1. Given that $y = x^{-4}$ is a solution of the following differential equation, find the general solution.

$$x^2 \frac{d^2 y}{dx^2} + 11x \frac{dy}{dx} + 24y = 0.$$ 

$$0. \ y = x^{-4}, \ y' = -4x^{-5}u + x^{-3}u'$$

$$y'' = 20x^{-6}u - 4x^{-5}u' - 4x^{-5}u' + x^{-4}u''$$

$$= x^{-4}u'' - 8x^{-5}u' + 20x^{-6}u$$

$$\Rightarrow x^{-2}u'' - 8x^{-3}u' + 20x^{-4}u + 11x^{-3}u + 24x^{-4}u = 0$$

$$\Rightarrow x^{-2}u'' + 3x^{-3}u' = 0 \Rightarrow ux'' + 3u' = 0$$

$$\Rightarrow w = u^{'}. \ \Rightarrow xw' + 3w = 0 \Rightarrow \int \frac{dw}{w} = \int -\frac{3dx}{x} \Rightarrow \omega = Cx^{-3}$$

$$\Rightarrow u = \int \omega \ dx = \frac{C_1}{x} + C_2$$

$$\Rightarrow y = \frac{C_1}{x^{-4}} + C_2 x^{-4}$$

2. Draw the phase line for $y' = (2+y)(10-y)$. Use the phase line to sketch some solutions on a separate set of axis. Which of the equilibrium solutions is stable?

3. The solution of a certain mass-spring system is given. What is the transient solution, the steady state solution? Is the system overdamped, critically damped, or underdamped? Does this system have resonance?

$$x(t) = 5e^{-2t} \sin t - 4e^{-2t} \cos t + 8 \cos 3t + \sin 3t.$$
1. Find the general solution of \( y'' + 6y' + 34y = 53\cos3t - 190\sin3t \)

- **Homogeneous**
  \[ m^2 + 6m + 34 = 0 \]
  \[ \Rightarrow (m+3)^2 + 25 = 0 \]
  \[ \Rightarrow m = -3 \pm 5i \]
  \[ y_h = e^{-3t}(C_1 \cos5t + C_2 \sin5t) \]

- **Particular Solution**
  \( y_p = A \cos3t + B \sin3t \)
  \[ \Rightarrow A = 5, \ B = -4 \]

\[ y = e^{-3t}(C_1 \cos5t + C_2 \sin5t) + 5 \cos3t - 4 \sin3t \]

2. Find the particular solution of the differential equation

- \( m = -1, n = -2 \)
  \[ y'' + 3y' + 2y = e^{-3t}(6t^2 - 18t - 16) \]

- **Guess**
  \[ y_p = e^{-3t}(At^2 + Bt + C) \]

- \[ \Rightarrow A = 3, \ B = 0, \ C = -11 \]

\[ y_p = e^{-3t}(3t^2 - 11) \]

3. Given that \( \lambda = 0 \) and \( \lambda = 4 \) are eigenvalues of the following matrix, find a corresponding eigenvector for each of these eigenvalues.

- \( \lambda = 0 \)
  \[
  \begin{bmatrix}
  1 & 0 & 1 \\
  3 & -5 & 4 \\
  7 & -15 & 10
  \end{bmatrix}
  \]
  \[ \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \]

- \( \lambda = 4 \)
  \[
  \begin{bmatrix}
  -3 & 0 & 1 \\
  3 & -7 & 4 \\
  7 & -15 & 6
  \end{bmatrix}
  \]
  \[ \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \]

4. Find the eigenvalues and a corresponding eigenvector for the matrix

- \( \lambda = 10 \)
  \[
  \begin{bmatrix}
  10 - \lambda & 37 \\
  -2 & -4 - \lambda
  \end{bmatrix}
  \]
  \[ \lambda^2 - (10 - \lambda)34 = 0 \]

- \( \lambda = 3 \pm 5i \)

- \( \lambda = 3 + 5i \)
  \[
  \begin{bmatrix}
  7 - 5i & 37 \\
  -2 & -7 - 5i
  \end{bmatrix}
  \]
  \[ x_1 = 0 \Rightarrow x = \begin{bmatrix} -37 \\ 7 - 5i \end{bmatrix} \]

- \( \lambda = 3 - 5i \)
  \[
  \begin{bmatrix}
  7 + 5i & 37 \\
  -2 & -7 + 5i
  \end{bmatrix}
  \]
  \[ x_2 = 0 \Rightarrow x = \begin{bmatrix} -37 \\ 7 + 5i \end{bmatrix} \]
1. An object of mass 1/2 slug is placed on a very long spring and it stretches the spring 8/5 feet. There is a damping force on the motion of the object of 2 times the velocity. There is an external force of 6 cos 3t lbs on the system. The object is set in motion from a point 3 ft below the equilibrium position and with an initial velocity of 4 ft/sec in the upward direction. Set up the initial value problem which describes the motion of this object using the assumption that up is positive.

\[
\begin{align*}
0 & \quad m y'' + c y' + k y = f_e, \\
\text{(a) } y(0) = -3, \quad y'(0) = 4, \\ 
1 & \quad f_e = 6 \cos 3t.
\end{align*}
\]

2. First find the general solution of \(y'' + 16y = 0\). Next solve the following boundary value problems:

a) \(y'' + 16y = 0, \quad y(0) = 2, \quad y(\pi/4) = -2\)

b) \(y'' + 16y = 0, \quad y(0) = 2, \quad y(\pi/8) = 3\)

c) \(y'' + 16y = 0, \quad y(0) = 2, \quad y(\pi/2) = 4\)

\[
\begin{align*}
\text{(a) } y''(0) = c_1 = 2, \\
y(\pi/4) = -c_1 = -2 \Rightarrow c_1 = 2 \\
y(t) = c_1 \cos 4t + (c_2 + 4) \sin 4t.
\end{align*}
\]

3. An LRC circuit contains a resistor of 20 ohms, a capacitor of 1/65 farad, and an inductor of 5 henrys. There is an input voltage of 200 \sin(8t) volts. The initial charge on the capacitor is 4 coulombs and the initial current is 10 amperes. Find the differential equation and initial conditions that describe the charge on the capacitor. You need not solve.

\[
\begin{align*}
0 & \quad LCQ'' + RQ' + \frac{1}{C} Q = E, \\
& \Rightarrow 55Q' + 70Q + 65Q = 200 \sin 8t. \\
Q(0) = 4, \quad Q'(0) = 10.
\end{align*}
\]

4. Given that the general solution of \(y'' - 2y' - 3y = 0\) is \(y = c_1 e^{3t} + c_2 e^{-t}\), what general form should we guess for the particular solution of each of the following differential equations:

a) \(y'' - 2y' - 3y = e^{3t} \quad A e^{3t}\)

b) \(y'' - 2y' - 3y = 8e^{5t} + \sin 3t \quad A e^{5t} + B e^{3t} + C \cos 3t + D \sin 3t\)

c) \(y'' - 2y' - 3y = 5t^2 + 8t + e^{6t} \quad A t e^{5t} + B e^{6t} + C t + D\)

d) \(y'' - 2y' - 3y = t \cos 3t \quad (A t + B)(C \cos 3t + D \sin 3t)\)

e) \(y'' - 2y' - 3y = 7te^{6t} \quad (A t + B) e^{6t}\)

\[
\begin{align*}
m^2 - 2m - 3 & = 0 \Rightarrow m = \frac{-3 \pm \sqrt{9}}{2} \\
m = -1.
\end{align*}
\]
1. Express \((-5/2)\cos 3t - 6\sin 3t\) in the form \(A\sin(\omega t + \phi)\) and also in the form \(\cos(\omega t - \phi)\). What are amplitude, period, and initial phase angle?

- Amplitude \(A = \sqrt{(\frac{5}{2})^2 + 6^2} = \frac{13}{2}\)
- Period \(T = \frac{2\pi}{3}\)
- \(A \sin(\omega t + \phi) = A \cos(\omega t - \phi)\) implies \(\tan \phi = \frac{6}{\frac{5}{2}} = \frac{12}{5}\)
  \(\Rightarrow\) \(\phi = \tan^{-1} \frac{12}{5} + \pi\)

2. Find the general solution of the differential equation \(y' = (2 + y)(10 - y)\). Solve for \(y\) in the solution. Find the solution of each of the following initial value problems.

a) \(y' = (2 + y)(10 - y), \ y(0) = 6\)

b) \(y' = (2 + y)(10 - y), \ y(0) = 16\)

c) \(y' = (2 + y)(10 - y), \ y(0) = -4\)

d) \(y' = (2 + y)(10 - y), \ y(0) = 10\)

e) \(y' = (2 + y)(10 - y), \ y(0) = -2\)

\(\Rightarrow\) \(y = 10\), stationary, stable \(t \in \mathbb{R}\)

\(\Rightarrow\) \(y = -2\), stationary, unstable \(t \in \mathbb{R}\)

3. Find the eigenvalues and eigenfunctions for the following boundary value problem:

\[ y'' + \lambda^2 y = 0, \quad y'(0) = 0, \quad \text{and} \quad y'(4) = 0. \]