7953 Review for Final I

Note that these review sheets do not cover the entire course. You should also consult all previous review sheets and handouts.

1. Find the general solution of the differential equation

\[ t \frac{dy}{dt} = 3y + 10t^4 \sin 5t \]

2. A very large tank contains 50 lbs of salt dissolved in 400 gallons of water. Brine that contains 4/5 lbs of salt per gallon of water enters the tank at the rate of 5 gal/min. The mixture leaves the tank at the lower rate of 3 gal/min. Find an expression for the amount of salt in the tank at time \( t \).

3. Consider the autonomous differential equation \( \frac{dy}{dt} = (2 + y)(8 - y) \).
   a) Determine the equilibrium solutions (points) and also determine which equilibrium solution is stable.
   b) Solve the initial value problem \( \frac{dy}{dt} = (2 + y)(8 - y) \) and \( y(0) = 10 \). For exactly what values of \( t \) is this a solution of the initial value problem.
1. Find the inverse of the following matrix using either of the two methods learned in class: (Make sure you can do both methods.)

\[
\begin{pmatrix}
2 & 8 & -5 \\
0 & -2 & 1 \\
-2 & -7 & 5
\end{pmatrix}
\]

2. Suppose we have a spring mass system with an object weighing 64 lbs. The system has a damping constant of 6 lbs sec/ft. A force of 60 lbs will stretch the spring 3 feet beyond its natural length. There is an external force applied to the system of \(8 \sin 3t\) lbs. The object is started in motion 4 feet above its equilibrium position with an initial velocity of 3 ft/sec in the downward direction. Set up the initial value problem which describes the motion of this object.

3. Are the following vectors linearly independent? Can the third vector be expressed as a linear combination of the first two? Justify your answer.

\[
\begin{pmatrix}
1 \\
4 \\
2 \\
-3
\end{pmatrix}, \quad \begin{pmatrix}
7 \\
10 \\
-4 \\
-1
\end{pmatrix}, \quad \begin{pmatrix}
-2 \\
1 \\
5 \\
-4
\end{pmatrix}
\]

4. Express \(f(t) = 6 \cos 3t - (9/2) \sin 3t\) in the form \(A \sin(wt+d)\) and in the form \(B \cos(wt-d)\).
1. First, show that the following differential equation is coefficient homogeneous and then solve the initial value problem:

\[ xy \frac{dy}{dx} = 3y^2 + 10x^2 \quad \text{and} \quad y(1) = 3. \]

2. During a 5 year period the widget company compiled the following data where \( x \) indicates the annual expenditure on advertising and \( y \) is the annual income (both measured in units of ten thousand dollars).

<table>
<thead>
<tr>
<th>( x )</th>
<th>10</th>
<th>12</th>
<th>15</th>
<th>18</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>70</td>
<td>80</td>
<td>90</td>
<td>110</td>
<td>115</td>
</tr>
</tbody>
</table>

a) Determine \( b \) and \( m \) where \( y = mx + b \) the least squares line for this data.

b) Use the result to predict the annual income when the amount spent on advertising is $160,000.

3. a) Find the eigenvalues of the following matrix.

b) Find one of the complex eigenvectors for this matrix. Hint. In order to solve for the eigenvector add \((-3)\) times row two to row one.

\[
\begin{pmatrix}
12 & -7 & 3 \\
14 & -4 & 1 \\
2 & 3 & -2
\end{pmatrix}
\]
7956 Review for Final IV

1. Find the general solution of the differential equation

\[ y'' + 2y' + 10y = 697e^{3t} \cos 2t. \]

2. Solve the following system of equations by starting with the augmented matrix and using Gaussian elimination.

\[
\begin{align*}
    x + 3y + 2z &= 0 \\
    3x + 11y + 8w &= -10 \\
    -2x - 10y + 9z - 13w &= 25 \\
    6y - 21z + 19w &= -41
\end{align*}
\]

3. Find the general solution of the following differential equation:

\[ y''(t) + 16y(t) = 12 \cos 4t. \]

4. A steel ball is heated to a temperature of 200°C and at time \( t = 0 \) is placed in water maintained at 20°C. At \( t = 5 \) minutes the temperature of the ball is 110°C. When is the temperature of the ball reduced to 65°C?
1. Solve the initial value problem

\[
\begin{pmatrix}
x' \\
y'
\end{pmatrix} = \begin{pmatrix}
2 & 13 \\
-1 & 6
\end{pmatrix} \begin{pmatrix}
x \\
y
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
x(0) \\
y(0)
\end{pmatrix} = \begin{pmatrix}
26 \\
-11
\end{pmatrix}.
\]

2. Let \( F(s) = \frac{5}{s} - \frac{2}{s^3} + e^{-2s} \left[ \frac{2}{s^3} + \frac{6}{s^2} \right] + e^{-5s} \left[ -\frac{2}{s^3} - \frac{4}{s} \right] \). Find \( f(t) = \mathcal{L}^{-1}\{F(s)\} \) in terms of the Heaviside unit step function. Express \( f(t) \) as a function in parts and sketch a graph of \( f(t) \).

3. Find the eigenvalues and eigenfunctions for the following boundary value problem.

\[ y'' + 2y' + (\lambda^2 + 1)y = 0, \quad y(0) = 0, \quad \text{and} \quad y(5) = 0. \]

4. Given that \( y(t) = t^3 \) is a solution of \( t^2 \frac{d^2y}{dt^2} - 5t \frac{dy}{dt} + 9y = 0 \), find the general solution of this differential equation.