This exam contains 4 numbered problems on 6 pages of paper. Check to see if any pages are missing.

A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit. NO CALCULATORS are permitted. good luck.

You may like to use the following trigonometric identities:

\[
\sin(A + B) = \sin A \cos B + \cos A \sin B
\]
\[
\cos(A + B) = \cos A \sin B - \sin A \cos B
\]
\[
\sin^2(A/2) = (1 - \cos A)/2
\]
\[
\cos^2(A/2) = (1 + \cos A)/2
\]
\[
\sin A \cos B = \frac{1}{2}[\sin(A - B) + \sin(A + B)]
\]
\[
\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]
\]
\[
\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]
\]

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<tr>
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<th>25 pts</th>
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1. (25 points) What is the distance traveled along the path \( x(t) = t^4 + \sin t, \ y(t) = t^7 + 1 \) from \( t = 1 \) to \( t = 4 \)? Write your answer as an integral. Show any general formulas you are using.

We want the arc length:

\[
S = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \ dt
\]

\[
= \int_{1}^{4} \sqrt{(4t^3 + \cos t)^2 + (7t^6)^2} \ dt
\]
2. (25 points) What is the solution of \( x^2y' = y - y' \) with \( y(0) = -2 \)? Write the answer as a function of \( x \). Show your work.

\[
0 \quad (x^2+1)y' = y \\
y' = \frac{y}{x^2+1} \\
2 \quad \frac{dy}{dx} = y \cdot \frac{1}{x^2+1} \\
\text{Separation of variables:} \\
\int \frac{dy}{y} = \int \frac{dx}{x^2+1} \\
\log |y| = \arctan x + C \\
|y| = Ce^{\arctan x} \\
3 \quad \text{Find } C': \quad 2 = |y(0)| = Ce^{\arctan 0} = C' \\
\Rightarrow \quad C' = 2 \\
\Rightarrow \quad |y| = 2e^{\arctan x} \\
4 \quad \text{Because } y(0) = -2, \\
y = -2e^{\arctan x}
3. (25 points) What is the centroid of the region lying between \( f(x) = x^2 + 6x - 5 \) and \( g(x) = 2x^2 \)? You can write your answer in terms of integrals. Write down any formulas you are using.

\begin{enumerate}
\item Find the intersections:
\[ f(x) = g(x) \]
\[ x^2 + 6x - 5 = 2x^2 \]
\[ x^2 - 6x + 5 = 0 \]
\[ (x-1)(x-5) = 0 \]
\[ x = 1 \text{ or } x = 5 \Rightarrow \text{domain: } x \in [1, 5] \]

\item For \( x \in [1, 5] \), \( g(x) - f(x) = (x-1)(x-5) \leq 0 \)
\[ \Rightarrow f(x) \geq g(x) \]

Given that, the area integral is:
\[ A = \int_{x_1}^{x_2} f(x) - g(x) = \int_{1}^{5} -x^2 + 6x - 5 \, dx \]

\item Moments:
\[ M_y = \int_{x_1}^{x_2} x \left[ f(x) - g(x) \right] dx = \int_{1}^{5} x (-x^2 + 6x - 5) \, dx \]
\[ M_x = \int_{x_1}^{x_2} \frac{1}{2} \left[ f^2(x) - g^2(x) \right] dx = \int_{1}^{5} \frac{1}{2} \left[ (x^2 + 6x - 5)^2 - (2x^2)^2 \right] \, dx \]

\item Finally, the centroid is given by
\[ (\bar{x}, \bar{y}) = \left( \frac{M_y}{A}, \frac{M_x}{A} \right) \]
4. (25 points) Find the area of the region that lies inside $r = 1 - \sin \theta$ and outside $r = 1$. Make a sketch of both curves on the graph provided below. Show any formulas you are using.

\[ \int_{\theta_0}^{\theta_1} \frac{1}{2} [r(\theta)]^2 d\theta \]

\[ A_1, A_2 \text{ be the area encircled by } \theta \in [-\pi, 0], r=1, r=1-\sin \theta \text{ respectively. Our region is just the difference of the two:} \]

\[ A = A_2 - A_1 = \int_{-\pi}^{0} \frac{1}{2} [(1-\sin \theta)^2 - 1^2] d\theta \]