Besides going over this batch of review problems for the final exam, make sure you are familiar with all the assigned homework. It is also a good idea to be familiar with all the basic maneuvers rehearsed in the low-numbered problems. And old midterms and old review materials for those midterms are useful.

**Problem 1**

Standing in line at the supermarket I see Alice, Bob and Carol ahead of me in the express check-out lane. Alice buys 2 bags of chips, 3 diet sodas, 1 bag of circus peanuts, and spends $5.54. Bob buys 3 bags of chips, 1 diet soda, 3 bags of circus peanuts and spends $7.13. Carol buys 1 bag of chips, 1 diet soda, 1 bag of circus peanuts and spends $2.97. Find the price for a bag of chips, for a diet soda and for a bag of circus peanuts. Set up the problem carefully. Then use your calculator to get hard numbers.

**Solution**

\[
\begin{bmatrix}
2 & 3 & 1 \\
3 & 1 & 3 \\
1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
\text{price of bag of chips} \\
\text{price of diet soda} \\
\text{price of bag of circus peanuts}
\end{bmatrix}
= 
\begin{bmatrix}
5.54 \\
7.13 \\
2.97
\end{bmatrix}
\]

After multiplying on both sides by the inverse of the coefficient matrix, we get

\[
\begin{bmatrix}
\text{price of bag of chips} \\
\text{price of diet soda} \\
\text{price of bag of circus peanuts}
\end{bmatrix}
= 
\begin{bmatrix}
0.79 \\
0.89 \\
1.29
\end{bmatrix}
\]

**Problem 2**

Are the vectors

\[
\begin{bmatrix}
1 \\
2 \\
3 \\
4
\end{bmatrix}
, 
\begin{bmatrix}
5 \\
6 \\
7 \\
8
\end{bmatrix}
, 
\begin{bmatrix}
9 \\
10 \\
11 \\
12
\end{bmatrix}
\]

linearly independent? Explain why or why not. If not, express one of the vectors as a linear combination of the others.

Date: April 25, 2011.
Solution

The question is whether the system of equations

\[
x \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + y \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} + z \begin{bmatrix} 9 \\ 10 \\ 11 \\ 12 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]

has more than one solution or not. Well,

\[
\text{rref} \begin{bmatrix} 1 & 5 & 9 & 0 \\ 2 & 6 & 10 & 0 \\ 3 & 7 & 11 & 0 \\ 4 & 8 & 12 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
\]

from which we infer that

\[
- \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + 2 \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} = \begin{bmatrix} 9 \\ 10 \\ 11 \\ 12 \end{bmatrix}.
\]

Problem 3

Express the vector \[
\begin{bmatrix} 2 \\ 0 \\ 5 \\ 6 \end{bmatrix}
\]
as a linear combination of the vectors

\[
\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},
\]

if possible; also express \[
\begin{bmatrix} 2 \\ 0 \\ 5 \\ 6 \end{bmatrix}
\]
as a linear combination of

\[
\begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 8 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix},
\]

if possible.

Solution

In the first case, the problem is to solve the system of equations

\[
x \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 5 \\ 6 \end{bmatrix}.
\]
Well, 
\[
\text{rref } \begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 5 \\ 1 & 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
\]
which rules out the possibility of any solution. In the second case we have to solve 
\[
x \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \end{bmatrix} + z \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 5 \\ 6 \end{bmatrix},
\]
and since 
\[
\text{rref } \begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 2 & -1 & 0 \\ 1 & 4 & 1 & 5 \\ 1 & 8 & -1 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3/2 \\ 0 & 0 & 0 & 0 \end{bmatrix},
\]
we find that \( x = -1/2, \ y = 1 \) and \( z = 3/2 \) solve the problem.

**Problem 4**

Evaluate the determinant 
\[
\begin{vmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \\ x & y & z & 1 \end{vmatrix}
\]

**Solution**

\[
\begin{vmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \\ x & y & z & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & y & 0 & 1 \\ 0 & 1 & z & -x & 0 \\ y & z & 1 & x & y \\ z & 1 & +y & 0 & 1 \\ 1 & z & +x & 0 & 1 \\ 1-z^2-y^2-x^2. \end{vmatrix}
\]

**Problem 5**

Evaluate the determinant 
\[
\begin{vmatrix} 30 & 56 & 60 & 40 \\ 56 & 110 & 111 & 70 \\ 60 & 111 & 145 & 90 \\ 40 & 70 & 90 & 100 \end{vmatrix}
\]
without too much work. Hint:
\[
\begin{bmatrix} 30 & 56 & 60 & 40 \\ 56 & 110 & 111 & 70 \\ 60 & 111 & 145 & 90 \\ 40 & 70 & 90 & 100 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 5 & 0 & 0 \\ 3 & 6 & 8 & 0 \\ 4 & 7 & 9 & 10 \end{bmatrix}.
\]
Justify the main steps of your calculation. Lengthy explanations are not needed. Just give a word or phrase to indicate at each step what rule you are using.

**Solution**

The hint tells us that the given determinant by the product rule is the product of the determinants of the two triangular matrices on the right. For each of the triangular matrices the determinants are gotten by taking the product of the diagonal entries. Thus the given determinant equals \((1 \cdot 5 \cdot 8 \cdot 10)^2 = 160000.\)

**Problem 6**

Determine the values of \(x\) for which the matrix

\[
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & x
\end{pmatrix}
\]

fails to be invertible.

**Solution**

The “bad” values of \(x\) are the ones for which the determinant equals zero. Well,

\[
\begin{vmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & x
\end{vmatrix}
= 7\begin{vmatrix}
2 & 3 \\
5 & 6
\end{vmatrix} - 8\begin{vmatrix}
1 & 3 \\
4 & 6
\end{vmatrix} + x\begin{vmatrix}
1 & 2 \\
4 & 5
\end{vmatrix}
= 7 \cdot (12 - 15) - 8 \cdot (6 - 12) + x(5 - 8)
= -21 + 48 - 3x = 27 - 3x.
\]

The last expression equals zero when \(x = 9.\)

**Problem 7**

Determine the values of \(\lambda\) for which the difference

\[
\begin{pmatrix}
1 & 2 \\
2 & 3
\end{pmatrix}
- \lambda \begin{pmatrix}
4 & 5 \\
5 & 7
\end{pmatrix}
\]

fails to be invertible.

**Solution**

The “bad” values of \(\lambda\) are the ones which make the determinant vanish. Well,

\[
\begin{vmatrix}
1 - 4\lambda & 2 - 5\lambda \\
2 - 5\lambda & 3 - 7\lambda
\end{vmatrix}
= (3 - 19\lambda + 28\lambda^2) - (4 - 20\lambda + 25\lambda^2)
= -1 + \lambda + 3\lambda^2.
\]

Thus \(\lambda = \frac{-1 \pm \sqrt{1 + 4 \cdot 3 \cdot 28}}{6} = \frac{-1 \pm \sqrt{341}}{6}\) are the bad values of \(\lambda.\)
Problem 8

Solve the following system of equations for \( x \) using Cramer’s rule:

\[
\begin{align*}
    w - x + 2y + 5z &= 10 \\
    -w + 2x - y + z &= 11 \\
    2w - 3x + y + z &= 12 \\
    3w + x - y - z &= 13 \\
\end{align*}
\]

You do not actually have to calculate \( x \). Just express \( x \) as a quotient of four-by-four determinants, getting all the numbers in the right places.

Solution

\[
x = \frac{\begin{vmatrix}
1 & 10 & 2 & 5 \\
-1 & 11 & -1 & 1 \\
2 & 12 & 1 & 1 \\
3 & 13 & -1 & -1 \\
\end{vmatrix}}{egin{vmatrix}
1 & -1 & 2 & 5 \\
-1 & 2 & -1 & 1 \\
2 & -3 & 1 & 1 \\
3 & 1 & -1 & -1 \\
\end{vmatrix}}
\]

Problem 9

Evaluate the determinant

\[
\begin{vmatrix}
-1 & 1 & 0 & 0 \\
0 & -1 & 1 & 0 \\
0 & 0 & -1 & 1 \\
1 & 2 & 3 & 4 \\
\end{vmatrix}
\]

by hand.

Solution

\[
\begin{align*}
    &\begin{vmatrix}
-1 & 1 & 0 & 0 \\
0 & -1 & 1 & 0 \\
0 & 0 & -1 & 1 \\
1 & 2 & 3 & 4 \\
\end{vmatrix} \\
    &= -\begin{vmatrix}
-1 & 1 & 0 \\
0 & -1 & 1 \\
2 & 3 & 4 \\
\end{vmatrix} \begin{vmatrix}
1 & 0 & 0 \\
-1 & 1 & 0 \\
0 & -1 & 1 \\
\end{vmatrix} \\
    &= -\begin{vmatrix}
-1 & 1 \\
3 & 4 \\
\end{vmatrix} \begin{vmatrix}
1 & 0 \\
-1 & 1 \\
\end{vmatrix} - 1 \\
    &= -4 - 3 - 2 - 1 = -10.
\end{align*}
\]
Problem 10

Find positive numbers $A$ and $\phi$ such that

$$A \cos(5t - \phi) = -3 \cos(5t) - 7 \sin(5t).$$

Remember that $\cos(u - v) = \cos(u) \cos(v) + \sin(u) \sin(v)$.

Solution

Using the trig identity of which we were reminded, we have

$$A \cos(5t - \phi) = A \cos(5t) \cos(\phi) + A \sin(5t) \sin(\phi).$$

To solve the problem we need to have

$$A \cos(\phi) = -3 \quad \text{and} \quad A \sin(\phi) = -7.$$

This only happens if $(A, \phi)$ is a representation in polar coordinates of the point $(-3, -7)$ in the cartesian plane. Thus

$$A = \sqrt{3^2 + (-7)^2} = \sqrt{58} = 7.6158 \quad \text{and} \quad \phi = \arctan(7/3) + \pi = 4.3075$$

do the job. We had to add $\pi$ because the point $(-3, -7)$ is in quadrant III rather than quadrant I.

Problem 11

Find positive numbers $A$ and $\phi$ such that

$$A \sin(4t + \phi) = 3 \sin(4t) + \cos(4t).$$

Remember that $\sin(u + v) = \sin(u) \cos(v) + \cos(u) \sin(v)$.

Solution

Using the trig identity of which we were reminded we have

$$A \sin(4t + \phi) = A \sin(4t) \cos(\phi) + A \cos(4t) \sin(\phi).$$

To solve the problem we need to have

$$A \cos(\phi) = 3 \quad \text{and} \quad A \sin(\phi) = 1.$$

This only happens if $(A, \phi)$ is a representation in polar coordinates of the point $(3, 1)$ in the cartesian plane. Thus

$$A = \sqrt{3^2 + 1^2} = \sqrt{10} = 3.1622 \quad \text{and} \quad \phi = \arctan(1/3) = 0.3218$$

do the job.

Problem 12

Fill in the blanks.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 3 & A & B & C & 4 \\ 4 & 7 & 9 & 8 & 8 \\ 1 & 1 & 0 & 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ -5 & 1 & 1 & 0 & 0 \\ -17 & 5 & 1 & 1 & 0 \\ 30 & -9 & -3 & -1 & 1 \\ -6 & 2 & 1 & 0 & -1 \end{bmatrix}$$

Find $A$, $B$ and $C$. Explain precisely where you got the equations you needed to solve for $A$, $B$ and $C$. 

Solution

Let us imagine what happens when we do the following matrix multiplication:

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 & 1 \\
3 & A & B & C & 4 \\
4 & 7 & 9 & 8 & 8 \\
1 & 1 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
-1 & 1 & 0 & 0 & 0 \\
-5 & 1 & 1 & 0 & 0 \\
-17 & 5 & 1 & 1 & 0 \\
30 & -9 & -3 & -1 & 1 \\
-6 & 2 & 1 & 0 & -1
\end{bmatrix}
\]

The last three entries of the third row of the product work out to be

\[4 + A + B - 3C, B - C, -4 + C\]

respectively, and these should equal

\[1, 0, 0\]

respectively. So we get equations

\[4 + A + B - 3C = 1, B - C = 0, -4 + C = 0\]

for \(A, B\) and \(C\). Solving these, we get \(B = C = 4\) and \(A = 5\).

Problem 13

Solve the initial value problem

\[y'' + 9y = 9H(t - \pi),\ y(0) = 0,\ y'(0) = 1\]

using Laplace transforms.

Solution

\[s^2Y - 1 + 9Y = \frac{9e^{-\pi s}}{s}\]

\[Y = \frac{1}{s^2 + 9} + \frac{9e^{-\pi s}}{s(s^2 + 9)}\]

\[\frac{9}{s(s^2 + 9)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 9}\]

\[9 = (A + B)s^2 + Cs + 9A \Rightarrow A = 1, B = -1, C = 0\]

\[\frac{9}{s(s^2 + 9)} = \frac{1}{s} - \frac{s}{s^2 + 9}\]

\[Y = \frac{1}{3} \frac{3}{s^2 + 9} + e^{-\pi s} \left( \frac{1}{s} - \frac{s}{s^2 + 9} \right)\]

\[y = \frac{1}{3} \sin(3t) + H(t - \pi)(1 - \cos(3(t - \pi)))\]

Problem 14

Solve the initial value problem \(y'' + y' = -32, y(0) = 200\) and \(y'(0) = 0\) by any method learned in class.
Solution

\[ y'' + y' = -32 \Rightarrow (e^t y')' = -32e^t \Rightarrow e^t y' = -32e^t + C \Rightarrow y' = -32 + C_1 e^{-t} \]

0 = y'(0) = -32 + C_1 \Rightarrow C_1 = 32

\[ y' = -32 + 32e^{-t} \]

\[ y = -32t - 32e^{-t} + C_2 \]

200 = y(0) = -32 + C_2 \Rightarrow C_2 = 232.

\[ y = -32t - 32e^{-t} + 232. \]

Problem 15

Solve the initial value problem

\[ \frac{dy}{dt} = -5(y - 2)(y + 1), \quad y(0) = 1. \]

Solution

\[ \frac{dy}{dt} = -5dt \]

\[ \frac{dy}{y - 2} - \frac{dy}{y + 1} = \frac{3dy}{(y - 2)(y + 1)} = -15dt \]

\[ \ln(y - 2) - \ln(y + 1) = -15t + \ln C \]

\[ \frac{y - 2}{y + 1} = Ce^{-15t} \]

\[ \frac{1 - 2}{1 + 1} = C = -\frac{1}{2} \]

\[ \frac{y - 2}{y + 1} = -\frac{1}{2} e^{-15t} \]

\[ y - 2 = -\frac{y}{2} e^{-15t} - \frac{1}{2} e^{-15t} \]

\[ 2y - 4 = -y e^{-15t} - e^{-15t} \]

\[ y(2 + e^{-15t}) = 4 - e^{-15t} \]

\[ y = \frac{4 - e^{-15t}}{2 + e^{-15t}} \]

Problem 16

At 3am in the morning the police discover a corpse in a park. The temperature in the park was 40°F. The temperature of the corpse at 3am was 85°F. Later, at 5am, the temperature was 75°F. Use Newton’s law of cooling to find a formula for the temperature of the corpse at time \( t \) in hours after 3am. Start from the differential equation. Then use the formula you found in order to estimate the time of death. Recall that normal (live) body temperature is 98.6°F.
Solution

Let $S$ be the temperature of the corpse at time $t$. Then

$$\frac{dS}{dt} = k(40 - S), \quad S(0) = 85, \quad S(2) = 75.$$

The problem can be rewritten

$$S' + kS = 40k, \quad S(0) = 85, \quad S(2) = 75.$$

We solve using the methods for first order linear equations.

$$(Se^{kt})' = 40ke^{kt} \Rightarrow Se^{kt} = 40e^{kt} + C \Rightarrow S = 40 + Ce^{-kt}$$

$S(0) = 85 = 40 + C \Rightarrow C = 45.$

$S(t) = 40 + 45e^{-kt}$

$S(2) = 40 + 45e^{-2k} = 75 \Rightarrow \frac{35}{45} = e^{2k} = \frac{7}{9} \Rightarrow e^{k} = \left(\frac{7}{9}\right)^{1/2}.$

$S(t) = 40 + 45 \left(\frac{7}{9}\right)^{TOD/2}$

$98.6 = 40 + 45 \left(\frac{7}{9}\right)^{TOD/2} \Rightarrow 48.6 = \frac{48.6}{45} = \left(\frac{7}{9}\right)^{TOD/2}$

$TOD = 2\ln(48.6/45)/\ln(7/9) = -.612 \text{ hours} = -37 \text{ minutes}$

Final answer: 2:23am

Problem 17

Solve the initial value problem $\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$ and $y(1/4) = 1/4.$

Solution

$$\frac{dy}{\sqrt{y}} = -\frac{dx}{\sqrt{x}}$$

$2\sqrt{y} = -2\sqrt{x} + C$

$\sqrt{x} + \sqrt{y} = C/2$

$\sqrt{1/4} + \sqrt{1/4} = 1 = C/2$

Therefore $\sqrt{x} + \sqrt{y} = 1$ and hence $y = (1 - \sqrt{x})^2.$

Problem 18

Find positive numbers $A$ and $\phi$ such that $-1 + 2i = Ae^{i\phi}.$

Solution

$(A, \phi)$ are the polar coordinates of the point with Cartesian coordinates $(-1, 2).$

Thus $A = \sqrt{(-1)^2 + 2^2} = \sqrt{5} = 2.2361$ and $\phi = \arctan(-2) + \pi = 2.0344.$
Problem 19

Find the general solutions of the following differential equations. Your answer
should not involve any complex numbers.
(i) \( y'' + 5y' + 4y = 0. \)
(ii) \( y'' + 6y' + 9y = 0. \)
(iii) \( y'' + 9y = 0. \)
(iv) \( y'' + 4y' + 29y = 0. \)

Solution
\[(i) \quad r^2 + 5r + 4 = (r + 1)(r + 4) = 0 \Rightarrow r = -1, -4, \text{ so } y = C_1 e^{-t} + C_2 e^{-4t} \]
\[(ii) \quad r^2 + 6r + 9 = (r + 3)^2 = 0 \Rightarrow r = -3 \text{ (double root), so } y = C_1 e^{-3t} + C_2 t e^{-3t} \]
\[(iii) \quad r^2 + 9 = 0 \Rightarrow r = \pm 3i, \text{ so } y = C_1 \cos(3t) + C_2 \sin(3t). \]
\[(iv) \quad r^2 + 4r + 29 = 0 \Rightarrow r = -2 \pm i\sqrt{5}, \text{ so } y = e^{-2t}(C_1 \cos(5t) + C_2 \sin(5t)). \]

Problem 20

A business development is expected to start paying off in 10 years. Once the
development starts paying off, it is expected to generate income at a rate of $35000
per year for 10 years, then $70000 per year for 10 years, and after that no income.
Use Laplace transforms to evaluate the present value of this business development
at a rate of interest \( r \).

Solution
The income stream \( f(t) \) described verbally above can be described in mathematical symbols as follows:
\[ f(t) = \begin{cases} 
0 & \text{for } 0 \leq t < 10, \\
35000 & \text{for } 10 \leq t < 20, \\
70000 & \text{for } 20 \leq t < 30, \\
0 & \text{for } t \geq 30. 
\end{cases} \]
In terms of Heaviside functions we have
\[ f(t) = 35000(H(t - 10) - H(t - 20)) + 70000(H(t - 20) - H(t - 30)). \]
Thus
\[ F(s) = \mathcal{L}(f(t)) = 35000 \frac{e^{-10s}}{s} + 35000 \frac{e^{-20s}}{s} - 70000 \frac{e^{-30s}}{s}. \]
The final answer is gotten by plugging in \( r \) for \( s \).
\[ 35000 \frac{e^{-10r}}{r} + 35000 \frac{e^{-20r}}{r} - 70000 \frac{e^{-30r}}{r}. \]

Problem 21

At a rate of interest of 18% per year compounded continuously, what steady
rate of payment on a loan of $10000 leaves you owing $15000 after five years? Use
differential equations to solve the problem.
Solution

Let \( y = y(t) \) be the amount you owe at time \( t \). The problem we want to solve is

\[
y' = .18y - M, \quad y(0) = 10000, \quad y(5) = 15000,
\]

where \( M \) is the rate of payment in dollars per year. Well,

\[
y' - .18y = -M \Rightarrow (e^{-0.18t}y)' = -Me^{-0.18t} \Rightarrow e^{-0.18t}y = Me^{-0.18t}/.18 + C
\]

\[
y(0) = M/.18 + C = 10000, \quad y(5) = M/.18 + Ce^{-.9} = 15000.
\]

Solving for \( M \) and \( C \), we get \( C(e^{-.9} - 1) = 5000 \) and hence \( C = 3425.59 \).

In turn \( M = .18(10000 - 3425.59) = 1183.39 \). The rate of payment is $1183.39 per year or, just about $100 per month.

Problem 22

Find the general solution of the system

\[
\begin{bmatrix} x'' \\ y'' \end{bmatrix} + \begin{bmatrix} 23 & 4 \\ 4 & 17 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 375 \\ 750 \end{bmatrix}
\]

by diagonalizing the coefficient matrix and uncoupling the system of differential equations.

Solution

\[
\begin{vmatrix} 23 - t \\ 417 - t \end{vmatrix} = t^2 - 40t + 23 \cdot 17 - 16 = 375 - 40t + t^2 = (t - 15)(t - 25).
\]

Let

\[
A = \begin{bmatrix} 23 & 4 \\ 4 & 17 \end{bmatrix}, \quad P = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 25 & 0 \\ 0 & 15 \end{bmatrix}.
\]

Then \( AP = PD \). We now substitute

\[
\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.
\]

\[
\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} u'' \\ v'' \end{bmatrix} + \begin{bmatrix} 23 & 4 \\ 4 & 17 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 375 \\ 750 \end{bmatrix}
\]

\[
\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} u'' \\ v'' \end{bmatrix} + \begin{bmatrix} 25 & 0 \\ 0 & 15 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 375 \\ 750 \end{bmatrix}
\]

\[
\begin{bmatrix} u'' \\ v'' \end{bmatrix} + \begin{bmatrix} 25 & 0 \\ 0 & 15 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 375 \\ 750 \end{bmatrix} = \begin{bmatrix} 300 \\ 225 \end{bmatrix}.
\]

The new equations are uncoupled:

\[
u'' + 25u = 300, \quad v'' + 15v = 225
\]

For these we know how to write down general solutions:

\[
u = C_1 \cos(5t) + C_2 \sin(5t) + 12, \quad v = C_3 \cos(\sqrt{15}t) + C_4 \sin(\sqrt{15}t) + 15.
\]

Then for the original problem the solution is

\[
\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C_1 \cos(5t) + C_2 \sin(5t) + 12 \\ C_3 \cos(\sqrt{15}t) + C_4 \sin(\sqrt{15}t) + 15 \end{bmatrix}.
\]
Laplace transforms

\[ f(t) \quad \mathcal{L}\{f(t)\} = F(s) \]

1 \[ \frac{1}{s} \]

\( t \) \[ \frac{1}{s^2} \]

\( t^n \) \[ \frac{n!}{s^{n+1}} \]

\( \sin bt \) \[ \frac{b}{s^2 + b^2} \]

\( \cos bt \) \[ \frac{s}{s^2 + b^2} \]

\( e^{at} \) \[ \frac{1}{s - a} \]

\( te^{at} \) \[ \frac{1}{(s - a)^2} \]

\( e^{at} \sin bt \) \[ \frac{b}{(s - a)^2 + b^2} \]

\( e^{at} \cos bt \) \[ \frac{s - a}{(s - a)^2 + b^2} \]

\( \delta(t) \) \[ 1 \]

\( \delta(t - a) \) \[ e^{-sa} \]

\( e^{at} f(t) \) \[ F(s - a) \]

\( f(t - a) H(t - a) \) \[ e^{-as} F(s) \]

\( f(t) H(t - a) \) \[ e^{-as} \mathcal{L}\{f(t + a)\} \]

\( H(t - a) \) \[ \frac{e^{-as}}{s} \]

\( f'(t) \) \[ sF(s) - f(0) \]

\( f''(t) \) \[ s^2 F(s) - sf(0) - f'(0) \]