This exam contains 4 numbered problems on 6 pages of paper. Check to see if any pages are missing.

A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit. NO CALCULATORS are permitted. Good luck.

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1. (25 points) Does the series \( \sum_{n=1}^{\infty} \left[ \frac{1}{\sqrt{n+1}} + \frac{1}{\sqrt{n^3}} \right] \) converge? Explain why, specifying any tests you are using.

\[
\sum_{n=1}^{\infty} \left( \frac{1}{\sqrt{n+1}} + \frac{1}{\sqrt{n^3}} \right) \geq \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}
\]

By integral test, \( \int_{1}^{\infty} \frac{1}{\sqrt{x+1}} \, dx = 2(x+1)^{1/2} \bigg|_{1}^{\infty} \) diverges,

we conclude \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}} \) diverges.

By comparison theorem the original series diverges.
2. (25 points) What is the Taylor series for $f(x) = (x - 1)^{-2}$ at $x = 0$?

\[
\begin{align*}
    f^{(0)}(0) &= 1 \\
    f^{(1)}(0) &= -2(x-1)^{-3} \bigg|_{x=0} = -2 \\
    f^{(2)}(0) &= (-2)(-3)(x-1)^{-4} \bigg|_{x=0} = 12 \\
    &\vdots \\
    f^{(n)}(0) &= (n+1)! \\
    \Rightarrow \quad f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\
    &= \sum_{n=0}^{\infty} \frac{(n+1)!}{n!} x^n = \sum_{n=0}^{\infty} \frac{(n+1)!}{n!} x^n
\end{align*}
\]
3. (25 points) Where does the series \( \sum_{n=2}^{\infty} \frac{(-1)^n(x+1)^n}{2n+3} \) converge?

\( \text{Ratio test:} \)

\[
\lim_{n \to \infty} \left| \frac{(-1)^n(x+1)^{n+1}}{2n+3} \right| = \lim_{n \to \infty} \frac{2n+1}{2n+3} |x+1|
\]

\( = |x+1| < 1 \implies x \in (-2, 0) \)

\( \implies \) The series converges absolutely in \((-2,0)\).

\( \text{End points:} \)

(a) \( x = -2 \). \( \sum_{n=2}^{\infty} \frac{1}{2n+1} \) diverges by integral test \( \int_{-2}^{\infty} \frac{dx}{2x+1} = \log(2x+1) \bigg|_{2}^{\infty} \)

(b) \( x = 0 \). \( \sum_{n=2}^{\infty} \frac{(-1)^n}{2n+1} \) is decreasing in \( n \) and \( \lim_{n \to \infty} \frac{1}{2n+1} = 0 \)

By alternating series test, it converges

So the power series converges in \((-2,0] \)
4. (25 points) Use series to approximate \( \int_{0}^{0.5} x^2 e^{-x^2} \, dx \) to within the accuracy \(|\text{error}| < 0.001\). You don’t need to do the final addition and subtraction on numbers, but you should show the terms needed for the indicated accuracy.

\[
0. \quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{converges \quad } \forall x \in \mathbb{R}
\]

\[
0.5 \quad x^2 e^{-x^2} = x^2 \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n!} \quad \text{converges absolutely \quad } \forall x \in \mathbb{R}.
\]

\[
\int_{0}^{0.5} x^2 e^{-x^2} \, dx = \int_{0}^{0.5} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n!} \, dx
\]

\[
= \sum_{n=0}^{\infty} \int_{0}^{0.5} \frac{(-1)^n x^{2n+2}}{n!} \, dx
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{(2n+3) \cdot n!} \bigg|_{0}^{0.5} = \sum_{n=0}^{\infty} \frac{(-1)^n \left( \frac{1}{2} \right)^{2n+3}}{(2n+3) \cdot n!}
\]

\[
= \frac{\left( \frac{1}{2} \right)^3}{3} - \frac{\left( \frac{1}{2} \right)^5}{5 \cdot 1} + \frac{\left( \frac{1}{2} \right)^7}{7 \cdot 2} - \frac{\left( \frac{1}{2} \right)^9}{9 \cdot 6} + \ldots
\]

We see \( \frac{\left( \frac{1}{2} \right)^7}{7 \cdot 2} = \frac{1}{256 \cdot 7} = \frac{1}{256 \cdot 7} < \frac{1}{1000} \)

The remainder is an alternating series, the sum is smaller than the first term.

So the error is bounded by \( \left| \frac{1}{256 \cdot 7} \right| < 0.001 \).

\[
\Rightarrow \int_{0}^{0.5} x^2 e^{-x^2} \, dx \approx \frac{1}{24} - \frac{1}{160} \quad \text{up to accuracy 0.001.}
\]