The quiz is double-sided and there are 3 problems with a total score of 20. You have 30 minutes to solve all the problems on your own. Calculators are allowed but should not be useful. Please print your name and section number at the top-right corner before you start.

Problem 1  (6 points) Find the following limit if it exists, or show that the limit does not exist:

$$\lim_{{(x,y)\to(0,0)}} \frac{xy}{\sqrt{x^2 + 4y^2}}$$

Solution 1 (Textbook approach):
We don’t know whether the limit exists or not, so we try the approaches suggested by the textbook. First, we try $x = ky$,

$$\lim_{y \to 0} \frac{ky^2}{\sqrt{k^2y^2 + 4y^2}} = \lim_{y \to 0} \frac{k}{\sqrt{k^2 + 4}}y = 0$$

which converges for all $k$.
Then we try to let the two terms in the denominator the same: $x^2 = 4y^2 \implies x = \pm 2y$, which is covered by the $x = ky$ case already.
After this, we should suspect convergence. We can prove convergence use the textbook trick. We have

$$x^2 + 4y^2 \geq x^2$$

which leads to

$$\left| \frac{xy}{\sqrt{x^2 + 4y^2}} \right| \leq \left| \frac{xy}{\sqrt{x^2}} \right| \leq |y|$$

which converges to 0. Then by squeezing principle the limit exists and equals 0. Solution 2 (Preferred):
Apply Cauchy inequality directly:

$$\left| \frac{xy}{\sqrt{x^2 + 4y^2}} \right| \leq \frac{xy}{\sqrt{4xy}} = \frac{1}{2} \sqrt{xy} \to 0$$

as $(x,y) \to O$. Thus the limit converges to 0 by squeezing principle.

Problem 2  (6 points) Determine the subset of $(x,y) \in \mathbb{R}^2$ in which the following function $f(x,y)$ is continuous:

$$f(x,y) = \log \left(\sqrt{x^2 + y^2} - 2\right)$$

Solution:
We see $f$ is a composition of elementary functions. Therefore, it is continuous in its domain. So we only need to find the domain.
(1) The outer most is log$(x)$, with domain $x > 0$. So $\sqrt{x^2 + y^2} - 2 > 0 \implies x^2 + y^2 > 4$
(2) Then we have the square root, which requires $x^2 + y^2 > 0$.
Combine (1) and (2), we have $f$ is continuous on $\{(x,y) \in \mathbb{R}^2 | x^2 + y^2 > 4\}$. Geometrically it is the exterior of the circle of radius 2 centered at $O$.

Problem 3  (8 points) Find the partial derivative $u_{qp}$ of the following function

$$u(p,q) = \frac{(1+p)^q}{q} + \int_{1-q^2}^{1+q^2} \frac{e^{x^2-x}}{\sqrt{2 + \sin(x)}} \, dx$$

Solution:
(1) Notice the second term doesn’t depend on $p$, signaling changing the order of differentiation.
(2) (Just to be rigorous, this part maybe is not required for this course.) By observation, we see $u_{qp}$ and $u_{pq}$ are elementary functions and their compositions, which are continuous in their domains. Therefore we can use $u_{qp} = u_{pq}$.
(3) $u_{qp} = u_{pq} = \partial_q \left( \partial_p \left( \frac{(1+p)^q}{q} \right) + 0 \right) = \partial_q \left( (1+p)^{q-1} \right) = (1+p)^{q-1} \log(1+p)$