HW 2

5 problems graded in Chapter 2: 1.1, 2.5, 3.4, 4.1, 9.26, each problem 15 points.
Completion 25 points.

Ex. 2.1.1. (a) $g$ is differentiable with $g'(x) = 1 + x^2/(1-x^2)^2 > 0$. Thus $g$ is increasing.
Also, the image of $g$ is $\mathbb{R}$. Since $g$ is increasing, $g^{-1}$ is defined and continuous, so $g$ is a homeomorphism. The restriction to $[0,1)$ maps onto $[0,\infty)$ and gives a homeomorphism there.

(b) Take the homeomorphism $h(x) = (x-y)/r$ with inverse $h^{-1}(z) = y + rz$.
(c) Use $g|[0,1)$ of part (a) to define $k : B(0,1) \to \mathbb{R}^n$ by $k(x) = g(\|x\|)x/\|x\|$ if $x \neq 0$ and $k(0) = 0$, with inverse $k^{-1}(y) = g^{-1}(\|y\|)y/\|y\|$ if $y \neq 0$ and $k^{-1}(0) = 0$.

Ex. 2.1.5. If there is a boundary point, then we can assume that a ngh is mapped homeomorphically onto $\mathbb{H}^n$ and the boundary points in the ngh are mapped onto $\partial \mathbb{H}^n = \mathbb{R}^{n-1}$. The the boundary points satisfy the definition of an $(n-1)$-manifold.

Ex. 2.1.6. The homeomorphism must send interior points to interior points and boundary points to boundary points. Thus it must restrict to a homeomorphism between $\partial M$ to $\partial N$ (as well as between the interiors).

Ex. 2.2.2 (a). $P$ is continuous and connected since $S^2$ is and $f$ is continuous. To show that the map $f$ is a local homeomorphism, for any $x \in S^2$, we just choose a ngh $U$ small enough so that it does not contain any pair of antipodal points.

Ex. 2.2.3. Start with $\mathbb{R}^2$ and identify $(x,y)$ with $(x,y + 2)$ to obtain the infinite cylinder. We further identify $(x,y)$ with $(x+2,-y)$ to form the Klein bottle.

Notice that, after these identifications, each point will be identified with a point in $D^1 \times D^1$, the upper and lower edges identified via $(x,-1) \sim (1,-y)$, and the right and left edges identified via $(-1,y) \sim (1,-y)$. The map from the plane to the quotient space is a local homeomorphism, so the quotient $K$ is a surface.

Ex. 2.2.5. The Disk Lemma says that this is true when there is one disk pair $D_i, D'_i$. Assume the result is true for $n$ disk pairs and suppose we have $n+1$ disk pairs $D_i, D'_i, i = 1,...,n+1$. Apply the induction hypothesis to get a homeomorphism $h : M \to M$ sending $D_i$ to $D'_i$ for $1 \leq i \leq n$. Let $D_{n+1}' = h(D_{n+1})$. Consider the manifold $N$ from $M$ by removing the union of the interior of $D_i'$ for $1 \leq i \leq n$. This is a manifold with boundary $\partial N = \partial M \cup \partial D'_1 \cup ... \cup \partial D'_{n+1}$. Then the Disk Lemma says that there is a homeomorphism $k : N \to N$ sending $D_{n+1}'$ to $D_{n+1}'$ and is the identity on the boundary. We then extend $k$ to a homeomorphism $k' : M \to M$ which sends the disks $D'_i$ identically to themselves for $1 \leq i \leq n$. The composition $k' \circ h$ is the desired homeomorphism.

Ex. 2.3.3. Notice that $f \circ r'$ is orientation preserving. Thus there is an isotopy $F_t : I \to I$ with $F_0 = id, F_1 = f \circ r'$. Then $G_t = F_t \circ r'$ gives an isotopy with $G_0 = r'$ and $G_1 = f r' r' = f$.

Ex. 2.3.4. (a) Given $g$, form $f : I \to I$ by $f = hgh^{-1}$. By Lemma 2.3.3 there is an isotopy $F_t$ with $F_0 = id, F_1 = f$. Then $G_t = h^{-1}F_t$ gives an isotopy with $G_0 = id, G_1 = g$. 

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(b) Apply (a) to $gr$ to get an isotopy $G_t$ between $id$ and $gr$. Then $G_tr$ is a desired isotopy.

Ex. 2.4.1. For the annulus, the complement is the disjoint union $S^1 \times [-1, 0) \cup S^1 \times (0, 1]$, which is separated.

Think of the Mobius band as $D^1 \times D^1 / (1, -y) \sim (-1, y)$. When we remove the center circle $D^1 \times \{0\} / (-1, 0) \sim (1, 0)$, the space can be divided into the equivalence classes of points $M_1$ with 2nd coordinate $> 0$ or $M_2$ with 2nd coordinate $< 0$, each of which is path connected from $D^1 \times D^1 / D^1 \times \{0\}$. But the equivalence relation says that some points in $M_1$ are in $M_2$, so the whole complement is path connected. In fact, the complement can be shown to be homeomorphic to $S^1 \times (0, 1]$.

Ex. 2.4.2. If the Mobius band were embedded in the plane, then a ngh of the image of the interior would be an open set by the Invariance of Domain. By the Jordan Curve Theorem, the image of the central circle would separate the plane, and in particular there would be points in the image of the interior of the Mobius band which lie in different components of the complement. Since there is only one path component in the complement of the center circle of the Mobius band and it si embedded we can find a path connecting points in different components, which is a contradiction.

Ex. 2.4.3. We could change orientations on each of the two handles. If we change orientations on both of them, then all arrows are reversed and the two orientations still disagree on the left edge of $h^0$ and agree on the right edge. If we change the orientation of one handle but retain the orientation of the other handle, then the orientations will now agree on the left edge of $h^0$ and disagree on the right edge.

Ex. 9.26. (a) Let $F_t : S^1 \to S^1$ be an isotopy between $f|_{S^1} : S^1 \to S^1$ and the identity with $F_0 = id, F_1 = f|_{S^1}$ given by Lemma 2.3.6. Extend $F_t$ to $G_t : D^2 \to D^2$ radially by defining $G_t(0) = 0, G_t(x) = rF_t(x/r)$, with $r = \|x\|$. This is the radial extension which identifies the circle of radius $r$ with the unit circle and maps it to itself the same way $F_t$ did. Then $g = G_1^{-1} \circ f$ will be isotopic to $f$ and is the identity on $S^1$.

We show that $g$ is isotopic to the identity by gradually extending the part of the disk on which $g$ is the identity to the whole disk and compressing the action of $g$ to smaller and smaller subdisks. Define $H_t : D^2 \to D^2$ by $H_t(x) = x$ if $\|x\| \geq 1 - t, t \neq 1$ or $t = 1$, and $H_t(x) = (1 - t)g(x/(1 - t))$ if $\|x\| \leq 1 - t, t \neq 1$.

(b) Composing $f$ with a reflection $r$ to give $g = fr$, then $g$ is isotopic to the identity with an isotopy $G_t$. Then $F_t = G_tr$ gives an isotopy between $r$ and $f$. 

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