READ THESE INSTRUCTIONS

Check to make sure that this test booklet has 13 pages: a cover page and 12 pages of problems. Make sure you have all the pages. Also make sure that you get a formula sheet (consisting mostly of Laplace transform formulas), which will be handed out separately from the exam.

Fill in all the blanks above. Put your name on every page of the booklet because sometimes pages become separated. Do all your writing in this booklet, using back sides of pages if necessary—only work written here will be credited. You have 3 hours to work on this examination. Point values of problems are as indicated. There 200 points in total on the exam. Books and notes are not permitted, but calculators are permitted. No laptops! All other electronic devices, for example, iPods and cellphones, must be turned off and stowed under your seat.

We expect exact answers in response to each exam question. (There are no questions for which the answers involve numerical approximations.) A correct final answer unsupported by any calculations, explanation, or algebraic work will receive no credit. An incorrect final answer supported by substantially correct calculations and explanation might still receive partial credit. For full credit you must show the correct final answer, you must give a reasonably neat and logical account of how you got that answer, and you must use the methods taught in this course. Miraculous answers, i.e., answer appearing with no visible calculations to support them, will receive no credit. Standard matrix operations on your calculator such as \texttt{rref}, determinant and matrix inverse are legal for solving all exam problems, provided that you indicate each time you have used such a command to get an answer—otherwise your answers will be considered miraculous and will not receive credit. More advanced commands, such as for example, \texttt{evec} for getting eigenvectors are not legal—they cannot be used to justify your work on the exam. (But no one is stopping you from using such commands to check your work.)

In summary: **Show all your work and justify your answers completely.**

Note: In Spring 2011 I edited out the extra white space so that this sample exam would print out in just a few pages. Ordinarily the problems are printed only one or two to a page.
**Problem 1 (10 points)**

Express the vector \[
\begin{bmatrix}
26 \\
2 \\
12 \\
11
\end{bmatrix}
\]
as a linear combination of the vectors \[
\begin{bmatrix}
5 \\
2 \\
2 \\
2
\end{bmatrix},
\begin{bmatrix}
0 \\
2 \\
1 \\
0
\end{bmatrix},
\begin{bmatrix}
2 \\
0 \\
1 \\
1
\end{bmatrix}.
\]

**Solution**

\[
\begin{bmatrix}
26 \\
2 \\
12 \\
11
\end{bmatrix} = 4 \begin{bmatrix}
5 \\
2 \\
2 \\
2
\end{bmatrix} - 6 \begin{bmatrix}
0 \\
2 \\
1 \\
0
\end{bmatrix} + 3 \begin{bmatrix}
2 \\
0 \\
1 \\
1
\end{bmatrix}.
\]

**Problem 2 (10 points)**

Consider the following four vectors in \(\mathbb{R}^3\):

\[
\begin{bmatrix}
6 \\
-1 \\
-2
\end{bmatrix},
\begin{bmatrix}
-1 \\
2 \\
-2
\end{bmatrix},
\begin{bmatrix}
16 \\
1 \\
-10
\end{bmatrix},
\begin{bmatrix}
1 \\
-9 \\
12
\end{bmatrix}.
\]

Which sets of three of these vectors are linearly independent?

**Solution**

We can use determinants to decide linear independence since the number of vectors to be tested and the number of their entries is the same. A nonzero determinant means independent; a zero determinant means dependent.

\[
\begin{vmatrix}
6 & -1 & 16 \\
-1 & 2 & 1 \\
-2 & -2 & -10
\end{vmatrix} = 0
\]

\[
\begin{vmatrix}
6 & -1 & 1 \\
-1 & 2 & -9 \\
-2 & -2 & 12
\end{vmatrix} = 12 \neq 0
\]

\[
\begin{vmatrix}
6 & 16 & 1 \\
-1 & 1 & -9 \\
-2 & -10 & 12
\end{vmatrix} = 24 \neq 0
\]

\[
\begin{vmatrix}
-1 & 16 & 1 \\
2 & 1 & -9 \\
-2 & -10 & 12
\end{vmatrix} = -36 \neq 0
\]

**Problem 3 (10 points)**

The graph of the equation 

\[6x^2 - 4xy + 9y^2 = 77\]
is an ellipse “tilted” with respect to the \(x\)- and \(y\)-axes. Find the points on the ellipse closest to the origin. Given an exact answer. Use eigenvalues and eigenvectors to get your answer.
Solution

All farthest and closest points are at the intersection points between the ellipse and its principal axes. So we will find those intersections and see which ones are closest.

\[ f(x, y) = 6x^2 - 4xy + 9y^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 6 & -2 \\ -2 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \]

\[ f(-t, 2t) = t^2 \begin{bmatrix} -1 & 2 \\ -2 & 9 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -10t^2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = 50t^2 \Rightarrow t = \pm \sqrt{\frac{77}{50}}, \]

and so two candidates for closest point are \( \pm \sqrt{\frac{77}{50}}(-1, 2). \) Similarly

\[ f(2s, s) = 25s^2 = 77 \Rightarrow s = \pm \sqrt{\frac{77}{25}}, \]

and so two more candidates for closest point are \( \pm \sqrt{\frac{77}{25}}(1, 2). \) Clearly, the first two of these four points are the closest. The astute reader will see that the closest points were associated to the largest eigenvalue.

Problem 4 (10 points)

Using the Euclidean algorithm as presented in class, find a two-by-two matrix \( U \) with integer entries such that

\[ \det U = \pm 1, \quad \text{and} \quad U \begin{bmatrix} 42 \\ 179 \end{bmatrix} = \begin{bmatrix} \text{greatest common divisor of 42 and 178} \\ 0 \end{bmatrix}. \]

Solution

\[ \begin{bmatrix} 42 & 1 & 0 \\ 178 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 42 & 1 & 0 \\ 10 & -4 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 17 & -4 \\ 10 & -4 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 17 & -4 \\ 0 & -89 & 21 \end{bmatrix}. \]

Check:

\[ \begin{bmatrix} 17 & -4 \\ -89 & 21 \end{bmatrix} \begin{bmatrix} 42 \\ 178 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad \det \begin{bmatrix} 17 & -4 \\ -89 & 21 \end{bmatrix} = 1. \]

Problem 5 (10 points)

Write out the inverse of the matrix

\[ \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix} \]

using the cofactor rule. Do not evaluate any determinants! Write out the answer as a three-by-three matrix filled with two-by-two determinants, all divided by a three-by-three determinant. The point is to get all the numbers in the right places and all the signs right so that I can see you know the rule.
Solution

\[
\begin{pmatrix}
+ & 5 & 6 & - & 4 & 6 & + & 4 & 5 \\
8 & 10 & - & 7 & 10 & + & 7 & 8 \\
- & 2 & 3 & + & 1 & 3 & - & 1 & 2 \\
8 & 10 & + & 7 & 10 & - & 7 & 8 \\
+ & 2 & 3 & - & 1 & 3 & + & 1 & 2 \\
5 & 6 & - & 4 & 6 & + & 4 & 5
\end{pmatrix}
\]

Problem 6 (10 points)

Here is the general solution of a three-by-three system \( x' = Ax \):

\[
C_1 e^{-t} \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} + C_3 e^t \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.
\]

Find the matrix \( A \).

Solution

The diagonalization of \( A \) must have been \( AP = PD \) with

\[
P = \begin{pmatrix} 3 & 2 & -1 \\ 2 & 2 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

and hence

\[
A = PDP^{-1} = \begin{pmatrix} 1 & -6 & -6 \\ 2 & -7 & -6 \\ -1 & 4 & 4 \end{pmatrix}.
\]

Problem 7 (15 points)

A brine tank of huge capacity initially holds 100 gallons of water with 23 pounds of salt dissolved in the water. Brine at a concentration of 2 pounds of salt per gallon is then piped into the tank at a rate of 5 gallons per minute. Meanwhile, well-mixed brine is drained from the tank at the rate of 2 gallons per minute. Find a formula for the amount of salt in the tank at time \( t \).

Solution

\[
S(0) = 23, \quad \frac{dS}{dt} = \left( \frac{2 \text{ lbs}}{\text{gal}} \right) \left( \frac{5 \text{ gal}}{\text{min}} \right) - \left( \frac{S}{100 + (5 - 2)t} \right) \left( \frac{2 \text{ lbs}}{\text{gal}} \right) \left( \frac{2 \text{ gal}}{\text{min}} \right)
\]

After “clean-up”:

\[
\frac{dS}{dt} + \frac{2S}{100 + 3t} = 10
\]

Integrating factor:

\[
I = \exp \left( \int \frac{2dt}{100 + 3t} \right) = \exp \left( \frac{2}{3} \log(100 + 3t) \right) = (100 + 3t)^{2/3}.
\]

\[
((100 + 3t)^{2/3}S)' = 10(100 + 3t)^{2/3}
\]
\[(100 + 3t)^{2/3} S = \frac{3}{5} \cdot 10(100 + 3t)^{5/3} + C = 2(100 + 3t)^{5/3} + C \]
\[C = 100^{2/3} \cdot 23 - 2 \cdot 100^{5/3} = -177 \cdot 100^{2/3}, \quad S = 2(100 + 3t) + C(100 + 3t)^{-2/3} \]

**Problem 8 (10 points)**

Solve the initial value problem
\[
\frac{dy}{dt} = -3(y - 1)(y - 4), \quad y(0) = 0.
\]

For what values of \(t\) is the solution defined?

**Solution**

Separate variables:
\[
\frac{dy}{(y - 1)(y - 4)} = -3dt
\]

Partial fractions by “cover-up”:
\[
\frac{1}{(y - 1)(y - 4)} = \frac{1/3}{y - 4} - \frac{1/3}{y - 1}
\]

Rewrite differential equation:
\[
\frac{1}{3} \frac{dy}{y - 4} - \frac{1}{3} \frac{dy}{y - 1} = -3dt
\]

Clear fractions:
\[
\frac{dy}{y - 4} - \frac{dy}{y - 1} = -9dt
\]

Integrate on both sides and exponentiate:
\[
\ln(y - 4) - \ln(y - 1) = -9t + \ln C \Rightarrow \frac{y - 4}{y - 1} = Ce^{-9t}.
\]

Plug in \(t = 0\) to get \(C = 4\).

Solve for \(y\):
\[
\frac{y - 4}{y - 1} = 4e^{-9t} \Rightarrow y - 4 = 4ye^{-9t} - 4e^{-9t} \Rightarrow y(1 - 4e^{-9t}) = 4 - 4e^{-9t} \Rightarrow y = \frac{4 - 4e^{-9t}}{1 - 4e^{-9t}}.
\]

Find when denominator vanishes: \(1 - 4e^{-9t} = 0 \Rightarrow t = \frac{\ln 4}{9}\).

Finally, the solution is defined only for \(t \neq \frac{\ln 4}{9}\). And according to some schools of thought we really should say that the solution exists only for \(t < \frac{\ln 4}{9}\) because the connection with the initial condition \(y(0) = 0\) is broken at \(t = \frac{\ln 4}{9}\).

**Problem 9 (9 points)**

Use Euler’s formula to evaluate the indefinite integral
\[
\int \sin(7t) \cos(9t) \sin(11t) dt.
\]

You do not have to simplify your final answer but you do have to complete the integration.
Solution
Let \( z = e^{it} \). Then the integrand equals
\[
\frac{1}{2i}(z^7 - z^{-7}) \frac{1}{2}(z^9 + z^{-9}) \frac{1}{2i}(z^{11} - z^{-11})
\]
\[
= \frac{1}{8}(-z^{27} + z^5 - z^9 + z^{13} + z^{-13} - z^{-9} + z^{-5} - z^{-27}).
\]
So the integral equals
\[
\frac{1}{8}(-\frac{z^{27}}{27i} + \frac{z^5}{5i} - \frac{z^9}{9i} + \frac{z^{13}}{13i} + \frac{z^{-13}}{-13i} - \frac{z^{-9}}{-9i} + \frac{z^{-5}}{-5i} - \frac{z^{-27}}{-27i})
\]
and after simplification (which was not requested) becomes
\[
-\frac{1}{108} \sin(27t) + \frac{1}{20} \sin(5t) - \frac{1}{36} \sin(9t) + \frac{1}{52} \sin(13t).
\]
(Plus, of course, the usual constant of integration.)

Problem 10 (10 points)

In this problem we consider the Fibonacci rabbit reproduction model with one change: we assume that each adult breeding pair produces two baby breeding pairs per time unit. Starting with one baby breeding pair and no adult breeding pairs at time \( n = 1 \), find the difference equation satisfied by the number \( y_n \) of rabbits at time \( n \) and find a closed form formula for \( y_n \). (If you are doing the problem correctly you will not see any ugly square roots. The formula for \( y_n \) is quite simple.)

Solution

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b_n )</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>6</td>
<td>10</td>
<td>22</td>
<td>42</td>
</tr>
<tr>
<td>( a_n )</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>11</td>
<td>21</td>
<td>43</td>
</tr>
<tr>
<td>( y_n )</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>11</td>
<td>21</td>
<td>43</td>
<td>85</td>
</tr>
</tbody>
</table>

\[ y_{n+2} = y_{n+1} + 2y_n, \quad y_1 = 1, \quad y_2 = 1. \]

\[ r^2 = r + 2 \Rightarrow r^2 - r - 2 = 0 \Rightarrow (r - 2)(r + 1) = 0 \Rightarrow r = 2, -1 \]

Guess:
\[ y_n = C_1 2^n + C_2 (-1)^n. \]

Impose initial conditions:
\[ 2C_1 - C_2 = 1, \quad 4C_1 + C_2 = 1 \Rightarrow C_1 = 1/3 \quad \text{and} \quad C_2 = -1/3. \]

Final answer:
\[ y_n = \frac{2^n - (-1)^n}{3}. \]

Problem 11 (10 points)

Find a particular solution of the differential equation
\[ y'' + 4y' + 13y = 40 \cos(3t) \]
by the method of undetermined coefficients. Then stop. You do not have to find the general solution.
Solution

We make the simplest type of guess, namely
\[ y_p = A \cos(3t) + B \sin(3t). \]

Then
\[ y'_p = 3B \cos(3t) - 3A \sin(3t), \]
\[ y''_p = -9A \cos(3t) - 9B \sin(3t) \]
\[ y''_p + 4y'_p + 13y_p = (-9A + 12B + 13A) \cos(3t) + (-9B - 12A + 13B) \sin(3t) \]
\[ = (4A + 12B) \cos(3t) + (-12A + 4B) \sin(3t) \]
Thus
\[ 4A + 12B = 40, \quad -12A + 4B = 0 \Rightarrow A = 1 \text{ and } B = 3. \]

Final answer:
\[ y_p = \cos(3t) + 3 \sin(3t). \]

Problem 12 (10 points)

Find the general solution of the system
\[ \begin{bmatrix} x'' \\ y'' \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ -4 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0. \]

Be careful! Note the second derivative. Note also that the matrix has one positive and one negative eigenvalue.

Solution

We have \( AP = PD \) with
\[ A = \begin{bmatrix} 3 & -4 \\ -4 & -3 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} -5 & 0 \\ 0 & 5 \end{bmatrix}. \]

Making the substitution
\[ \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix} \]
we get
\[ \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} u'' \\ w'' \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ -4 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix} = 0 \]
equivalently
\[ \begin{bmatrix} u'' \\ w'' \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & -4 \\ -4 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix} = 0 \]
and after multiplying out
\[ \begin{bmatrix} u'' \\ w'' \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix} = 0. \]

The transformed system is uncoupled and its general solution can be gotten by using what we already know about homogeneous linear second order differential equations with constant coefficients. We have
\[ \begin{bmatrix} u \\ w \end{bmatrix} = \begin{bmatrix} C_1 e^{\sqrt{5}t} + C_2 e^{-\sqrt{5}t} \\ C_3 \cos(\sqrt{5}t) + C_4 \sin(\sqrt{5}t) \end{bmatrix}. \]
Thus
\[
\begin{bmatrix}
  x \\
  y \\
  z
\end{bmatrix} = \begin{bmatrix} 1 & 2 \\
  2 & -1 \end{bmatrix} \begin{bmatrix} C_1 e^{\sqrt{5}t} + C_2 e^{-\sqrt{5}t} \\
  C_3 \cos(\sqrt{5}t) + C_4 \sin(\sqrt{5}t) \end{bmatrix}
\]
is the general solution of the given system.

**Problem 13 (10 points)**

Solve the initial value problem
\[
\begin{bmatrix}
  x' \\
  y' \\
  z'
\end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 \\
  0 & -2 & 1 \\
  1 & 1 & -3 \end{bmatrix} \begin{bmatrix} x \\
  y \\
  z \end{bmatrix}, \quad \begin{bmatrix} x(0) \\
  y(0) \\
  z(0) \end{bmatrix} = \begin{bmatrix} 3 \\
  6 \\
  9 \end{bmatrix}.
\]

To speed up your calculations, here is the diagonalization of the coefficient matrix:
\[
\begin{bmatrix}
  -2 & 0 & 1 \\
  0 & -2 & 1 \\
  1 & 1 & 3
\end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\
  1 & -1 & 1 \\
  1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\
  1 & -1 & 1 \\
  1 & 0 & -2 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\
  0 & -2 & 0 \\
  0 & 0 & -4 \end{bmatrix}.
\]

**Solution**

\[
\begin{bmatrix}
  x \\
  y \\
  z
\end{bmatrix} = \exp \left( t \begin{bmatrix} -2 & 0 & 1 \\
  0 & -2 & 1 \\
  1 & 1 & 3 \end{bmatrix} \right) \begin{bmatrix} 3 \\
  6 \\
  9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\
  1 & -1 & 1 \\
  1 & 0 & -2 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 & 0 \\
  0 & e^{-2t} & 0 \\
  0 & 0 & e^{-4t} \end{bmatrix}^{-1} \begin{bmatrix} 3 \\
  6 \\
  9 \end{bmatrix}.
\]

This formula could be simplified further but it is simplified enough.

**Problem 14 (10 points)**

Solve the initial value problem
\[
y'' + 6y' + 8y = 198 e^{7t}, \quad y(0) = 2, \quad y'(0) = 3
\]
by the method of undetermined coefficients.

**Solution**

\[
r^2 + 6r + 8 = (r + 2)(r + 4) = 0 \Rightarrow r = -2, -4.
\]

Thus
\[
y_h = C_1 e^{-2t} + C_2 e^{-4t}.
\]

Guess \( y_p = Ae^{7t} \).

\[
y_p'' + 6y_p' + 8y_p = (7^2 + 6 \cdot 7 + 8)y_p = 99Ae^{7t}
\]

Must have 99A = 198 and hence A = 2. General solution:
\[
y = 2e^{7t} + C_1 e^{-2t} + C_2 e^{-4t}.
\]

Impose initial conditions:
\[
y(0) = 2 + C_1 + C_2 = 2 \Rightarrow C_1 + C_2 = 0.
\]
\[
y'(0) = 14 - 2C_1 - 4C_2 = 3 \Rightarrow -2C_1 - 4C_2 = -11.
\]
Thus $C_1 = -\frac{11}{2}$ and $C_2 = \frac{11}{2}$. Final answer:
\[
y = 2e^{7t} - \frac{11}{2}e^{-2t} + \frac{11}{2}e^{-4t}.
\]

Problem 15 (10 points)

Find the inverse Laplace transform $f(t)$ of
\[
F(s) = \frac{1}{s^2} - \frac{e^{-3s}}{s^2 + 9}.
\]
Express $f(t)$ as a function “in pieces”. Finally, graph $f(t)$.

Solution

\[
\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = t,
\mathcal{L}^{-1}\left(\frac{e^{-3s}}{s^2 + 9}\right) = H(t-3)\sin(3t)
\]

\[
f(t) = t + (3-t)H(t-3) + H(t-5)\sin(3(t-5)).
\]

The graph is omitted from this solution.

Problem 16 (10 points)

Solve the initial value problem
\[
\begin{bmatrix}
x'
\end{bmatrix} = \begin{bmatrix} 5 & -3 \\ 6 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}.
\]

To speed things up for you a bit, take for granted that the eigenvalues are $2 \pm 3i$. Your final answer should not involve any complex numbers.

Solution

To crack the problem we need an eigenvector for the eigenvalue $2 + 3i$. Subtracting this eigenvalue from the diagonal of the coefficient matrix we get
\[
\begin{bmatrix} 3 - 3i & -3 \\ 6 & -3 - 3i \end{bmatrix}
\]
from which we get
\[
\begin{bmatrix} 3 \\ 3 - 3i \end{bmatrix}
or more simply
\[
\begin{bmatrix} 1 \\ 1 - i \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix}
\]
as an eigenvector. Let
\[
Q = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}.
\]
Then we have
\[
Q^{-1} \begin{bmatrix} 5 & -3 \\ 6 & -1 \end{bmatrix} Q = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}
\]
and hence
\[
\begin{bmatrix} 5 & -3 \\ 6 & -1 \end{bmatrix} = Q \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix} Q^{-1}.
\]
Finally, the solution is obtained as follows:

\[
\begin{bmatrix}
  x \\
  y
\end{bmatrix}
= \exp\left(t \begin{bmatrix}
  5 & -3 \\
  6 & -1
\end{bmatrix}\right) \begin{bmatrix}
  5 \\
  10
\end{bmatrix}

= \exp\left(tQ \begin{bmatrix}
  2 & 3 \\
  -3 & 2
\end{bmatrix} Q^{-1}\right) \begin{bmatrix}
  5 \\
  10
\end{bmatrix}

= Q \exp\left(t \begin{bmatrix}
  2 & 3 \\
  -3 & 2
\end{bmatrix} Q^{-1}\right) \begin{bmatrix}
  5 \\
  10
\end{bmatrix}

= Q e^{t \begin{bmatrix}
  \cos(3t) & \sin(3t) \\
  -\sin(3t) & \cos(3t)
\end{bmatrix} Q^{-1} \begin{bmatrix}
  5 \\
  10
\end{bmatrix}}.
\]

The solution could be simplified more but this is enough, because the matrix \( Q \) has been worked out explicitly, and the whole mess could be plugged into a computer to yield hard numbers.

**Problem 17 (10 points)**

Find the general solution of the inhomogeneous second order linear differential equation

\[ x^2 y'' - 2xy' + 2y = 1 \]

given that \( y = x^2 \) is a solution of the associated homogeneous equation.

**Solution**

This an application of the technique of reduction of order. We substitute \( y = ux^2 \) and get what we hope will be a simpler differential equation for \( u \).

\[
x^2(ux^2)'' - 2x(ux^2)' + 2ux^2 = 1
\]

\[
x^2(u''x^4 + 4u'x^3 + 2u) - 2x(u'x^2 + 2ux) + 2ux^2 = 1
\]

\[
u''x^4 + 4u'x^3 + 2ux^2 - 2u'x^3 - 4ux^2 + 2ux^2 = 1
\]

Happy Cancellation!

\[
u''x^4 + 2u'x^3 = 1
\]

\[
u'' + 2u'x^{-2} = x^{-4}
\]

The correct integrating factor here is \( \exp(\int 2x^{-2}dx) = \exp(2\log x) = x^2 \).

\[
(u'x^2) = x^{-2}
\]

\[
u'x^2 = -x^{-1} + C_1
\]

\[
u' = -x^{-3} + C_1x^{-2}
\]

\[
u = \frac{1}{2}x^{-2} - C_1x^{-1} + C_2
\]

Recalling that \( y = ux^2 \), the final answer (which gets rid of \( u \)) is

\[
y = \frac{1}{2} - C_1x + C_2x^2.
\]

The answer

\[
y = \frac{1}{2} + C_1x + C_2x^2
\]

would be okay too since \( C_1 \) is arbitrary anyhow.
Problem 18 (10 points)

Find the inverse Laplace transform of

\[
\frac{68s}{(s - 5)(s^2 + 6s + 13)}.
\]

Solution

\[
\frac{68s}{(s - 5)(s^2 + 6s + 13)} = \frac{A}{s - 5} + \frac{Bs + C}{s^2 + 6s + 13}
\]

\[
68s = (A + B)s^2 + (6A - 5B + C)s + (13A - 5C).
\]

\[
\begin{bmatrix}
1 & 1 & 0 \\
6 & 5 & 1 \\
13 & 0 & -5
\end{bmatrix}
\begin{bmatrix}
A \\
B \\
C
\end{bmatrix}
= \begin{bmatrix}
0 \\
68 \\
0
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
A \\
B \\
C
\end{bmatrix}
= \begin{bmatrix}
5 \\
-5 \\
13
\end{bmatrix}.
\]

Finally:

\[
L^{-1}\left(\frac{68s}{(s - 5)(s^2 + 6s + 13)}\right) = 5e^{5t} - 5e^{-3t}\cos(2t) + 14e^{-3t}\sin(2t).
\]

Problem 19 (15 points)

Solve the initial value problem

\[
y'' + 7y' + 12y = 12H(t - 3), \quad y(0) = 4, \quad y'(0) = -5
\]

using Laplace transforms.

Solution

\[
s^2Y - 4s + 5 + 7(sY - 4) + 12Y = \frac{12e^{-3s}}{s}
\]

\[
(s^2 + 7s + 12)Y - 4s - 23 = \frac{12e^{-3s}}{s}
\]

\[
Y = \frac{4s + 23}{(s + 3)(s + 4)} + \frac{12e^{-3s}}{s(s + 3)(s + 4)}
\]

Partial fractions by “coverup”.

\[
Y = \frac{11}{s + 3} - \frac{7}{s + 4} + e^{-3s}\left(\frac{1}{s} - \frac{4}{s + 3} + \frac{3}{s + 4}\right)
\]

Final answer:

\[
y = 11e^{-3t} - 7e^{-4t} + H(t - 3)(1 - 4e^{-3(t-3)} + 3e^{-4(t-3)}).
\]