat{X} = \text{spt} (f \ast g) \subseteq \text{spt} f \cap \text{spt} g \) — compact
- \( \hat{X}(\xi) = |\hat{X}_0(\xi)|^2 \Rightarrow \hat{X}(\theta) = \hat{X}(0) = \int_{-\infty}^{\infty} X(x) dx = 1 \).

By the property of "Fourier transform & convolution", \((f \ast g) = \hat{F} \hat{G} \)

\[
\hat{\mathcal{U}}(t, x, y) = \mathcal{F}_{x \to t} \int e^{i(x-\mu)} d\mu e(x, y, \mu)
\]  
(21.4)

使之後可封存取「四議轉換」

In order to get more refined estimate, we need the theory of Fourier Integral Operator, i.e.

**Theorem 20.1** (e^{-|l|t} 急用! 小時是 FIO)

If \( \varepsilon > 0 \) is sufficiently small, then for \( |l| < \varepsilon \)

\[
\mathcal{U}(t) := \exp(-itA)
\]
can be represented in the form of a sum of an operator with a smooth kernel in \( t, x, y, \xi \) and an FIO, given by the phase function

\[
\mathcal{S}(t, x, y, \xi) = \mathcal{S}(x, y, \xi) - \text{tan}(y, \xi)
\]
(20.8)

and by an amplitude \( p(t, x, y, \xi) \) which is a classical symbol of order 0, smooth in \( t \), and such that the following estimate holds

\[
|\frac{d}{d\xi} \mathcal{S}(x, y, t p(t, x, y, \xi))| \leq C_\alpha \delta < \delta >^{-1/2}
\]
(20.9)
Remark. (Important Idea of proving this theorem).

The operator $U(t)$ satisfies the conditions

\[
\begin{aligned}
    (D_t + A) U(t) &= 0 \\
    U(0) &= I,
\end{aligned}
\]

Our goal is to construct the operator $Q(t)$, which approximates $U(t)$ and is an FIO of the form

\[
Q(t)f(x) = \int \int q(t, x, y, z) e^{i y(t, x, y, z)} f(y) dy dz
\]

More precisely, we'll try to choose $Q(t)$ satisfying the conditions

\[
\begin{aligned}
    (D_t + A) Q(t) &\in L^{-\infty}(M) \\
    Q(0) - I &\in L^{-\infty}(M)
\end{aligned}
\]

where the left side of (20,13) will also be a smooth function of $t$ with values in $L^{-\infty}(M)$.

And this can be achieved by considering the asymptotic expansion for the symbol $a(x,y,z)$, put $x,y,z$ and solving some Cauchy problems from the asymptotic coefficients which are determined by (20,13), (20,14).

Choose \( q(t,x,y,z) = \phi(x,y,z) - t a(x,y,z) \), where \( \phi(x,y,z) \) satisfies the equation

\[
\begin{aligned}
    a_1(x,y,z) \phi(x,y,z) &= a_2(y,z) \\
    \phi(x,y,z) |_{x-y} = 0 \\
    \frac{\partial}{\partial x} \phi(x,y,z) |_{x=y} = \delta.
\end{aligned}
\]

\[ (20.19) \]

\[ (20.20) \]

\[ (20.21) \]

\[ (20.22) \]
Let now \( Q(t, x, y) \) be the distribution kernel of the FIO \( Q(t) \) for small \( t \). In view of Theorem 2.1, we have

\[
U(t, x, y) - Q(t, x, y) \in C^\infty_c (-\varepsilon, \varepsilon) \times M \times M.
\]

(2.1.5)

**Question**

Why not directly use \( U(t, x, y) \), rather than \( Q(t, x, y) \)?

**Answer**

"\( Q(t, x, y) \) in \( 1/t \) small represents FIO \( \Rightarrow \) forms factorization!"

Substitute \( U(t, x, y) \) in (2.1.4) by \( Q(t, x, y) \), we'll obtain

**Lemma 21.1**

\[
\int \mathbf{1}(\lambda - \mu) \, d\nu(x, y, \mu) - \int_{-\lambda}^{\lambda} \{ \hat{\mathbb{E}}(t) Q(t, x, y) \}.
\]

(21.6)

is a smooth function in all variables, tending to 0 faster than any power of \( t \) as \( \lambda \to +\infty \), uniformly in \( x, y \in M \).

\[
U(t, x, y) - Q(t, x, y) = G(t, x, y) \in C^\infty_c (-\varepsilon, \varepsilon) \times M \times M \quad \text{good!}
\]

\[
\Rightarrow \int \mathbf{1}(\lambda - \mu) \, d\nu(x, y, \mu) - \hat{\mathbb{E}}(t) Q(t, x, y) = \hat{\mathbb{E}}(t) G(t, x, y) \in C^\infty_c (-\varepsilon, \varepsilon)
\]

(21.6)

\[
\Rightarrow \text{Lemma 21.1 holds by taking inverse Fourier transform.}
\]

Since \( Q(t, x, y) \) is the distribution kernel of the FIO \( Q(t) \),

the term \( \int_{-\lambda}^{\lambda} \{ \hat{\mathbb{E}}(t) Q(t, x, y) \} \) can be represented in a more "pleasant" way:

This can be easily done, thanks to the linearity in \( t \) of the phase function

\[
g(t, x, y, z) = g(x, y, z) - g(x, y, z) \cdot t
\]

in the definition of the operator \( Q(t) \).
In the notation of Theorem 20.1 we have
\[ Q(t, x, y) = \int q(t; x, y, \beta) e^{i(f(x, y, \beta) - t \alpha(x, y, \beta))} \, d\beta \]  
(21.7)

Hence,
\[ \mathcal{F}_{t \to x} \{ \hat{Q}(t; x, y, \beta) \} = (2\pi)^{-\frac{3}{2}} \int \hat{Q}(t; x, y, \beta) e^{it(x, y, \beta)} \, e^{-i\beta} \, dt \]  
(21.8)
\[ = \int R(\lambda; x, y, \beta) e^{it(x, y, \beta)} \, d\beta \]  
(21.10)

Set \[ R(\lambda; x, y, \beta) := (2\pi)^{-\frac{3}{2}} \int \hat{Q}(t; x, y, \beta) e^{it(x, y, \beta)} \, dt. \]  
(21.9)

Therefore, by rewriting Lemma 21.1, we see that we have proven:

**Lemma 21.2.**

For any \( N > 0 \), we have
\[ \| \mathcal{F}_{t \to x} \{ \hat{Q}(t; x, y, \beta) \} \|_N \leq C_N (1 + |\lambda|)^{-N} \]  
(21.12)

where \( C_N \) is independent of \( x \) and \( y \).

**Remark.**

We compute formally in (21.7) → (21.10), but actually, this can be studied rigorously by the theory of oscillatory integral:

1. \( Q(t; x, y, \beta) \) is classical symbol
2. \( R(\lambda; x, y, \beta) \in S(\mathbb{R}) \) and \( R \) is a smooth function in all variables
3. \( \int dx \int dy \int d\beta \left| R(x; x, y, \beta) \right| \leq C \exp(-\beta^2) (|\lambda|^{-\frac{1}{2}})^N \), for all \( N > 0 \)
and $R$ admits an asymptotic expansion into homogeneous functions in $\xi$.

This implies \((21.10)\) is absolutely convergent.

Moreover, the transition from \((21.8)\) to \((21.10)\) is understood by performing the standard passage to the limit, as in the definition of oscillatory integrals.*

Now an important question arose:

**Question 1**

How to relate \(\int (x, y, \mu) \) to our desired term \(e(x, y, \mu)\)?

**Observation**

\[
\int e(\sigma, \mu) d\mu e(x, y, \mu) = \int e(\sigma, \mu) e(x, y, \mu) d\mu.
\]

\[
\Rightarrow \int_{-\infty}^{\infty} e(\sigma, \mu) d\mu e(x, y, \mu) d\sigma = \int e(\sigma, \mu) e(x, y, \mu) d\mu.
\]

By Fubini Theorem.

\[
\int e(x, y, \lambda) - \int_{-\infty}^{\infty} \left[ \int e(\sigma, \mu) d\mu e(x, y, \mu) \right] d\sigma.
\]

\[
= e(x, y, \lambda) - \int e(\lambda, \mu) e(x, y, \mu) d\mu
\]

\[
= \int e(\lambda, \mu) [ e(x, y, \lambda) - e(x, y, \mu) ] d\mu.
\]
\[
\int \hat{e}(\mu) \left[ e(x, y, \lambda) - e(x, y, \mu + \lambda) \right] d\mu.
\]

Namely,
\[(*)\]
\[e(x, y, \lambda) - \int_0^\lambda \int_0^\lambda e(\sigma - \mu) d\nu e(x, y, \mu) d\sigma = \int \hat{e}(\mu) [e(x, y, \lambda) - e(x, y, \mu + \lambda)] d\mu.
\]

Therefore, if we are able to look for a suitable estimate for
\[|e(x, y, \mu + \lambda) - e(x, y, \mu)|
\]
uniform in \(x, y\), as the function of variables \(\lambda, \mu\).

Then we can connect the information of "\(e(x, y, \lambda)\)"
with "\(\int \hat{e}(\sigma - \mu) d\nu e(x, y, \mu)\)".

\[
\begin{bmatrix}
\text{In fact, we have}
\end{bmatrix}
\]

\[\text{(1) } |e(x, y, \mu + \lambda) - e(x, y, \lambda)| \leq C (1 + 12\lambda + \mu)^{n-1}\]

\[\text{Lemma 2.1.6}
\]

and the above directly imply
\[\text{(2) } e(x, y, \mu) - \int_0^\lambda \int_0^\lambda e(\sigma - \mu) d\nu e(x, y, \mu) d\sigma \leq C (1 + 12\lambda)^{n-1}\]

\[\text{Lemma 2.1.8}
\]
It's helpful to remark here the relationships we have obtained until now:

\[ e(x, y, \lambda) = \int_0^\lambda R(x, y, z) \, dz \]

**Lemma 21.3**

\[ e(x, y, \lambda) - \int_0^\lambda \left( e(x, y, z) - e(x, y, \lambda) \right) \, dz \leq (1 + \sigma_1)^{1/2} \]

**Key Point**

\[ |e(x, y, \lambda + \mu) - e(x, y, \lambda)| \leq (1 + x^2 + 1)^{1/2} + (1 + \mu) \]

**Question 2.**

How to obtain the estimate of \( |e(x, y, \lambda + \mu) - e(x, y, \lambda)| \)?

**Study.** (In short, Lemma 21.3 \(\rightarrow\) Corollary 21.1 \(\rightarrow\) Lemma 21.4.)

**Step 1.**

Look at the simplified case first, i.e., whether we can control the following term:

\[ |e(x, x, \lambda + 1) - e(x, x, \lambda)| \]

**Advantage:** \( e(x, x, \cdot) \) is an non-decreasing function & nonnegative
Observe that since $L(u)$ is positive and non-zero on $[-2, 2]$, we have

$$\int L(u) d\mu(x, x, \mu) \geq C \left( \inf_{\lambda \geq 0} \left\{ \int_{-2}^{2} \left( \int_{-2}^{2} \varphi_{\lambda}(x, y, \mu) \right) d\mu(x, x, \mu) \right\} \right)$$

where $C > 0$.

This implies once after we've found an estimate for $\int L(u) d\mu(x, x, \mu)$, then we find an estimate for

$$|e(x, x, \lambda + 1) - e(x, x, \lambda)|.$$

Actually, we have

**Corollary 21.1**

$$\left| \int L(u) d\mu(x, y, \mu) \right| \leq C \left( 1 + |\lambda| \right)^{-1},$$

where $C > 0$ is independent of $x, y, \lambda$.  \hspace{1cm} \text{(21.1b)}

**Remark.**

**Corollary 21.1** $\Rightarrow$ $|e(x, x, \lambda + 1) - e(x, x, \lambda)| \leq C \left( 1 + |\lambda| \right)^{-1}$ \hspace{1cm} \text{(21.17)}
Proof.

To prove this corollary, we first claim that:

$$\left| \int R(\lambda-a(y,3), x, y, 3) e^{i\phi(x,y,3)} d\xi \right| \leq C (1+|\lambda|)^{-N}$$ \hspace{1cm} (21.13)

Then by combining Lemma 21.2 and (21.13), we can get the desired result.

Now back to look at our claim (21.13):

Observe a crucial equality:

$$\int R(\lambda-a(x, y, 3), x, y, 3) e^{i\phi(x,y,3)} d\xi = \int R(\lambda-\sigma, x, y, 3) e^{i\phi(x,y,3)} dV_y(\sigma)$$ \hspace{1cm} (21.16)

[ Zygmund ]

Let \( \phi \) be continuous on \((-\infty, \infty)\). If \( \phi(f) \in L(E) \), then \( \int_{\infty}^{\infty} \phi(x) d\mu(x) \) exists, and

$$\int_{E} \phi(f) = \int_{\infty}^{\infty} \phi(x) d\mu(x), \text{ where } \mu(x) = m(\{ f < x \}).$$

In our case, take

$$\phi(\sigma) := R(\lambda-\sigma, x, y, 3) e^{i\phi(x,y,3)}$$

$$f_y(\sigma) := a(y, 3), \quad \exists \in K^n$$

$$\mu(x) := m(\{ a(y, 3) < x \}) = (2\pi)^n V_y(x).$$

Therefore,

$$\left| \int R(\lambda-a(y,3), x, y, 3) e^{i\phi(x,y,3)} d\xi \right|$$

$$\leq C_N \int_{\sigma}^{\infty} (1 + |\lambda-\sigma|)^{-N} dV_y(\sigma) \hspace{1cm} (By \ (21.16) \ & \ (21.11))$$

$$= C_N' \int_{\sigma}^{\infty} (1 + |\lambda-\sigma|)^{-N} \sigma^{-N} d\sigma \hspace{1cm} (\because V_y(x) = V_y(1) \sigma^n)$$
\[
\leq (1+\lambda_1)^{n-1} \int_0^\infty (1+\lambda_1-\sigma)^{-N+n-1} d\sigma
\]

\[
= C \cdot (1+\lambda_1)^{n-1}
\]

from which (21.13) follows, and hence, so does Corollary 21.1.

**Step 2.**

Next, we want to look for an estimate for \( |e(x,y,\lambda+1) - e(x,x,\lambda)| \)

with second variable different from the first one:

\[
e(x,y,\lambda+1) - e(x,y,\lambda) = \sum_{\lambda \ll j \ll \lambda+H} \overline{g_j(x)} y_j(y)
\]

\[
\Rightarrow |e(x,y,\lambda+1) - e(x,y,\lambda)| \leq \left[ \sum_{\lambda \ll j \ll \lambda+H} |g_j(x)|^2 \right]^{\frac{1}{2}} \cdot \left[ \sum_{\lambda \ll j \ll \lambda+H} |g_j(y)|^2 \right]^{\frac{1}{2}}
\]

\[
= \left[ e(x,x,\lambda+1) - e(x,x,\lambda) \right]^{\frac{1}{2}} \cdot \left[ e(y,y,\lambda+1) - e(y,y,\lambda) \right]^{\frac{1}{2}}
\]

\[
\leq C \cdot (1+\lambda_1)^{n-1} \quad \text{--- (21.18)}
\]

**Step 2.**

For our final step, we decompose

\[
e(x,y,\lambda+\mu) - e(x,y,\lambda)
\]

into a sum of the form as in Step 2, which are at most \(1+\mu_1\) terms and can be estimated as in (21.18).
(i) When $\mu \geq 1$,

\[ |e(x, y, \lambda + \mu) - e(x, y, \lambda)| \]
\[ \leq |e(x, y, \lambda + \mu_1) - e(x, y, \lambda + \mu_1)| \leq (1 + 1\lambda_1 + \mu_1 - 1)^{n-1} \leq (1 + 1\lambda_1 + 1\mu_1)^{n-1} \]
\[ \vdots \]
\[ + |e(x, y, \lambda + (\mu - \lfloor \mu \rfloor + 1) - e(x, y, \lambda + (\mu - \lfloor \mu \rfloor))| \]
\[ + |e(x, y, \lambda + (\mu - \lfloor \mu \rfloor)) - e(x, y, \lambda)| \]
\[ \leq C (1 + 1\lambda_1 + 1\mu_1)^{n-1} \cdot (1 + 1\mu_1) \]

(ii) When $\mu < 1$,

\[ |e(x, y, \lambda + \mu) - e(x, y, \lambda)| \]
\[ \leq \left| e(x, y, \lambda + \mu_1) - e(x, y, \lambda + \mu) \right| + \left| e(x, y, \lambda + \mu_1) - e(x, y, \lambda) \right| \]
\[ \leq \text{by (21.17)} C (1 + 1\lambda_1 + \mu_1)^{n-1} \]
\[ \leq \text{by } \mu \geq 1 \text{ case } C (1 + 1\lambda_1 + 1\mu_1 + 1)^{n-1} \cdot (1 + 1\mu_1 + 1) \]
\[ \leq \text{by } C (1 + 1\lambda_1 + 1\mu_1)^{n-1} \cdot (1 + 1\mu_1) \]

In conclusion, we've proved the following:

**Lemma 21.6**

\[ |e(x, y, \lambda + \mu) - e(x, y, \lambda)| \leq C \cdot (1 + 1\lambda_1 + 1\mu_1)^{n-1} \cdot (1 + 1\mu_1) \]

\[ \text{independent of } x, y, \lambda, \mu \]
which is very crucial in the proof of Lemma 2.18 & Proposition 2.1.

Now, we want to deal with our latest question:

**Question 3**

How to transform \( \int_{\alpha(x,y)} \int R(\lambda-a, y, z) \, e^{i \phi(x,y,z)} \, dz \) into \( \int_{\alpha(x,y)} \int e^{i \phi(x,y,z)} \, dz \) and further into \( \int_{\alpha(x,y)} e^{i \phi(x,y,z)} \, dz \)?

**Study**

**Step 1.** \( \int_{\alpha(x,y)} \int R(\lambda-a, y, z) \, e^{i \phi(x,y,z)} \, dz \) \( \approx \) \( \int_{\alpha(x,y)} \int e^{i \phi(x,y,z)} \, dz \)

\[
\int_{\alpha(x,y)} \int R(\lambda-a, y, z) \, e^{i \phi(x,y,z)} \, dz \\
= \int_{\alpha(x,y)} \int R(\lambda-a, y, z) \, e^{i \phi(x,y,z)} \, dz \\
= \int_{\alpha(x,y)} \int L_{\alpha(x,y)} \, e^{i \phi(x,y,z)} \, dz \\
+ \int_{\alpha(x,y)} \int R(\sigma-a, y, z) \, e^{i \phi(x,y,z)} \, dz \\
= \int_{\alpha(x,y)} \left( L_{\alpha(x,y)} \right) e^{i \phi(x,y,z)} \, dz \\
+ \int_{\alpha(x,y)} \int R(\sigma-a, y, z) \, e^{i \phi(x,y,z)} \, dz \\
I(x,y,z)
\]

**Advantage:** \( \phi(x,y,z) \bigg|_{y=x} = 1 \Rightarrow \int_{\alpha(x,y)} e^{i \phi(x,y,z)} \, dz = V(x) \)
Therefore,

\[(*) = \int_{a(y,3)<\lambda} I(x,y,3) e^{i f(x,y,3)} d\bar{z} \]

\[+ \int_{a(y,3)<\lambda} \int_{\sigma_0}^{\lambda} R(\sigma, x, y, 3) d\sigma e^{i f(x,y,3)} d\bar{z} + \int_{a(y,3) > \lambda} \int_{\sigma_0}^{\lambda} R(\sigma, x, y, 3) d\sigma e^{i f(x,y,3)} d\bar{z} \]

\[= \int_{a(y,3)<\lambda} I(x,y,3) e^{i f(x,y,3)} d\bar{z} \]

\[+ \int_{a(y,3)>\lambda} \int_{\sigma_0}^{\lambda} R(\sigma, x, y, 3) d\sigma e^{i f(x,y,3)} d\bar{z} + \int_{a(y,3) > \lambda} \int_{\sigma_0}^{\lambda} R(\sigma, x, y, 3) d\sigma e^{i f(x,y,3)} d\bar{z} \]

\[= \int_{a(y,3)<\lambda} I(x,y,3) e^{i f(x,y,3)} d\bar{z} + \int_{a(y,3) > \lambda} R(\lambda-a(y,3), x, y, 3) e^{i f(x,y,3)} d\bar{z} \]

Here, we set

\[R_1(\zeta, x, y, 3) := \left\{ \begin{array}{ll}
\int_{-\infty}^{\zeta} R(\sigma, x, y, 3) d\sigma, & \zeta < 0 \\
\int_{-\infty}^{\zeta} R(\sigma, x, y, 3) d\sigma - I(x,y,3) (= -\int_{\zeta}^{\infty} R(\sigma, x, y, 3) d\sigma), & \zeta > 0
\end{array} \right. \]

Namely, we have rewritten \[\int_{a(y,3)} R(\sigma-a(y,3), x, y, 3) e^{i f(x,y,3)} d\sigma d\bar{z}\] as

\[\int \int_{\sigma<\lambda} R(\sigma-a(y,3), x, y, 3) e^{i f(x,y,3)} d\sigma d\bar{z} \]

\[= \int_{a(y,3)<\lambda} I(x,y,3) e^{i f(x,y,3)} d\bar{z} + \int_{a(y,3) > \lambda} R(\lambda-a(y,3), x, y, 3) e^{i f(x,y,3)} d\bar{z} .\]
Remark that

From the estimate for \( R_1(x,x,y,3) \) (21.11), we have the similar estimate for \( R_1(x,x,y,3) \):

\[
| \frac{\partial^2}{\partial x \partial y} R_1(x,x,y,3) | \leq C_{d_3} <\lambda^{-1/3} \lambda^{-N} \tag{21.24}
\]

for any \( N > 0 \).

Hence, one can show by analogy with (21.13) that

\[
| \int R_1(x-a_1(y,3),x,y,3) e^{i\chi(x,y,3)} dx \| \leq C \cdot (1+|\lambda|)^{-N} \]

Therefore, bycombining the above inequality and Proposition 21.1, we see that the following holds:

\[
\text{Lemma 21.9}
\]

\[
| e(x,y,\lambda) - \int \begin{array}{c} I(x,y,3) e^{i\chi(x,y,3)} dx \end{array} | \leq C (1+|\lambda|)^{-N} \tag{21.25}
\]

where \( C \) is independent of \( x, y, \lambda \).

This accomplishes our first step of the transformation!

Remark.

\[
\int_{\sigma} R_1(\sigma, x,y,3) d\sigma = Q(0,x,y,3) = I(x,y,3). \tag{21.23}
\]
Step 2. \( \int_{\alpha(y, z) < \lambda} \to \int_{\alpha(y, z) < \lambda} e^{i 2\pi (x, y, z)} \, dx \)

We need to estimate:

\[
\left| \int_{\alpha(y, z) < \lambda} (I(x, y, z) - 1) e^{i 2\pi (x, y, z)} \, dx \right|
\]

Actually, from the proof of \textbf{Theorem 19.1}, \textbf{Theorem 20.1}, for nearby the diagonal \( x = y \), we have

\[
|I(x, y, z) - 1| \leq C (1 + |z|)^{-1}
\]

\text{(20.26)}

\( I(x, y, z) \in CS^s \), which is supported on an arbitrary small nbhd.
of the diagonal, and satisfies:

\[
I(x, y, z) - 1 \in CS^{s'} (\text{near } x = y, \mathbb{R}^n)
\]

and

\[
\int I(x, y, z) e^{i 2\pi (x, y, z)} f(y) \, dy \, dz = \frac{1}{2\pi} \int f(x), f \in C^0
\]

smoothing operator

Moreover, in the proof of \textbf{Theorem 20.1}, to make that the operator \( Q(t) \) for \( t \to \infty \)
shall represent the identity operator (mod smoothing operator), we must pose the boundary condition

\[
I(x, y, z) = g(0, x, y, z)
\]

\text{(20.24)}
Therefore,
\[ \left| \int_{\alpha(y, 3) \leq x} (I(x, y, 3) - 1) \cdot e^{ixy, 3} \, d\beta \right| \]
\[ \leq C \int_{\alpha(y, 3) \leq x} (1 + 1 \alpha, 1) \, d\beta \]
\[ \leq C \int_{\alpha(y, 3) \leq x} (1 + 1 \alpha, 1) \, d\beta \]
\[ = C \int_{\mu \leq x} (1 + 1 \mu, 1) \, d\gamma(y, \mu) \]
\[ \leq C \int_{\mu = 0}^{x} (1 + 1 \mu, 1) \, d\mu \]
\[ \leq C \int_{\mu = 0}^{x} (1 + 1 \mu, 1) \, d\mu \]
\[ \leq C \left( 1 + 1 \alpha, 1 \right)^{-1} \cdot \mu \cdot \mu. \]

Namely, we obtain the following

\[ \text{Lemma 21.10} \]

For nearly \( x, y \), we have
\[ \left| \int_{\alpha(y, 3) \leq x} I(x, y, 3) \cdot e^{i\beta(x, y, 3)} \, d\beta - \int_{\alpha(y, 3) \leq x} e^{i\beta(x, y, 3)} \, d\beta \right| \leq C \left( 1 + 1 \alpha, 1 \right)^{-1} \]

where \( C \) does not depend on \( x, y, \alpha \).

And finally, our main theorem follows!