1. Spatial Filters in Practice

Choose your favorite image and use the Matlab commands \textit{fspecial}, \textit{imfilter} and \textit{imshow} to create the following enhancements:

a) Apply $3 \times 3$, $5 \times 5$, $10 \times 10$, and $20 \times 20$ averaging to the image.

b) Apply Gaussian filters, with various mask sizes and choices of $\sigma$.

c) Apply $3 \times 3$ Laplacian filters, with various choices of the parameter $\alpha$.

d) Apply Laplacian of Gaussian filters with various mask sizes.

e) Apply the Sobel filter for differentiating both in the $x$ and $y$ direction. Then apply filters corresponding to both $l_1$ and $l_2$ norms of the gradient.

f) Choose one filter from the ones you used in part c) and one from part d). For each one of those filters apply a high-boost filter with two choices of the parameter $k$ (note that $k = 1$ coincides with the filter \textit{unsharp}).

2. Properties of the Fourier Transform

Assume that $f$ is a function in $L_1(\mathbb{R})$ (or in the Schwartz space $\mathcal{S}(\mathbb{R})$ if you prefer). Establish the following properties.

a) $\hat{f(x+h)} = \hat{f}(\xi) e^{2\pi i h \xi}$.

b) $f(x) e^{-2\pi i x h} = \hat{f}(\xi + h)$.

c) $\hat{f(\delta x)} = \delta^{-1} \hat{f}(\delta^{-1} \xi)$.

d) Formulate the analogs of the above properties for a function $f \in L_1(\mathbb{R}^2)$.

3. More Properties of the Fourier Transform

You may solve only 4 of the next following 5 subquestions.

Assume that $f$ is a function in the Schwartz space $\mathcal{S}(\mathbb{R})$ (or any reasonable space where you can prove the following identities).

a) Prove that $\hat{f'(x)} = 2\pi i \xi \hat{f}(\xi)$.

b) Prove that $-2\pi i x \hat{f}(x) = \frac{df(\xi)}{d\xi}$.

c) Formulate an analog of the above two properties when replacing $f'(x)$ by $f^{(n)}(x)$, the $n$-th
derivative of \( f(x) \) in part a) above and when replacing the derivative of \( \hat{f}(\xi) \) by its \( n \)-th derivative in part b) above.

d) Show that if \( f \in S(\mathbb{R}) \), then \( \hat{f} \in S(\mathbb{R}) \). Hint: use your result in part c).
e) Formulate the analogs of the above properties for a function \( f \in S(\mathbb{R}^2) \)

4. Computation of the Continuous Fourier Transform

Let \( a \) be a real positive number. Compute 3 of the Fourier transforms of the following 4 functions (in parts 2 and 4 you need to compute the Fourier transform in two different ways):

1. \( f(x) = \chi_{[-a,a]}(x) \), where \( \chi_{[-a,a]} \) is the indicator function obtaining the value 1 on the interval \([-a,a]\) and 0 outside it.

2. \( f(x) = (1 - |x/a|) \chi_{[-a,a]}(x) \). Compute the Fourier transform of this function in two different ways. First, by direct calculation. Second, by expressing this function as \( g \ast g \) for some function \( g \) (verify your claim) and then using properties of the Fourier transform (Hint: you may use a scaled version of the function in 1 above, or look at exercise 4.7 in the textbook).

3. \( f(x) = e^{-a|x|} \).

4. \( f(x) = a^{-1/2} e^{-\pi x^2 / a} \). Compute the Fourier transform in two different ways. First, by direct calculation (see e.g., online solution of the textbook problem 4.31*). Then calculate it in the following way: assume that \( a = 1 \) and show that in this case

\[
\frac{df(\xi)}{d\xi} = -2\pi \xi \hat{f}(\xi)
\]

and that \( \hat{f}(0) = 1 \). Conclude the form of \( \hat{f}(\xi) \) when \( a = 1 \). Use properties of the Fourier transform to conclude the Fourier transform of \( f \) for any \( a \in \mathbb{R} \).

5. Questions from the textbook

Solve problems 4.1, 4.5, 4.14, 4.20

6. Bonus Problem: Fixed points of the Fourier transform

Suggest an infinite sequence of functions \( \{f_n(x)\}_{n \in \mathbb{N}} \) such that \( \hat{f}_n = f_n \) and moreover no function in the sequence is a linear combination of other functions in the sequence. Explain your answer.

7. Bonus Problem: Demonstration of the “General Principle”

Assume that \( f \) is a function in \( L_1(\mathbb{R}) \) whose Fourier transform satisfies

\[
\hat{f}(\xi) = O\left(\frac{1}{|\xi|^{1+\alpha}}\right) \quad \text{as} \quad |\xi| \to \infty
\]

for some \( 0 < \alpha < 1 \). Prove that \( f \) satisfies a Hölder condition of order \( \alpha \), that is

\[
|f(x + h) - f(x)| \leq M|h|^\alpha \quad \text{for some} \quad M > 0 \quad \text{and all} \quad x, h \in \mathbb{R}.
\]