Group Testing:
How to find out what’s important in life

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Outline

1. Group Testing - Overview
2. Group Testing - Example
3. Group Testing - Strongly Selective Matrices
4. Group Testing - More examples and Applications
5. Compressed Sensing - Escaping the Binary World
Group Testing - Overview

- Find a small number of interesting items hidden in a large set.
  - Syphilis Testing [Dorfman 1943]

- Industrial Experiment Design [Sobel and Groll, 1959]

Test Requirements...
- Test large arbitrary groups of objects
- Tests must be sensitive to defectives isolated with (many) other non-defective items
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  - Test large arbitrary groups of objects
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Encode the problem in a **binary array**

\[ [0, 1, 0, 0, 1, ...] \]

Find the nonzero entries by **testing subsets** of the array

- Boolean $K \times N$ measurement matrix $M$
- Boolean signal $a \in \{0, 1\}^N$ containing $k$ ones
- All arithmetic Boolean ($+$ = OR, $*$ = AND)
- Identify the location of $k$ ones using $y = Ma$ measurements
- How small can we make $K$ and still recover $a$ using $y$?
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  ![Binary Array](image)

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  ![Testing Subsets](image)

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\[
\begin{align*}
&\text{Boolean } K \times N \text{ measurement matrix } M \\
&\text{Boolean signal } a \in \{0, 1\}^N \text{ containing } k \text{ ones} \\
&\text{All arithmetic Boolean (+ = OR, * = AND)} \\
&\text{Identify the location of } k \text{ ones using } y = Ma \text{ measurements} \\
&\text{How small can we make } K \text{ and still recover } a \text{ using } y? 
\end{align*}
\]
Group Testing - Overview

- Encode the problem in a **binary array**
- Find the nonzero entries by **testing subsets** of the array

![Diagram](image)

- Boolean $K \times N$ measurement matrix $\mathcal{M}$
- Boolean signal $\mathbf{a} \in \{0, 1\}^N$ containing $k$ ones
- All arithmetic Boolean ($+ = \text{OR}, \times = \text{AND}$)
- Identify the location of $k$ ones using $y = \mathcal{M}\mathbf{a}$ measurements
- How small can we make $K$ and still recover $\mathbf{a}$ using $y$?
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- **Boolean** $K \times N$ measurement matrix $\mathcal{M}$
- **Boolean signal** $\mathbf{a} \in \{0, 1\}^N$ containing $k$ ones
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**Group Testing - Example**

- \( \mathcal{M} \) is \( 5 \times 30 \), \( \mathbf{a} \) contains 1 nonzero entry.

<table>
<thead>
<tr>
<th>Bit</th>
<th>Matrix</th>
<th>Result</th>
</tr>
</thead>
</table>
| 0th   | \[
0 1 0 1 0 1 0 1 0 \\
0 0 1 1 0 0 1 1 0 \\
0 0 0 0 0 0 0 0 0 \\
0 0 0 0 0 0 0 0 0 \\
0 0 0 0 0 0 0 0 0 \\
\]
|       | \[
0 0 0 0 0 0 0 0 0 \\
0 0 0 0 0 0 0 0 0 \\
0 0 0 0 0 0 0 0 0 \\
0 0 0 0 0 0 0 0 0 \\
0 0 0 0 0 0 0 0 0 \\
\]
|       | \[
0 0 0 0 0 0 0 0 0 \\
0 0 0 0 0 0 0 0 0 \\
0 0 0 0 0 0 0 0 0 \\
0 0 0 0 0 0 0 0 0 \\
0 0 0 0 0 0 0 0 0 \\
\] | \[
0 0 0 0 0 0 0 0 0 \\
0 0 0 0 0 0 0 0 0 \\
0 0 0 0 0 0 0 0 0 \\
0 0 0 0 0 0 0 0 0 \\
0 0 0 0 0 0 0 0 0 \\
\] |
Example

Group Testing - Example

- $\mathcal{M}$ is $5 \times 30$, $\mathbf{a}$ contains 1 nonzero entry.

\[
\begin{align*}
\text{0th bit} & \quad \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & \ldots \end{pmatrix} \\
\text{1st bit} & \quad \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & \ldots \end{pmatrix} \\
\text{2nd bit} & \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & \ldots \end{pmatrix} \\
\text{3rd bit} & \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \end{pmatrix} \\
\text{4th bit} & \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \end{pmatrix}
\end{align*}
\]
Recovery is simple: The result is the position of 1 in binary.

QUIZ: Can we do better if we let our measurement matrix contains arbitrarily large integers?
**Group Testing - Example**

\[
\begin{pmatrix}
0 & 1 & 0 & 1 & 0 & 1 & 0 & \ldots \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & \ldots \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots 
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
0 \\
1 \\
0 \\
\ldots 
\end{pmatrix}
= 
\begin{pmatrix}
1 \\
1 \\
0 \\
0 \\
0 \\
\ldots 
\end{pmatrix}
\]

\(\leftarrow 0^\text{th} \text{ bit } = 1\)
\(\leftarrow 1^\text{st} \text{ bit } = 1\)
\(\leftarrow 2^\text{nd} \text{ bit } = 0\)
\(\leftarrow 3^\text{rd} \text{ bit } = 0\)
\(\leftarrow 4^\text{th} \text{ bit } = 0\)

- **Recovery is simple:** The result is the position of 1 in binary.
- **QUIZ:** Can we do better if we let our measurement matrix contains arbitrarily large integers?
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\[
\begin{pmatrix}
0 & 1 & 0 & 1 & 0 & 1 & 0 & \cdots \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & \cdots \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
0 \\
1 \\
\vdots
\end{pmatrix} =
\begin{pmatrix}
1 \\
1 \\
0 \\
0 \\
0 \\
\end{pmatrix}
\]

\( \leftarrow 0^{th} \text{ bit } = 1 \)
\( \leftarrow 1^{st} \text{ bit } = 1 \)
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- **Recovery is simple:** The result is the position of 1 in binary.
- **QUIZ:** Can we do better if we let our measurement matrix contains arbitrarily large integers?
Example

Group Testing - Example

\[
\begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & \ldots \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & \ldots \\
\end{pmatrix} = 3
\]

- Recovery is simple: The result is the position of 1 in binary.
- QUIZ: Can we do better if we let our measurement matrix contains arbitrarily large integers?
- YES!!!
Measurement Matrix Construction

A binary matrix $M$ is \textit{k}-strongly selective if for any column, $x$, and subset of columns containing at most $k$ elements, $X$, there exists a row in $M$ with a 1 in column $x$ and zeros in all of the other $X - \{x\}$ columns.
Measurement Matrix Construction

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$$
\begin{array}{ccccccc}
0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 1
\end{array}
$$
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$$
\begin{array}{cccccccc}
H & H & H & H & H \\
0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 \\
\end{array}
$$
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\begin{array}{cccccccc}
H & H & H & H & H & H & H & H \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
* & * & * & * & * & * & * & * \\
1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
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A binary matrix $M$ is \textit{k-strongly selective} if for any column, $x$, and subset of columns containing at most $k$ elements, $X$, there exists a row in $M$ with a 1 in column $x$ and zeros in all of the other $X - \{x\}$ columns.

- \textbf{Simple Recovery}: For each $k$-strongly selective test that evaluates to a 0 (i.e., All Healthy)...

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- **Simple Recovery**: For each $k$-strongly selective test that evaluates to a 0 (i.e., All Healthy)...
- Mark all individuals tested in that test as Healthy.
- If there are at most $k$ sick individuals, we will find them all!
Theorem 1

Let \( \mathbf{a} \in \{0, 1\}^N \) be a binary vector containing \( k \) nonzero entries. Furthermore, let \( \mathcal{M} \) be a \( k \)-strongly selective binary matrix. Then, the positions of all \( k \) nonzero entries in \( \mathbf{a} \) can be recovered using only the result of \( \mathcal{M} \mathbf{a} \).

Theorem 2

There exist explicitly constructible \((\min\{k^2 \cdot \log N, N\}) \times N \) \( k \)-strongly selective binary matrices. And, they are optimal in the number of rows.\(^a\)

\(^a\)See Porat and Rothschild’s paper “Explicit Non-Adaptive Combinatorial Group Testing Schemes”.

Theorem 1

Let \( a \in \{0, 1\}^N \) be a binary vector containing \( k \) nonzero entries. Furthermore, let \( M \) be a \( k \)-strongly selective binary matrix. Then, the positions of all \( k \) nonzero entries in \( a \) can be recovered using only the result of \( Ma \).

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Group Testing - Another example

Error Detection

Suppose we want to transmit a binary vector $a \in \{0, 1\}^N$ through a noisy environment. How can we tell if we received the real message?

- Used for DVD, CD, and other media devices in your house!
- Basic Methods: Parity and Checksums
- A bit stronger: Use a strongly selective matrix!
  - Transmit (or read) both $a$ and $Ma$
  - The receiver gets (or reads) $a' = a + \epsilon$
  - Check to see if $Ma = Ma'$
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- Basic Methods: Parity and Checksums
- A bit stronger: Use a strongly selective matrix!
  - Transmit (or read) both \( \mathbf{a} \) and \( \mathcal{M}\mathbf{a} \)
  - The receiver gets (or reads) \( \mathbf{a}' = \mathbf{a} + \epsilon \)
  - Check to see if \( \mathcal{M}\mathbf{a} = \mathcal{M}\mathbf{a}' \)
We have seen strongly selective matrices with $O(k^2 \log N)$ rows allow us to find all $k$ nonzero entries in $a \in \{0, 1\}^N$ when we are allowed to compute ONE round of sampling.

What if we can adaptively sample $a \in \{0, 1\}^N$ several times?

**ANSWER**: We can use at most $\log(N)$ matrices with at most $2k + 1$ rows each! The total number of inner products is now only $O(k \log N)$!
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\textbf{ANSWER}: We can use at most \( \log(N) \) matrices with at most \( 2k + 1 \) rows each! The total number of inner products is now only \( O(k \log N) \)!
Suppose \( a \in \mathbb{R}^N \) contains \( k \) nonzero entries all of which are positive. Can we still identify all \( k \) locations of the nonzero entries using the group testing methods we have seen?

**YES!**

**HOMEWORK:** Write out the step of the adaptive group testing example just discussed in class to find the two nonzero entries in the vector

\[
(0, 0, 0, 0, 0, .4, 0, 23, 0, 0, 0, 0, 0, 0, 0, 0, 0).
\]

How many inner products does it take? How many inner products would it have taken to use a strongly selective binary matrix (see Theorem 2)? Which method requires fewer tests?
Group Testing - A Question

Suppose \( a \in \mathbb{R}^N \) contains \( k \) nonzero entries all of which are positive. Can we still identify all \( k \) locations of the nonzero entries using the group testing methods we have seen?

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Suppose $\mathbf{a} \in \mathbb{R}^N$ contains $k$ nonzero entries all of which are positive. Can we still identify all $k$ locations of the nonzero entries using the group testing methods we have seen?

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**HOMEWORK:** Write out the step of the adaptive group testing example just discussed in class to find the two nonzero entries in the vector $(0, 0, 0, 0, 0, .4, 0, 23, 0, 0, 0, 0, 0, 0, 0, 0, 0)$.

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Compressed Sensing - Overview

- $K \times N$ measurement matrix $M$ with complex entries
- An $N \times N$ complex matrix $\Psi$ which induces sparsity in signals of interest.
- Signal $a \in \mathbb{C}^N$ which is sparse under $\Psi$ (i.e., $\Psi a$ contains $k$ nonzero entries).
- Identify the location of $k$ ones using $y = M\Psi a$ measurements
- How small can we make $K$ and still recover $\Psi a$ – and therefore $a$ – using only $y$?
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Bit Testing - Group Testing for Compressed Sensing

- $B$ is $4 \times 6$, $a$ contains 1 nonzero entry.

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
0 \\
-3.5 \\
0 \\
0
\end{bmatrix}
$$

- Sparsity pattern gives nonzero location in binary.
- We get the entry value at least once.
- $B$ is called a bit testing matrix.

IN THIS EXAMPLE WE SAVED 2 TESTS!
Bit Testing - Group Testing for Compressed Sensing

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\begin{pmatrix}
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0 & 0 & 0 & 0 & 1 & 1
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0 \\
-3.5 \\
0 \\
0 \\
0
\end{pmatrix}
\]

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\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
-3.5 \\
0 \\
0 \\
0
\end{pmatrix}
= 
\begin{pmatrix}
-3.5 \\
0 \\
-3.5 \\
0
\end{pmatrix}
\]

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0 \\
-3.5 \\
0 \\
0 \\
0
\end{pmatrix}
= 
\begin{pmatrix}
-3.5 \\
0 \\
-3.5 \\
0
\end{pmatrix}
\]

- Sparsity pattern gives nonzero location in binary.
- We get the entry value at least once.
- $B$ is called a bit testing matrix.

IN THIS EXAMPLE WE SAVED 2 TESTS!
Bit Testing - Group Testing for Compressed Sensing

- $B$ is $4 \times 6$, $a$ contains 1 nonzero entry.

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
-3.5 \\
0 \\
0 \\
0
\end{pmatrix} =
\begin{pmatrix}
-3.5 \\
0 \\
-3.5 \\
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$$
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1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
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IN THIS EXAMPLE WE SAVED 2 TESTS!
Suppose \( \mathbf{a} \in \mathbb{C}^N \) contains \( k \) nonzero values.

- We will say that binary vector \( \mathbf{m} \in \{0, 1\}^N \) is a good mask for \( \mathbf{a} \) if the component-wise multiple of \( \mathbf{m} \) and \( \mathbf{a} \) contains only one isolated nonzero value.
- Anytime this happens we can identify the isolated value!
- There will be at least one row in \( \mathcal{M} \) that contains each nonzero value in \( \mathbf{a} \) isolated from all the others.
- Thus, \( \mathcal{M} \) will contain a good mask for every nonzero value!
Suppose $a \in \mathbb{C}^N$ contains $k$ nonzero values.

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Compressed Sensing - More Than One Nonzero Value

Strongly Selective Matrices

A binary matrix $M$ is $k$-strongly selective if for any column, $x$, and subset of columns containing at most $k$ elements, $X$, there exists a row in $M$ with a 1 in column $x$ and zeros in all of the other $X - \{x\}$ columns.

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Theorem 3
Let \( \mathbf{a} \in \mathbb{C}^N \) be a complex valued vector containing \( k \) nonzero entries. Furthermore, let \( \mathcal{M} \) be a \( k \)-strongly selective binary matrix and \( \mathcal{B} \) be a bit testing matrix. Then, all \( k \) nonzero entries in \( \mathbf{a} \) can be recovered using only the result of \((\mathcal{M} \otimes \mathcal{B})\mathbf{a}\).

Theorem 4
We can explicitly construct binary measurement matrices, \( \mathcal{M} \otimes \mathcal{B} \), of size \( \left( \min \left\{ k^2 \cdot \left( \log_2 N + \log_2 N \right), N \right\} \right) \times N. \)
Compressed Sensing - More Than One Nonzero Value

Theorem 3

Let $a \in \mathbb{C}^N$ be a complex valued vector containing $k$ nonzero entries. Furthermore, let $M$ be a $k$-strongly selective binary matrix and $B$ be a bit testing matrix. Then, all $k$ nonzero entries in $a$ can be recovered using only the result of $(M \star B)a$.

Theorem 4

We can explicitly construct binary measurement matrices, $M \otimes B$, of size

$$\left( \min \left\{ k^2 \cdot \left( \log_2 N + \log_2 N \right), N \right\} \right) \times N.$$