Questions 3, 4, 9 are not graded, so solutions are given. For graded questions, only comments are given. Questions are ordered as on the website.

Question 1: Let $p_s, p_n : S^d \to \mathbb{R}^d$ be the stereographic projections. One need to show $p_s \circ p_n^{-1}$ and $p_n \circ p_s^{-1}$ are smooth and it is true because the domains of these two functions don’t include $0 \in \mathbb{R}^d$. Also, one cannot say $p_s$ or $p_n$ is differentiable since we have not established $S^d$ to be a differentiable manifold yet.

Question 2: $\mathcal{F} \neq \mathcal{F}_1$ because $\phi : t \mapsto \sqrt[3]{t}$ is not in $\mathcal{F}$. But $\phi$ defines a diffeomorphism between the two.

Question 3: (a) For $(x, v) \in \mathbb{R}^{2d}$, we have $\tilde{\psi} \circ \tilde{\varphi}^{-1}(x, v) = (\psi \circ \varphi^{-1}(x), d\psi \circ d\varphi^{-1}(v))$ is smooth.

(b) The collection is clearly a cover. To show it is a basis, let $W_1, W_2$ be open subset of $\mathbb{R}^{2d}$ and $(U, \varphi), (V, \psi)$ two coordinate charts such that $\tilde{\varphi}^{-1}(W_1) \cap \tilde{\psi}^{-1}(W_2)$ is nonempty. Take $v \in \tilde{\varphi}^{-1}(W_1) \cap \tilde{\psi}^{-1}(W_2)$. Then there exists chart $(A, \phi)$ such that $\pi(v) \in A \subset U \cap V$ and open set $W \subset \mathbb{R}^{2d}$ such that $v \in \tilde{\phi}^{-1}(W) \subset \tilde{\varphi}^{-1}(W_1) \cap \tilde{\psi}^{-1}(W_2)$.

(c) We already know $\tilde{\psi} \circ \tilde{\varphi}^{-1}$ is smooth. $\{\pi^{-1}(U)\}$ is obviously an open cover.

Question 4: (a) Take the product chart $(U \times V, \psi \times \varphi)$ on $M \times N$ and a chart $(W, f)$ on $\tilde{M}$. Clearly $(\psi \times \varphi) \circ \alpha \circ f^{-1} : \mathbb{R}^p \to \mathbb{R}^{m+n}$ is smooth if and only if $\psi \circ \alpha \circ f^{-1} : \mathbb{R}^p \to \mathbb{R}^m$ and $\varphi \circ \alpha \circ f^{-1} : \mathbb{R}^p \to \mathbb{R}^n$ are smooth.

(b) It is easy to see $v \to (d\pi_1(v), d\pi_2(v))$ is a linear map between two linear spaces of the same dimension. Suppose $d\pi_1(v) = d\pi_2(v) = 0$. Let $\gamma : (-\varepsilon, \varepsilon) \to M \times N$ be an integral curve with $\dot{\gamma}(0) = v$. Note that $\gamma = (\gamma_1, \gamma_2)$ and thus $v = \dot{\gamma} = (\dot{\gamma}_1, \dot{\gamma}_2) = (d\pi_1(v), d\pi_2(v)) = 0$. So this map is injective and thus isomorphism.

(d) Let $\gamma, \gamma_1, \gamma_2$ be as in (b), but at point $(m_0, n_0)$. Then $v(f) = \frac{d}{dt} \bigg|_{t=0} f(\gamma(t)) = \frac{\partial f}{\partial m}(m_0, n_0) \frac{d\gamma_1}{dt}(0) + \frac{\partial f}{\partial n}(m_0, n_0) \frac{d\gamma_2}{dt}(0)$. Let $f_1 = f \circ i_{m_0} \in C^\infty(M)$ and $f_2 = f \circ i_{n_0} \in C^\infty(N)$.
\[ C^\infty(N). \text{ Then } v_1(f_1) + v_2(f_2) = \left. \frac{d}{dt} \right|_{t=0} f_1(\gamma_1(t)) + \left. \frac{d}{dt} \right|_{t=0} f_2(\gamma_2(t)) = \partial f \frac{d\gamma_1}{dt}(0) + \frac{\partial f}{\partial n}(m_0, n_0) \frac{d\gamma_2}{dt}(0). \]

**Question 5:** For diffeomorphism \( f : M \to N \), the differential \( df_m \) is isomorphism between tangent spaces. So \( M, N \) must have equal dimension.

**Question 6:** Common mistake is to define \( f : TS^1 \to S^1 \times \mathbb{R} \) by \((x, y \frac{d}{dx}) \mapsto (x, y)\), since we don’t have a global coordinate on \( TS^1 \). One need to take the two charts \( U_1, U_2 \) on \( S^1 \) and define \( f \) on the two trivializations \( U_i \times \mathbb{R} \). Note then \( f \) would be \((x, y \frac{d}{dx}) \mapsto (x, y)\) and \((x, y \frac{d}{dx}) \mapsto (x, -y)\) respectively on the two charts so that they can be patched together.

The proof using \( TS^1 \subset \mathbb{R}^2 \times \mathbb{R}^2 \) is a bit cheating but is also accepted.

**Question 7 and 10:** Practically the same question. If \( f \) is everywhere non-singular, then it is local diffeomorphism and thus an open map. So the image of \( f \) has to be open and compact, contradicting Heine-Borel.

**Question 8:** \( f(x, y) = (e^x \cos y, e^x \sin y) \) would work.

**Question 9:** Compute the differential \( df = \left( 3x^2 + y \quad x + 3y^2 \right) \)

Note that it is singular at \((0, 0)\) and \((-1/3, -1/3)\). So not a submanifold there.

For \( p = (1/3, 1/3) \), we have \( df \) nonsingular. Also \( f(1/3, 1/3) \neq f(0, 0) \neq f(-1/3, -1/3) \), so \( df \) is nonsingular at every point of \( f^{-1}(f(p)) \). So a submanifold.