Problem 1  (10 points) Let $S$ be the surface generated by rotating $y = x^2$, $x \in [0,1]$ around the x-axis. Calculate the following surface integral:

$$\int \int_{S} \sqrt{4x^2 + 1} \, dS.$$ 

Solution:
Note that the surface is generated by rotating the curve around the x-axis. For surface of revolution, it is always convenient to use cylindrical coordinates. We use the x-axis as the azimuthal axis. So:

$$x = x, \quad y = r \cos \theta, \quad z = r \sin \theta.$$ 

The radius $r$ should be the distance between the surface and the x-axis, which is just $x^2$. So our parametrization of the surface should be:

$$\Phi : \quad x = x, \quad y = x^2 \cos \theta, \quad z = x^2 \sin \theta, \quad x \in [0,1], \quad \theta \in [0,2\pi].$$

Then we are ready to evaluate the integral:

$$\int \int_{S} \sqrt{4x^2 + 1} \, dS = \int_{0}^{2\pi} \int_{0}^{1} \sqrt{4x^2 + 1} \| \Phi_x \times \Phi_\theta \| \, dx \, d\theta$$

$$= \int_{0}^{2\pi} \int_{0}^{1} \sqrt{4x^2 + 1} \left\| \begin{bmatrix} 1 \\ 2x \cos \theta \\ 2x \sin \theta \end{bmatrix} \times \begin{bmatrix} 0 \\ -x^2 \sin \theta \\ x^2 \cos \theta \end{bmatrix} \right\| \, dx \, d\theta$$

$$= \int_{0}^{2\pi} \int_{0}^{1} \sqrt{4x^2 + 1} (x \sqrt{4x^2 + 1}) \, dx \, d\theta$$

$$= \int_{0}^{2\pi} \int_{0}^{1} (4x^2 + 1) \, dx \, d\theta$$

$$= 2\pi \int_{0}^{1} (4x^2 + 1) \, dx = 14\pi/3.$$