Exam 2 Review

Solve the differential equations. Simplify your answers.

1. \( y' = e^{x+y} \)

Re-writing the differential equation as \( \frac{dy}{dx} = e^{x+y} \), we see that it is separable, so \( e^{-y}dy = e^x \, dx \). Integrating both sides, we have

\[
-e^{-y} = e^x + C \Rightarrow -\frac{1}{y} = \ln(e^x + C) \Rightarrow y = -\frac{1}{\ln(e^x + C)}.
\]

2. \( y' + 2xy = x \)

First we re-write the equation as \( \frac{dy}{dx} = x - 2xy = x(1 - 2y) \). We can see from this that the equation is separable, so

\[
\frac{dy}{1 - 2y} = x \, dx \Rightarrow \int \frac{dy}{1 - 2y} = \int x \, dx \Rightarrow -\frac{1}{2} \ln(1 - 2y) = \frac{x^2}{2} + C.
\]

Thus

\[
\ln(1 - 2y) = -x^2 + C \Rightarrow 1 - 2y = Ke^{-x^2} \Rightarrow y = \frac{1}{2} - Ke^{-x^2}
\]

Alternatively, we can treat this as a first order linear differential equation, in which case the integrating factor is \( e^{x^2} \). Multiplying both sides by the integrating factor, we get

\[
\left( e^{x^2} y \right)' = xe^{x^2} \Rightarrow \int \left( e^{x^2} y \right)' = \int xe^{x^2} \Rightarrow e^{x^2} y = \frac{1}{2} e^{x^2} + C.
\]

Solving for \( y \), we get the same answer as above, where \( C = -K \).

3. \( xy' = \left( \frac{\ln x}{y} \right)^2 \), with \( y(1) = 0 \)

\[
\frac{dy}{dx} = \frac{(\ln x)^2}{x} \cdot \frac{1}{y^2}
\]

So the equation is separable.

\[
\int y^2 \, dy = \int \frac{(\ln x)^2}{x} \, dx
\]
Using a substitution $u = \ln x$ for the integral on the right hand side, we get

$$\frac{y^3}{3} = \frac{(\ln x)^3}{3} + C$$

Plugging in the initial condition, we find $C = 0$, so the solution is $y^3 = (\ln x)^3$, or $y = \ln x$.

4. $y' + 2y = e^{5x}$

This equation is not separable, but it is a first order linear equation. The integrating factor is $e^{2x}$. Multiplying both sides by the integrating factor, we get

$$(e^{2x}y)' = e^{7x} \Rightarrow \int (e^{2x}y)' = \int e^{7x} \Rightarrow e^{2x}y = \frac{1}{7} e^{7x} + C$$

So $y = \frac{1}{7} e^{5x} + Ce^{-2x}$.

5. $y - x^2y' = 1$

We first re-write the equation as $x^2 \frac{du}{dx} = y - 1$, and so $\frac{du}{dx} = \frac{y-1}{x^2}$, and we have a separable equation.

$$\int \frac{dy}{y-1} = \int \frac{dx}{x^2} \Rightarrow \ln(y - 1) = -\frac{1}{x} + C$$

Solving for y, we get $y = Ke^{-\frac{1}{x}} + 1$.

Alternatively, we can treat this as a first order linear equation. We must first put it in standard form:

$$y' - \frac{1}{x^2}y = -\frac{1}{x^2}$$

The integrating factor is $e^{\frac{1}{x}}$, so multiplying both sides by the integrating factor we get

$$(e^{\frac{1}{x}}y)' = -\frac{e^{\frac{1}{x}}}{x^2}$$

Using the substitution $u = \frac{1}{x}$ to integrate the right hand side, we get

$$\int \left( e^{\frac{1}{x}}y \right)' = e^{\frac{1}{x}}y = \int -\frac{e^{\frac{1}{x}}}{x^2} = e^{\frac{1}{x}} + C$$
So $y = 1 + Ce^{-\frac{1}{x}}$ as before.

6. $y' + y = \sec^2(e^x)$
   This is a first order linear equation, with integrating factor $e^x$. So,
   $$ (e^x y)' = e^x \sec^2(e^x) \Rightarrow \int (e^x y)' = \int e^x \sec^2(e^x) = \tan(e^x) + C $$
   Where we used the substitution $u = e^x$ to integrate the integral on the right. Thus, $y = e^{-x} \tan(e^x) + Ce^{-x}$.

7. A vat with 500 gallons of beer contains 10% alcohol (by volume). Beer with 4% alcohol is pumped into the vat at a rate of 10 gal/min and the mixture is pumped out at the same rate. What is the percentage of alcohol (a) after $t$ minutes and (b) after 20 minutes?

   Let $A$ be the number of gallons of alcohol in the vat. The number of gallons of alcohol we start out with is 10% of 500, i.e. 50 gallons. The amount of alcohol going into the vat per minute is $0.04 \times 10 = 0.4$ gal/min, and the amount leaving is $\frac{A}{500} \times 10$ gal/min, where $\frac{A}{500}$ is the concentration of alcohol (the volume doesn’t change) and 10 is the rate at which the mixed liquid leaves the vat. So, the differential equation is
   $$ \frac{dA}{dt} = \text{rate in} - \text{rate out} = 0.4 - \frac{A}{50} = \frac{20 - A}{50} $$
   This is a separable differential equation, so
   $$ \int \frac{dA}{20 - A} = \int \frac{1}{50} dt \Rightarrow -\ln(20 - A) = \frac{1}{50} t + C $$
   Solving for $A$, we get
   $$ \ln(20 - A) = -\frac{1}{50} t + C \Rightarrow 20 - A = Ke^{-\frac{t}{50}} \Rightarrow A = 20 - Ke^{-\frac{t}{50}} $$
   Plugging in the initial condition, which is $A(0) = 50$, we find that $K = -30$, i.e. $A = 20 + 30e^{-\frac{t}{50}}$

   To solve part b, we let $t = 20$ and find that $A = 20 + 30e^{-\frac{2}{5}}$, which we could put into decimal form with a calculator.

8. $$ \begin{cases} x &= t^2 \\ y &= \sin t \end{cases} $$
a. Determine at least one point where the curve intersects itself (there are many).
b. Find the two tangent lines at this point.
c. Find the point(s) on the curve where the tangent is horizontal or vertical
d. Sketch the curve.

To find an intersection point, we need to find times \( t_1 \) and \( t_2 \) with \( t_1^2 = t_2^2 \) and \( \sin t_1 = \sin t_2 \). The first condition means \( t_1 = \pm t_2 \), so plugging this into the second condition we get \( \sin t_1 = \sin(-t_1) = -\sin(t_1) \). This means that \( \sin t_1 = 0 \), or \( t_1 = k\pi \), where \( k \) is any integer. So, one point where the curve intersects itself corresponds to \( t_1 = \pi, t_2 = -\pi \), which in cartesian coordinates is \( (\pi^2, 0) \).

The equation for the slope of the tangent line is

\[
\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{\cos t}{2t}
\]

When \( t = \pi \), the slope is \( -\frac{1}{2\pi} \), so (using point slope form) the equation for the tangent line is \( y = -\frac{1}{2\pi}(x - \pi^2) \). When \( t = -\pi \), the slope is \( \frac{1}{2\pi} \), so the equation for the tangent line is \( y = \frac{1}{2\pi}(x - \pi^2) \).

Horizontal tangents occur when \( \frac{dy}{dt} = 0 \), which happens when \( \cos t = 0 \), i.e. when \( t = \frac{\pi}{2} + k\pi \), again \( k \) is any integer. Vertical tangents occur when \( \frac{dx}{dt} = 0 \), which happens when \( 2t = 0 \), i.e. \( t = 0 \).

The graph looks like this:

9.

\[
\begin{cases}
  x = t^2 \\
  y = \frac{1}{4}t^4 - \frac{1}{2}t^2
\end{cases}
\]

a. Sketch the curve. Indicate direction with an arrow.
b. Are there horizontal tangents? If so, at what points?
c. Are there vertical tangents? If so, at what points?
d. Find the general equation for the tangent line at a given \( t \) value.

The curve looks like this:
Horizontal tangents occur when \( \frac{dy}{dt} = 0 = t^3 - t = t(t+1)(t-1) \), so the possibilities are \( t = 0, t = 1 \), and \( t = -1 \). However, at \( t = 0 \), \( \frac{dx}{dt} = 0 \) also, so the slope at that point is indeterminate. In fact, if we look at the general formula for the tangent, \( \frac{dy}{dx} = \frac{t^3 - t}{\frac{1}{2}t^2} = \frac{1}{2}(t^2 - 1) \), we see that there is no horizontal tangent at \( t = 0 \).

Vertical tangents occur when \( \frac{dx}{dt} = 0 = 2t \), so the only option is \( t = 0 \). However, from the general formula, we see again that the slope is finite at \( t = 0 \), so there are no vertical tangents.

We use point slope form to find the general equation for the tangent line at a given \( t \) value.

\[
y - y_0 = m(x-x_0) \Rightarrow y - \left( \frac{1}{4}t^4 - \frac{1}{2}t^2 \right) = \frac{1}{2}(t^2 - 1)(x-t^2) \Rightarrow y = \left( \frac{1}{2}t^2 - 1 \right)x - \frac{1}{4}t^4 + \frac{1}{2}t^2
\]

10. Calculate the arc length of the parametric curve \( x = e^t - t \), \( y = e^{\frac{t^2}{2}} \) with \( 0 \leq t \leq 1 \).

\[
L = \int_0^1 \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \, dt
\]

\[
= \int_0^1 \sqrt{(e^t - 1)^2 + (2e^\frac{t^2}{2})^2} \, dt = \int_0^1 \sqrt{e^{2t} - 2e^t + 1 + 4e^t dt} = \int_0^1 \sqrt{e^{2t} + 2e^t + 1} \, dt
\]

\[
= \int_0^1 (e^t + 1) \, dt = \left. e^t + t \right|_0^1 = e + 1 - 1 = e
\]

11. Calculate the surface area if the curve in #10 is rotated around the x axis.
\[ A = \int_0^1 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt \]

\[ = \int_0^1 2\pi(4e^{t/2})(e^t + 1)dt = 8\pi \int_0^1 e^{\frac{3t}{2}} + e^t \, dt = 8\pi \left[ \frac{2}{3} e^{\frac{3t}{2}} + 2e^t \right]_0^1 \]

12. Identify the curve by finding a cartesian equation for the curve: \( r = 2\sin \theta + 4\cos \theta. \)

\( r = 2\sin \theta + 4\cos \theta \Rightarrow r^2 = 2r\sin \theta + 4r\cos \theta \Rightarrow x^2 + y^2 = 2y + 4x \)

Completing the square, we get

\[ x^2 - 4x + 4 + y^2 - 2y + 1 = 5 \Rightarrow (x - 2)^2 + (y - 1)^2 = 5 \]

which is a circle centered at \((2, 1)\) with radius \(\sqrt{5}.\)

13. Graph the function given in polar coordinates: \( r = 1 + 3\cos \theta \)

14. Graph the function given in polar coordinates: \( r = \cos(3\theta) \)
15. Find the area inside both \( r = 2 + \cos \theta \) and \( r = 2 \).
First we graph the curves:

The area is the area inside the limacon between \( \frac{\pi}{2} \) and \( \frac{3\pi}{2} \), plus the area of the half circle, which is \( \frac{1}{2} \left( \frac{1}{2} \pi (2)^2 \right) = \pi \). So,

\[
A = \frac{1}{2} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (2 + \cos \theta)^2 d\theta + \pi = \\
\frac{1}{2} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 4 + 4 \cos \theta + (1 + \cos (2\theta)) d\theta + \pi = \frac{1}{2} \left[ 6\theta + 4 \sin \theta + \sin 2\theta \right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = \frac{1}{2} \left[ 6\pi + 4(-1 - 1) \right] = 3\pi - 4
\]

16. Find the area inside the loop of \( r = 2 + 4 \cos \theta \).
First we plot the curve:
To find the bounds of integration, we need to find the points where \( r = 0 \), i.e. \( 0 = 2 + 4 \cos \theta \Rightarrow \cos \theta = -\frac{1}{2} \Rightarrow \theta = \frac{2\pi}{3}, \frac{4\pi}{3} \). So, the area is

\[
A = \frac{1}{2} \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (2 + 4 \cos \theta)^2 d\theta = \frac{1}{2} \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} 4 + 16 \cos \theta + 8 + 8 \cos 2\theta d\theta
\]

\[
= \frac{1}{2} \left[ 12\theta + 16 \sin \theta + 4 \sin 2\theta \right]_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} = 4\pi - 2\sqrt{3}
\]

17. Find the area inside both \( r = \cos \theta \) and \( r = 2 \sin \theta \).

Again we plot the curves:

We need to find the point of intersection of the curves, which occurs when \( \cos \theta = 2 \sin \theta \Rightarrow \tan \theta = \frac{1}{2} \). For convenience, we’ll call this point \( \theta_0 \). The area inside both curves is the area inside the cosine curve from 0 to \( \theta_0 \), plus the area inside the sine curve from \( \theta_0 \) to \( \frac{\pi}{2} \).

\[
A = \frac{1}{2} \left[ \int_0^{\theta_0} \cos^2 \theta d\theta + \int_{\theta_0}^{\frac{\pi}{2}} 4 \sin^2 \theta d\theta \right]
\]
\[ A = \frac{1}{4} \left[ \int_0^{\theta_0} \frac{1}{2} (1 + \cos 2\theta) \, d\theta + \int_{\theta_0}^{\frac{\pi}{2}} 2 - 2 \cos 2\theta \, d\theta \right] \]

\[ \begin{align*}
&= \frac{1}{2} \left[ \left. \left( \theta + \frac{1}{2} \sin 2\theta \right) \right|_0^{\theta_0} + \left. \left( \theta - \frac{1}{2} \sin 2\theta \right) \right|_{\theta_0}^{\frac{\pi}{2}} \right] \\
&= \frac{1}{4} \left[ (\theta + \sin \theta \cos \theta)|_{0}^{\theta_0} + (\theta - \sin \theta \cos \theta)|_{\theta_0}^{\frac{\pi}{2}} \right] \\
&= \frac{1}{4} (\theta + \sin \theta \cos \theta)|_{0}^{\theta_0} + (\theta - \sin \theta \cos \theta)|_{\theta_0}^{\frac{\pi}{2}} \\
&= \frac{3}{4} \tan^{-1} \left( \frac{1}{2} \right) - \frac{3}{4} \left( \frac{2}{\sqrt{5}} \right) + \frac{\pi}{2} \approx 0.923
\]

Now, just like when we did trig substitution in chapter 7, we can use a right triangle to find \( \sin \theta_0 \) and \( \cos \theta_0 \). If \( \tan \theta_0 = \frac{1}{2} \), \( \sin \theta_0 = \frac{1}{\sqrt{5}} \) and \( \cos \theta_0 = \frac{2}{\sqrt{5}} \). So,

\[ A = \frac{1}{4} \left( \tan^{-1} \left( \frac{1}{2} \right) + \left( \frac{1}{\sqrt{5}} \right) \left( \frac{2}{\sqrt{5}} \right) \right) + \frac{\pi}{2} - \tan^{-1} \left( \frac{1}{2} \right) - \left( \frac{1}{\sqrt{5}} \right) \left( \frac{2}{\sqrt{5}} \right) \]

\[ = -\frac{3}{4} \tan^{-1} \left( \frac{1}{2} \right) - \frac{3}{4} \left( \frac{2}{\sqrt{5}} \right) + \frac{\pi}{2} \approx 0.923 \]