15.12:

\[
\begin{array}{ccccccc}
(0,1,0) & (0,1,1) & (0,1,2) & (1,1,0) & (1,1,1) & (1,1,2) \\
(0,0,0) & (0,0,1) & (0,0,2) & (1,0,0) & (1,0,1) & (1,0,2) \\
(0,1,0) & (0,1,1) & (0,1,2) & (1,1,0) & (1,1,1) & (1,1,2) \\
(0,0,0) & (0,0,1) & (0,0,2) & (1,0,0) & (1,0,1) & (1,0,2) \\
(0,1,0) & (0,1,1) & (0,1,2) & (1,1,0) & (1,1,1) & (1,1,2) \\
(0,0,0) & (0,0,1) & (0,0,2) & (1,0,0) & (1,0,1) & (1,0,2) \\
(1,1,0) & (1,1,1) & (1,1,2) & (1,1,0) & (1,1,1) & (1,1,2) \\
(1,1,0) & (1,1,1) & (1,1,2) & (1,1,0) & (1,1,1) & (1,1,2) \\
(1,1,0) & (1,1,1) & (1,1,2) & (1,1,0) & (1,1,1) & (1,1,2) \\
(1,1,0) & (1,1,1) & (1,1,2) & (1,1,0) & (1,1,1) & (1,1,2) \\
\end{array}
\]

\[\langle (1,1,1) \rangle = \{(1,1,1), (0,1,2), (1,1,0), (0,0,2), (1,0,1), (0,0,1)\}\]

Thus \(\mathbb{Z}_2 \times \mathbb{Z}_3\) is cyclic.

15.13:

\[\langle (1,2,3), (2,4) \rangle = \{(1,2,3), (1,3)(2,4), (1,3), (1,4)(2,3), (1,2)(3,4), (2,4), (1,2)(3,4), (1)\}\]

15.22:

a). \(\langle (2,2), (2,2) \rangle = \{(2,2), (2,2), (2,2), (2,2)\}\)

b). \(\langle (2,2) \rangle = \{(2,2), (2,2), (2,2), (2,2)\}\) for left \(\mathbb{Z}_2\) side \(\langle (2,2) \rangle\)

\(\langle (2,2) \rangle = \{(2,2), (2,2), (2,2), (2,2)\}\) for right \(\mathbb{Z}_2\) side \(\langle (2,2) \rangle\)

\[\langle (2,2) \rangle \times \langle (2,2) \rangle = \{(2,2), (2,2), (2,2), (2,2)\}\]

16.2:

Let \(H = \langle (2,2) \rangle\). \Rightarrow\) Right cosets are:
\[ H = \{ e, [01], [37], [67], [97], [3] \} \]

\[ H \oplus [11] = \{ e, [11], [41], [71], [101] \} \]

\[ H \oplus [21] = \{ e, [27], [57], [87], [111] \} \]

16.13

\[ G = S_3, \quad H = \{ e, (12) \} \]

\[ S_3 = \{ e, (12), (13), (23), (123), (132) \} \]

So left cosets are:

- \((1) H = \{ e, (12) \} = (12) H\)
- \((13) H = \{ e, (123) \} = (123) H\)
- \((23) H = \{ e, (132) \} = (132) H\)

For the right cosets,

\[ H(123) = \{ (123), (23) \}, \] which

is not equal to any one of the left cosets.

Thus they are distinct.

17.11

Subgroups of \( \mathbb{Z}_6 \): \( \langle [11] \rangle, \langle [22] \rangle, \langle [33] \rangle, \langle [66] \rangle = \mathbb{Z}_6 \)

\( \langle [44] \rangle = \langle [22] \rangle = \{ [22], [44], [00] \} \)

\( \langle [55] \rangle = \{ [33], [11] \} \)

and \( \langle [00] \rangle = \{ [00] \} \)

17.12


\( \langle [12] \rangle = \{ e, [11], [12] \} \)

\( \langle [13] \rangle = \{ e, [11], [13] \} \)

\( \langle [23] \rangle = \{ e, [11], [23] \} \)

\( \langle [123] \rangle = \{ e, [11], [123], [132], [23] \} \)

and \( S_3 \)

\( S_3 \)
17. 24.
By Lagrange's theorem we know
\[\forall a \in G, \quad |\langle a \rangle| = \frac{|G|}{\text{ord}(a)} = p^2\]

\[\Rightarrow |\langle a \rangle| = 1, \quad p, \quad \text{or} \quad p^2.\]

\[\text{if } |\langle a \rangle| = 1 \quad \text{or} \quad p \Rightarrow a^p = e.\]

\[\text{for } p^2, \quad \text{if } |\langle a \rangle| = p^2 = |G| \Rightarrow G = \langle a \rangle \Rightarrow G \text{ is cyclic} \Rightarrow e.\]

\[\text{thus } \forall a \in G, \quad a^p = e. \quad \Box\]

17. 28:
\[\langle (1 \ 2 \ 3), (1 \ 2) (3 \ 4) \rangle = \{ (1 \ 2 \ 3), (1 \ 3 \ 2), (1 \ 2 \ 3 \ 4), (1 \ 3 \ 2 \ 4), (1 \ 4 \ 2 \ 3), (2 \ 4 \ 1), (2 \ 3 \ 4 \ 1), (1 \ 4 \ 3 \ 2), (1 \ 4 \ 2 \ 3) \}\]

\[\langle (1 \ 2 \ 3) \rangle \neq 1 \Rightarrow \langle (1 \ 3 \ 2) \rangle \neq 1 \Rightarrow \ldots \Rightarrow \langle (2 \ 4 \ 1) \rangle \neq 1 \Rightarrow 1 = 3\]

\[\langle (1 \ 2) (3 \ 4) \rangle = \langle (1 \ 3) (2 \ 4) \rangle = \langle (1 \ 4) (3 \ 2) \rangle = 2\]

\[\langle (1 \ 1) \rangle = 1\]

\[\Rightarrow \text{more have order 4.}\]

17-30:
We know \[|H| = |aH| = |Ha| \quad \forall a \in G.\]

If \(a \in H\), \(\Rightarrow aH = H = Ha\).

If \(a \not\in H\), \(\Rightarrow aH \neq H\) (since \(a \in H\))

and since we have \(aH \cap H = \emptyset\)

if \(aH \neq H\), then we know \(aH = G \setminus H\),

since \(|H| = \frac{|G|}{2}\)

Similarly, \(Ha = G \setminus H\), \(\Rightarrow aH = Ha\),

thus \(\forall a \in G, \quad aH = Ha. \quad \Box\).
18.8: since these are the identities

\[ \theta(a) = b \]
\[ \theta(b) = a \]
\[ \theta(c) = c \]

so \( \theta(b \# b) = \theta(c) = c = a \# a = \theta(b) \# \theta(b) \)
\( \theta(b \# c) = \theta(a) = b = a \# c = \theta(b) \# \theta(c) \)
\( \theta(c \# c) = \theta(b) = a = c \# c = \theta(c) \# \theta(c) \)

Thus \( \theta \) is an isomorphism.