HW 5


Ex. 8.3. The proof of (b) with the subspace the empty space gives (a). The identity map gives a homotopy equivalence of \((X, A)\) to itself. If \(f : (X, A) \to (Y, B)\) with homotopy inverse \(g : (Y, B) \to (X, A)\), there are homotopies \(F : (X, A) \times I \to (X, A)\), \(G : (Y, B) \times I \to (Y, B)\) with \(F_0 = 1_{(x,a)}, F_1 = g f, G_0 = 1_{(Y,B)}, G_1 = f g\). These same maps then show that there is a homotopy equivalence fro \((Y, B)\) to \((X, A)\). Thus the relation is symmetric. Supposing that there is also a homotopy equivalence of pairs \(h : (Y, B) \to (Z, C)\) with homotopy inverse \(k\) and homotopies \(H, K\), then the map \(h f : (X, A) \to (Z, C)\) will have homotopy inverse \(g k : (Z, C) \to (X, A)\). The homotopy from \((g k)(h f) = g(kh)f\) will be given by first using \(H_t\) to homotope \(kh\) to the identity and thus homotoping \(g(kh)f\) to \(gf\) and then using \(F_t\) to homotope \(gf\) to the identity. We similarly use \(G_t\) and \(K_t\) to homotope \((h f)(g k) = h(f g)k\) to the identity.

Ex. 8.5. From the deformation retract we push out radially from the missing point to send the complement of the points inside the teardrops to the wedge. For points outside the teardrops, we push in radially toward the wedge point, stopping when we hit a circle. We choose the parametrization to hit the circle at time 1. For an exterior point \(x\), for example, we will be taking the straight line from \(x\) to the wedge point \(w\) and finding the first point \(y\) on one of the circles (which might be \(w\)) that it hits. During the homotopy from the identity to the retraction, this point will move along the path \((1 - t)x + ty\).

Ex. 8.8(a). We can reduce the problem to describing a homotopy equivalence between a rectangle \(D^1 \times D^1\) and \(\{\pm 1\} \times D^1 \cup D^1 \times \{0\}\). But this can come from deformation retraction from \(D^1 \times [0, 1]\) onto \(\{\pm 1\} \times [0, 1] \cup D^1 \times \{0\} = L\) and a similar deformation retraction on the bottom half. To describe the deformation retraction for the top piece, we take the point \(p = (0, 2)\) and consider straight line radiating from \(p\). Such a line intersects \(D^1 \times [0, 1]\) in a line segment joining \(x \in D^1 \times \{1\}\) to a point \(y \in L\), except for the case of \((\pm 1, 1)\) where it intersects in the point itself. Our deformation retraction sends this segment \(xy\) onto \(y\) with the standard linear deformation retraction so at time \(t\) the image is the line segment joining \((1 - t)x + ty\) to \(y\).

Ex. 9.1. For the wedge \(W_k\), we prove the result by induction on \(k\). It is true for \(k = 1\) by computation of the fundamental group of \(S^1\). Assuming it is true for \(W_k\), we write \(W_{k+1} = A \cup B\), where \(A\) is \(W_k\) together with a small arc into the \(k + 1\)-th circle about the wedge point to make the set open. This deformation retracts to \(W_k\) and so has fundamental group \(F_k\). Similarly, let \(B\) equal the last circle plus small arcs in the other \(k\) circles in \(W_k\) about the wedge point. This also deformation retracts to the last circle, so its fundamental group is \(F_1\). The intersection deformation retracts to the wedge point, so has trivial fundamental group. Hence the Seifert-van Kampen theorem says that the fundamental group of \(W_{k+1}\) is \(F_k \ast F_1 = F_{k+1}\).

Ex. 9.4. (a) The homeomorphism \(f : S^1 / x \sim e^{2\pi i / k} x \to S^1\) is induced by the map \(g : S^1 \to S^1, g(z) = z^k\). The map is continuous and surjective and the only points going
to the same point are identified in the quotient space. Thus it induces homeomorphism of
the quotient space to $S^1$.

(b) Following the hint and using the fact that the complement of the center point
deformation retracts to the boundary circle, we get $\pi_1(A, x) = \mathbb{Z}$. Since the interior of the
disk is contractible, we have $\pi_1(B, x) = \{e\}$. The intersection deformation retracts to the
circle and the map $\pi_1(A \cap B, x) \to \pi_1(A, x)$ is multiplication by $k$. Thus the fundamental
group is $\mathbb{Z}_k$.

Ex. 9.6. All are non-orientable. Part (a) has two boundary circles, and part (b) has
one boundary circle. In part (a) the abelianized fundamental group is $\mathbb{Z} \oplus \mathbb{Z}$, while in part
(b) it is $\mathbb{Z}$.

For part (a) use the computation that the abelianization of $F_k$ is $k\mathbb{Z}$, since the abelian-
ization of the free group $F_k$ is free Abelian.

For part (b), trace the arrows to get a relation between the generators of $\pi_1$ for the
surfaces in part (a). Then you can do direct calculation by quotient out THIS relation in
the abelianization of the part (a).

The effect of adding the 2–handle in part (b) is to add the relation $a = b^2$.

The space is $P_{(2)}$ in part (a) and is $P_{(1)}$ in part (b).

Ex. 9.7. Both surfaces are non-orientable. There is one boundary circle in part (a)
and none in part (b). In part (a) the abelianized fundamental group is $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ and
the space is $P_{(4)}^{(1)}$.

In part (b) adding the 2–handle introduces $a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2$, which gives the
abelianized fundamental group $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2$.

Here is an example how you do the abelinization: If you have a relation $aba^{-1}b$ in
$<a, b>$, then the abelianization is $\mathbb{Z} \oplus \mathbb{Z}_2$, since $aba^{-1}b^{-1}$ is trivial in the ‘ab’ quotient, so $aba^{-1}b$
becomes $b^2$ in the ‘ab’ quotient.

The space is $T\#K = P_{(4)}^{(1)}$.

Ex. 11.22. (a) There is a point $C$ in the center of the star $S$ so that every straight
line from $x \in S$ to $C$ lies in $S$. We then use the straight line homotopy.

(b) Let $F : X \times I \to X$ with $F(0, x, 0) = x, F(x, 1) = x_0$. Then if $x \in X$, the path
$f(t) = F_t(x)$ is a path connecting $x$ to $x_0$.

(c) Let $F$ be the map in (b) and suppose that $f : (I, \{0, 1\}) \to (X, x)$ is a loop at
x. Let $G : I \times I \to X$ be $G(s, t) = F(F(s), t)$. Then $G_0(s) = f(s), G_1(s) = x_0$ and
$G^I_{(0,1)} \times I = \alpha$ where $\alpha(t) = F(x, t)$. By composing $G$ with the map from the square to
itself which is determined by dividing the square into triangles and mapping them affine
linearly suing the map given by

$$(0, 0) \to (0, 0), (1, 0) \to (1, 0), (0, 1) \to (0, 0), (1/3, 1) \to (0, 1),$$

$$(2/3, 1) \to (1, 1), (1, 1) \to (1, 0), (1/2, 1/2) \to (1/2, 1/2),$$

we get a homotopy between $f$ and $\alpha * c_{x_0} * \bar{\alpha}$. This latter map is homotopic to a constant
map at $x$ since we can first absorb the $c_{x_0}$ part and then use the argument that $\alpha * \bar{\alpha}$ is
homotopic to the constant map $c_x$.  

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Ex. 11.23(a). This just uses the deformation retraction of $S^1_b \setminus \{-1\}$ to the point 1 which can be written as $F(e^{is}, t) = e^{ist}$ for $s \in [-\pi, \pi], t \in [0, 1]$. On $T$ we use $G(x, y, t) = (x, F(y, t))$.

Ex. 11.25. We take circles $C_1 = S^1 \times \{-1\}$ and $C_2 = S^1 \times \{1\}$. Then in $T \# T$ we remove $C_1$ from the first copy of $T$ and remove $C_2$ from the second copy. We then form the connected sum near $S^1 \times \{1\}$ in the first copy and $S^1 \times \{-1\}$ in the second copy. The repeating the deformation retraction from the last exercise, we get a deformation retraction of $T \# T \setminus (C_1 \cup C_2)$ onto $A \# A \simeq S(4)$.

Ex. 11.39. We regard the circle $C$ as $D^1 / -1 \sim 1$. Then the map $f : H \to D^1 / -1 \sim 1$ by sending $h^0$ to $[1]$ and sending $h^1 = D^1 \times D^1$ via projection onto the first coordinate. We map $g : C \to H$ by sending $D^1$ to the union of $D^1 \times \{0\} \subset h^1$ and the line segments joining the attaching points to the center. The map $fg : C \to C$ is induced from a map $D^1 \to D^1$ which sends $[-1, -1/2]$ to 0, sends $[1/2, 1]$ to 1, and expands $[-1/2, 1/2]$ to $[-1, 1]$ via an affine linear map. This is homotopic to the identity by a homotopy at time $t$ which sends $[-1/2, -t/2]$ to $-1$, sends $[1 - t/2, 1]$ to 1, and expands $[-t/2, t/2]$ to $[0, 1]$. This is consistent with the identifications to give a homotopy on the quotient space $C = D^1 / \sim$.

The map $gf$ sends $h^0$ to the center point and sends $D^1$ to the union of $D^1 \times \{0\}$ and the line segments joining the attaching points to the center. During the homotopy the image of $h^0$ at time $t$ will be the smaller disk of radius $1 - t$ within $D^2 = h^0$. On $h^1$ we will have already determined how the attaching regions are moved during the homotopy. They will move within triangular wedges joining the attaching regions to the center. At time $t$ the image of the handle and the wedge will just move within itself so that the attaching region moves to radius $1 - t$. At the end of this stage of the homotopy the handle will have moved within the wedge so that the handle itself now occupies the whole region. The handle at this stage will be embedded except for the attaching region which will be collapsed to the center point. The final stage of the homotopy will just deformation retract along the second factor $D^1$ to 0 so that we end up with the map $gf$. 