Homework #8

Solutions

25.2: \[ \mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_5, \mathbb{Z}_6, \mathbb{Z}_8 \]

25.6: \[ x = \frac{a}{b}, \quad a \in \mathbb{Z}, \; b \in \mathbb{Z} \setminus \{0\} \]

25.10: Suppose \( \mathbb{Z}_p \) is not an integral domain. \( \Rightarrow \exists [a], [b] \in \mathbb{Z}_p \) s.t.
\( [a][b] = [0], \) with \( [a], [b] \neq [0] \)
but then either \( a \cdot b = 0 \) or \( p | a \cdot b \).

The first case cannot happen since \( a, b \neq 0 \).

The second case would imply \( p | a \) or \( p | b \),
which also cannot happen since \( a, b < p \).
\( \Rightarrow \), thus \( \mathbb{Z}_p \) is an integral domain.

26.12: "\( \Rightarrow \)" If \( D \) is a field, \( \Rightarrow \) given
any \( ax = b \), \( \exists ! a^{-1} \in D \) s.t. \( a^{-1}a = 1 \).
\( \Rightarrow a^{-1}ax = a^{-1}b \)
\( = x = a^{-1}b \)
since \( a^{-1} \) is unique, \( \Rightarrow x \) is unique.

"\( \Leftarrow \)" If \( D \) is not a field, but
is an integral domain, \( \Rightarrow D \) has some non-zero element \( a \) s.t.
\( a \) does not have an \( a^{-1} \) inverse.
\( \Rightarrow \) the equation \( ax = b \) has
no solution.
26.14:  
- $0 + 0 \sqrt{2} = 0 \in \mathbb{Q}[\sqrt{2}]
- (a+b\sqrt{2}) + (c+d\sqrt{2}) = (a+c) + (b+d)\sqrt{2} \in \mathbb{Q}[\sqrt{2}]
- (a+b\sqrt{2})(c+d\sqrt{2}) = (ac + 2bd) + (ad + bc)\sqrt{2} \in \mathbb{Q}[\sqrt{2}]
- \frac{1}{a+b\sqrt{2}} = \frac{(a-b\sqrt{2})}{(a+b\sqrt{2})(a-b\sqrt{2})} = \left(\frac{a}{a^2-2b^2}\right) + \left(\frac{-b}{a^2-2b^2}\right)\sqrt{2} \in \mathbb{Q}[\sqrt{2}]

where for the last line, $a \neq 0, b \neq 0$ and we know $a, b \in \mathbb{Q} \Rightarrow a-b\sqrt{2} \neq 0$

thus, $\mathbb{Q}[\sqrt{2}]$ is a subfield of $\mathbb{R}$

27. 1:  
\[ \forall a \in \mathbb{R}, \quad ea = ae = a \]
- $e(au) = e(ae) = e(a) = e(a) = e(a) = e(a)$
\[ \Rightarrow \text{by definition } e \text{ is the unity of } S. \]

27. 14:  
if $x = -x \neq x \in \mathbb{R}$, then in particular $e = -e$
\[ \Rightarrow e + e = 0 \]
\[ = 2e = 0 \]
and since $1e = i2 \neq 0$
\[ \Rightarrow 2 \text{ is the characteristic of } \mathbb{R}. \]

if $e = 0$, then $1$ is the characteristic of $\mathbb{R}$. only when $e \neq 0.$
27.21: \( xy = (xy)^2 = xy \cdot xy \)
and \( xy = x^2 y^2 = xx yy \)
\( \implies yx = xy \quad \forall x, y \in R \)
also, \( -x = (-x)^2 = x^2 = x \)
\( \implies x + x = 2x = 0 \quad \forall x \in R. \)
\( \text{one such ring is } \mathbb{Z}_2 \)

27.22: Since \( R \) is finite,
\( \langle e \rangle = \{ \exists x \in R \mid x = n \cdot e \text{ for some } n \in \mathbb{Z} \} \)
\( = \{ \exists x \in R \mid x = n \cdot e \text{ for some } n \in \mathbb{N} \} \)
thus \( |\langle e \rangle| = \text{ characterize of } R. \)
Then by Lagrange's Theorem,
\( |\langle e \rangle| \mid |R| \)
\( \implies \text{char}(R) \mid |R|. \)