§ The Structure of the Complex Powers of an Elliptic Operators

Goal To find the "classical asymptotic expansion of A^z

Prop. 10.1

$$A_z := \frac{i}{2\pi}\int_\Gamma \lambda^z \boxed{(A-\lambda I)^{-1}}\, d\lambda, \quad \text{for } \mathrm{Re}\, z<0$$

Thm 10.1

$$A^z := A^k A_{z-k}, \quad \text{for } z\in\mathbb{C} \text{ with } \mathrm{Re}\, z < k\in\mathbb{Z}$$

Prop 11.1

• "Parametrix" of A_z, $\mathrm{Re}\, z<0$ (with singularity)

$$b^{(z),0}_{mz}(x,\xi) := \frac{i}{2\pi}\int_\Gamma \lambda^z \left(a_m(x,\xi)-\lambda\right)^{-1} d\lambda \quad \left(= a_m^z(x,\xi)\right)$$

$$b^{(z),0}_{mz-j}(x,\xi) := \frac{i}{2\pi}\int_\Gamma \lambda^z\, b^0_{-m-j}(x,\xi)\, d\lambda, \quad x\in X,\ \xi\neq 0$$

classical symbol

• "Parametrix" of A^z ($\mathrm{Re}\, z<0$) (removing singularity)

$$b^{(z)}_{mz-j}(x,\xi) := \theta(\xi)\, b^{(z),0}_{mz-j}(x,\xi) \leftrightarrow B^{(z)}_{mz-j} \leftrightarrow B^{(z)}_{(N)} := \sum_{j=0}^{N-1} B^{(z)}_{mz-j}$$

symbol — ΨDO of order $mz-j$ — ΨDO of N-th approximation

• "Parametrix" of $(A-\lambda I)$ (with singularity):

$$a_m(x,\xi,\lambda)\, b^0_{-m}(x,\xi,\lambda) = 1$$

$$a_m(x,\xi,\lambda)\, b^0_{-m-j}(x,\xi,\lambda) + \sum_{\substack{k+l+|\alpha|=j \\ l<j}} \frac{\partial_\xi^\alpha a_{m-k}(x,\xi,\lambda)\, D_x^\alpha b^0_{-m-l}(x,\xi,\lambda)}{\alpha!} = 0$$

$(j=1,2,\cdots)$

classical symbol

• Parametrix of $(A-\lambda I)$ (removing singularity):

$$b_{-m-j}(x,\xi,\lambda) := \theta(\xi,\lambda)\, b^0_{-m-j}(x,\xi,\lambda) \longleftrightarrow B_{-m-j}(\lambda) \longleftrightarrow B_{(N)}(\lambda) := \sum_{j=0}^{N-1} B_{-m-j}(\lambda)$$

symbol — ΨDO of $-m-j$ — ΨDO of N-th approximation

Proposition 11.2

1°. Deal with the symbol of A^k ($k\in\mathbb{Z}$):

(i) When $k\in\mathbb{Z}$, $k\geq 0$:

$$a^{(k)}(x,\xi) := \sum_{j=0}^{mk} a_j^{(k)}(x,\xi)$$

the symbol of A^k — the homogeneous component (of deg. j) of $a^{(k)}(x,\xi)$

(ii) When $k\in\mathbb{Z}$, $k<0$:

$a_j^{(k)}(x,\xi)$: the homogenous components of the symbol of A^{-k}
$\qquad\qquad \|$
the homog. components of the symbol of the parametrix of A^k.

$$a^{(-k)}_{-mk}(x,\xi)\cdot a^{(k)}_{mk}(x,\xi) = 1$$

$$a^{(-k)}_{-mk}(x,\xi)\, a^{(k)}_{mk-j}(x,\xi) + \sum_{\substack{p+q+|\alpha|=j \\ q<j}} \frac{\partial_\xi^\alpha a^{(-k)}_{-mk-p}(x,\xi)\cdot D_x^\alpha a^{(k)}_{mk-q}(x,\xi)}{\alpha!} = 0$$

2°. Definition of the symbol of A^z ($z\in\mathbb{C}$):

When $z\in\mathbb{C}$, choose $k\in\mathbb{Z}$ s.t. $\mathrm{Re}\, z<k$. Then define

$$b^{(z),0}_{mk-j}(x,\xi) := \sum_{p+q+|\alpha|=j} \boxed{\partial_\xi^\alpha a^{(k)}_{mk-p}(x,\xi)} \cdot \boxed{D_x^\alpha b^{(z-k),0}_{m(z-k)-q}(x,\xi)} \Big/ \alpha!$$

By (ii) — By ▨

$(j=0,1,2,\cdots)$

Thm 11.1

Thm 11.2