MATH 8302 HW 2 Solutions and comments

February 24, 2020

Questions 2,9 are not graded, so solutions are given. For graded questions, only comments are given. Questions are ordered as on the website.

Question 1: Need to check $d\psi_I$ is surjective. Locally one has $d\psi_I(A) = \lim_{h \to 0} \frac{\psi(I + hA) - \psi(I)}{h} = A + A^t$. So dim $Im(d\psi_I) = d^2 -$ dim $Ker(d\psi_I) = \frac{d(d+1)}{2}$ is surjective.

Question 2: (a) Locally write $X = \sum a_i \partial_{x_i}$ and $Y = \sum b_i \partial_{x_i}$, where a_i, b_i are smooth functions. Then $[X,Y] = \sum_i \sum_j (a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i}) \partial_{x_j}$, which is a smooth vector field.

(b) One can check in local coordinates. Coordinate freely, one can write $[fX, gY] = (fX)(gY) - (gY)(fX) = f((Xg)Y + gXY) - g((Yf)X + fYX) = fg[X,Y] + f(Xg)Y - g(Yf)X$.

(c)(d) Trivial computation.

Question 3: Easy computation.

Question 4: Because of compactness, we can get a lower bound ε such that locally at any point, an integral curve exists in time $(-\varepsilon, \varepsilon)$. So suppose a maximal integral curve at p is $\gamma : (a,b) \to M$, one can extend the curve at $\gamma(b - \varepsilon/4)$ to $\tilde{\gamma} : (a, b + \varepsilon/2)$, contradicting the maximality.

Question 5: For $f : \mathbb{R}^2 \to \mathbb{R}$, df is either identically 0 or rank 1. In the former case, f is constant map. In the latter case, f must have 1 dimensional preimage.

Question 6: Easy computation.

Question 7: Take $det : \mathbb{R}^4 \backslash \{0\} \to \mathbb{R}$ to be the determinant. Then one can show det is submersion and $det^{-1}(0)$ is exactly the space of rank 1 matrices.

Question 8: It's easy to show it is immersion and injective. To show density, one can show $Im(f) \cap \{x\} \times S^1$ is dense for all x. However, this is not an embedding because for any open U in $S^1 \times S^1$, $Im(f) \cap U$ is not path connected. So f is not homeomorphic onto the image.

Question 9: For $v \in P_m$, let γ be a integral curve in a neighborhoood of m in P such that $\gamma(0) = m$ and $\dot{\gamma}(0) = v$. Then we have $d\psi(di(v)) = \left.\frac{d}{dt}\right|_{t=0} (\psi(i(\gamma(t)))) = 0$ because $\psi \circ i \circ \gamma$ is a constant map.