1 5.12.  

K"unneth formula for compact cohomology. The K"unneth formula for compact cohomology states that for any manifolds $M$ and $N$ having a finite good cover,

$$H^c_\ast(M \times N) = H^c_\ast(M) \otimes H^c_\ast(N).$$

(a) In case $M$ and $N$ are orientable, show that this is a consequence of Poincaré duality and the K"unneth formula for de Rham cohomology.

(b) Using the Mayer-Vietoris argument, prove the K"unneth formula for compact cohomology for any $M$ and $N$ having a finite good cover.
Solution. (a) Since $M, N$ have finite good covers, it follows that their cohomologies and compact cohomologies are finite-dimensional, whence Poincaré duality does tell us that

$$H^q_c(X) \simeq H^{n-q}(X), \quad \text{for each } q \in \mathbb{N}_0,$$

where $X = M, N$. Let $m$ be the dimension of $M$ and $n$ the dimension of $N$. Then, for each integer $k = 0, \ldots, m + n$, we have that

$$H^k_c(M \times N) = \left( H^{m+n-k}(M \times N) \right)^* = \left( \bigoplus_{p+q=m+n-k} H^p(M) \otimes H^q(N) \right)^*$$

$$= \bigoplus_{p+q=m+n-k} (H^p(M))^* \otimes (H^q(N))^* = \bigoplus_{p+q=m+n-k} H^{m-p}_c(M) \otimes H^{n-q}_c(N)$$

$$= \bigoplus_{s+t=k} H^s_c(M) \otimes H^t_c(N),$$

where we used Poincaré duality, then the Kunneth formula for De Rham cohomology, then the commutativity of the dual operator $(\cdot)^*$ with direct sum and tensor product, then Poincaré duality, and finally a change of variables $s = m - p, t = n - q$.

(b) We follow the Mayer-Vietoris argument. The natural projections $\pi : M \times N \to M$ and $\rho : M \times N \to N$ give rise to a map on forms with compact support

$$\omega \otimes \phi \mapsto \pi^* \omega \wedge \rho^* \phi.$$

We have that $\pi^* \omega \wedge \rho^* \phi$ has compact support in $M \times N$. Hence we have the pushforward map in compact cohomology

$$\psi : H^*_c(M) \otimes H^*_c(N) \to H^*_c(M \times N).$$

We are done as soon as we show that $\psi$ is an isomorphism, which we now intend to prove. Let $U$ and $V$ be open sets in $M$ and let $n$ be a fixed integer. From the Mayer-Vietoris sequence

$$\cdots \to H^p_c(U \cap V) \to H^p_c(U) \oplus H^p_c(V) \to H^p_c(U \cup V) \to \cdots$$

we get an exact sequence by tensoring with $H^{n-p}_c$,

$$\cdots \to H^p_c(U \cap V) \otimes H^{n-p}_c(N) \to (H^p_c(U) \oplus H^p_c(V)) \otimes H^{n-p}_c(N)$$

$$\to H^p(U \cup V) \otimes H^{n-p}_c(N) \to \cdots,$$

since tensoring with a vector space preserves exactness. Summing over all integers $p$
yields the exact sequence

\[ \cdots \rightarrow \bigoplus_{p=0}^{n} H_p^c(U \cap V) \otimes H_c^{n-p}(N) \]

\[ \rightarrow \bigoplus_{p=0}^{n} \left( (H_p^c(U) \otimes H_c^{n-p}(N)) \oplus (H_p^c(V) \otimes H_c^{n-p}(N)) \right) \]

\[ \rightarrow \bigoplus_{p=0}^{n} H_p^c(U \cup V) \otimes H_c^{n-p}(N) \rightarrow \cdots . \]

The following diagram is commutative

\[ \bigoplus_{p=0}^{n} H_p^c(U \cap V) \otimes H_c^{n-p}(N) \rightarrow \bigoplus_{p=0}^{n} \left( (H_p^c(U) \otimes H_c^{n-p}(N)) \oplus (H_p^c(V) \otimes H_c^{n-p}(N)) \right) \rightarrow \bigoplus_{p=0}^{n} H_p^c(U \cup V) \otimes H_c^{n-p}(N) \]

Since \( M \) is an \( m \)-manifold with finite good cover, each of \( U, V, U \cap V \) is diffeomorphic to \( \mathbb{R}^m \). Note that \( H^k_c(\mathbb{R}^m) \simeq 0 \) for all \( k \neq m \), and \( H^m_c(\mathbb{R}^m) \simeq \mathbb{R} \) (see p.46). Hence, if \( n \geq m \), then

\[ \bigoplus_{p=0}^{n} H_p^c(\mathbb{R}^m) \otimes H_c^{n-p}(N) \cong \mathbb{R} \otimes H_c^{n-m}(N) \cong H_c^{n-m}(N) \cong H_c^n(\mathbb{R}^m \times N), \]

where we used Proposition 4.7 in the last step. Hence the Kunneth formula is verified for \( U, V, U \cap V \). By the Five lemma, then the Kunneth formula is also true for \( U \cup V \). Enough to show that \( \psi \) is an isomorphism on \( U, V, U \cap V \). The Kunneth formula now follows by induction on the cardinality of a good cover, as in the proof of Poincaré duality. \( \square \)

2 5.16.

The ray and the circle in \( \mathbb{R}^2 \setminus \{0\} \). Let \( x, y \) be the standard coordinates and \( r, \theta \) the polar coordinates on \( \mathbb{R}^2 \setminus \{0\} \).

(a) Show that the Poincaré dual of the ray \( \{(x,0) : x > 0\} \) in \( H^1(\mathbb{R}^2 \setminus \{0\}) \) is \( d\theta/2\pi \) in \( H^1(\mathbb{R}^2 \setminus \{0\}) \).

(b) Show that the closed Poincaré dual of the unit circle in \( H^1(\mathbb{R}^2 \setminus \{0\}) \) is 0, but the compact Poincaré dual is the nontrivial generator \( \rho(r) dr \) in \( H^1_c(\mathbb{R}^2 \setminus \{0\}) \) where \( \rho(r) \) is a bump function with total integral 1.
Solution. (a) Let $M = \mathbb{R}^2 \setminus \{0\}$ and $S = \{(x,0) : x > 0\}$, which is a closed oriented submanifold of dimension 1. Let $i : S \to M$ be the inclusion map. We need to show that for any $\omega \in H^1_c(M)$, we have that

$$\int_S i^* \omega = \int_M \omega \wedge \frac{d\theta}{2\pi}.$$ 

So let $\omega \in H^1_c(M)$, so that there exist $f, g \in C_c^\infty(M)$ such that $\omega = f(r, \theta) dr + g(r, \theta) d\theta$. Now, $d\omega = 0$ because $\omega$ must be closed, and hence it follows that $\frac{\partial f}{\partial \theta} = \frac{\partial g}{\partial r}$. Integrating this identity over $r$ from 0 to $\infty$ yields easily that the quantity $\int_0^\infty g(r,\theta) d\theta$ is a constant in $r$. Since for all $r$ and all $\theta$ large enough, $g(r,\theta) \equiv 0$ since $g$ is compactly supported, we conclude that $\int_0^\infty g(r,\theta) d\theta = 0$, for some (and hence, for every) $r > 0$. Consequently,

$$\int_S i^* \omega = \int_S f \mid_S dx = \int_S i^* \omega,$$

as claimed.

(b) Let $M = \mathbb{R}^2 \setminus \{0\}$ and $S$ is the unit circle, which is a closed oriented submanifold of dimension 1. Let $i : S \to M$ be the inclusion map. To show that 0 is the closed Poincaré dual of $S$, we have to prove that for any $\omega \in H^1_c(M)$, we have that

$$\int_S i^* \omega = 0.$$ 

So let $\omega \in H^1_c(M)$, so that there exist $f, g \in C_c^\infty(M)$ such that $\omega = f(r, \theta) dr + g(r, \theta) d\theta$. Now, $d\omega = 0$ because $\omega$ must be closed, and hence it follows that $\frac{\partial f}{\partial \theta} = \frac{\partial g}{\partial r}$. Integrating this identity over $\theta$ from 0 to $2\pi$ yields easily that the quantity $\int_0^{2\pi} g(r,\theta) d\theta$ is a constant in $r$. Since for all $r$ and all $\theta$ large enough, $g(r,\theta) \equiv 0$ since $g$ is compactly supported, we conclude that $\int_0^{2\pi} g(r,\theta) d\theta = 0$, for some (and hence, for every) $r > 0$. Consequently,

$$\int_S i^* \omega = \int_0^{2\pi} g(1,\theta) d\theta = 0,$$

as claimed.

We now purport to show that $\rho(r) dr$ is the compact Poincaré dual of $S$, where $\rho(r)$ is a bump function such that $\int_0^\infty \rho(r) dr = 1$. To do so, we have to prove that for any $\omega \in H^1(M)$, we have that

$$\int_S i^* \omega = \int_M \omega \wedge (\rho(r) dr).$$
So let $\omega \in H^1(M)$, so that there exist $f, g \in C^\infty(M)$ such that $\omega = f(r, \theta)dr + g(r, \theta)d\theta$. Now, $d\omega = 0$ because $\omega$ must be closed, and hence it follows that $\frac{\partial f}{\partial \theta} = \frac{\partial g}{\partial r}$. Integrating this identity over $\theta$ from $0$ to $2\pi$ yields easily that the quantity $\int_0^{2\pi} g(r, \theta) d\theta$ is a constant in $r$. Thus, we observe that

$$
\int_S i^*\omega = \int_0^{2\pi} g(1, \theta) d\theta = \left( \int_0^{2\pi} g(1, \theta) d\theta \right) \left( \int_0^\infty \rho(r) dr \right)
$$

$$
= \int_0^\infty \rho(r) \left( \int_0^{2\pi} g(1, \theta) d\theta \right) dr = \int_0^\infty \rho(r) \left( \int_0^{2\pi} g(r, \theta) d\theta \right) dr
$$

$$
= \int_M \left[ f(r, \theta)dr + g(r, \theta)d\theta \right] \wedge (\rho(r) dr) = \int_M \omega \wedge (\rho(r) dr),
$$

where in the second equality we used the fact that the integral of $\rho$ is 1, in the fifth equality we used that $d\omega = 0$ and Fubini's Theorem which is applicable since $\rho$ is non-negative, smooth, and has bounded support in $M$ (since it is a bump function), and $|g|$ is bounded in the support of $\rho$. The claim follows. \hfill $\square$

3 6.2.

Show that two vector bundles on $M$ are isomorphic if and only if their cocycles relative to some open cover are equivalent.

Solution. (Only if). Let $(E, \pi), (E', \pi')$ be two vector bundles over $M$ which are isomorphic, so that there is a vector bundle isomorphism $f : E \to E'$. Let $(U_\alpha, \phi_\alpha)$ be the open cover of $M$ with the corresponding trivializations for $E$, afforded by its definition. Then $(U_\alpha, \phi_\alpha \circ f^{-1})$ is an open cover of $M$ together with trivializations $\phi'_\alpha := \phi_\alpha \circ f^{-1} : E'|_{U_\alpha} \to U_\alpha \times \mathbb{R}^n$ for some $n$. Fix $\alpha, \beta$ and $x \in U_\alpha \cap U_\beta$. Note that, in this case via our construction,

$$
g'_{\alpha\beta}(x) = \phi'_\alpha \phi'^{-1}_\beta(x) = \phi_\alpha f^{-1} f\phi'^{-1}_\beta(x) = \phi_\alpha \phi'^{-1}_\beta(x) = g_{\alpha\beta}(x),
$$

so that $g_{\alpha\beta}$ and $g'_{\alpha\beta}$ are equivalent, but we are not technically done yet because $(E', \pi')$ may be a priori endowed with different trivializations than $\phi'_\alpha$. So let $\{\phi''_\alpha\}$ be any collection of trivializations with which $E'$ is endowed over the open cover $U_\alpha$. Then we may use Lemma 6.1 to see that $g''_{\alpha\beta}$ is equivalent with $g'_{\alpha\beta}$. Since equivalence is transitive, we thus have that $g_{\alpha\beta}$ is equivalent with $g''_{\alpha\beta}$, as desired.

(If). Now fix an open cover $\{U_\alpha\}$ of $M$. Let $(E, \pi), (E', \pi')$ be two vector bundles over $M$, let $\phi_\alpha, \phi_\alpha'$ be the respective trivializations over $\{U_\alpha\}$, and let $g_{\alpha\beta}, g'_{\alpha\beta}$ be the respective cocycles. By hypothesis, there exist invertible maps $\lambda_\alpha : U_\alpha \to GL(n, \mathbb{R})$ such that

$$
g_{\alpha\beta} = \lambda_\alpha g'_{\alpha\beta} \lambda_\beta^{-1}, \quad \text{on } U_\alpha \cap U_\beta,
$$
(here, $\lambda_{\beta}^{-1}$ is the inverse matrix to $\lambda_{\beta}$, not the inverse map of $\lambda_{\beta}$). For each $U_\alpha$, let $f_\alpha : E|_{U_\alpha} \to E'|_{U_\alpha}$ be the map given by

$$f_\alpha := \phi_\alpha^{-1} \circ (\lambda_\alpha^{-1} \cdot \phi_\alpha).$$

It is instructive to chase the map of $f_\alpha$. Let $x \in U_\alpha$ and $\zeta \in \pi^{-1}(x)$. We use the notation $\overrightarrow{\phi}$ for the second component of the map $\phi$ (the one that maps into $\mathbb{R}^n$). Then

$$\zeta \xrightarrow{\phi_\alpha} (x, \overrightarrow{\phi_\alpha} \zeta) \xrightarrow{\lambda_\alpha^{-1}} (x, \lambda_\alpha^{-1}(x) \overrightarrow{\phi_\alpha} \zeta) \xrightarrow{\phi_\alpha^{-1}} \pi'^{-1}(x, \overrightarrow{\phi_\alpha^{-1} \lambda_\alpha^{-1}(x) \overrightarrow{\phi_\alpha} \zeta}).$$

Thus we see that $f_\alpha$ is a fiber-preserving smooth map that is linear on corresponding fibers. Now, if $x \in U_\alpha \cap U_\beta$ and $\zeta \in \pi^{-1}(x)$, then $f_\alpha(\zeta)$ is given above, while similarly we have that

$$f_\beta(\zeta) = \pi'^{-1}(x, \overrightarrow{\phi_\beta^{-1} \lambda_\beta^{-1}(x) \overrightarrow{\phi_\beta} \zeta}).$$

Hence we see that $f_\alpha(x) = f_\beta(x)$ if and only if

$$\overrightarrow{\phi_\alpha^{-1} \lambda_\alpha^{-1} \overrightarrow{\phi_\alpha}} = \overrightarrow{\phi_\beta^{-1} \lambda_\beta^{-1} \overrightarrow{\phi_\beta}},$$

which in turn occurs if and only if

$$\overrightarrow{\phi_\alpha} \overrightarrow{\phi_\beta^{-1}} = \lambda_\alpha \overrightarrow{\phi_\alpha} \overrightarrow{\phi_\beta^{-1}} \lambda_\beta^{-1},$$

which is guaranteed by the hypothesis. It follows that the map

$$f := f|_{U_\alpha}, \quad \text{on each } U_\alpha$$

is a well-defined vector bundle isomorphism. \hfill \square

4 6.10.

Compute $\text{Vect}_k(S^1)$.

**Solution.** Recall that $\text{Vect}_k(S^1)$ is the isomorphism classes of rank $k$ real vector bundles over $S^1$. Let $(E, \pi)$ be a vector bundle over $S^1$, and let $f : [0, 1] \to S^1$ be given by $t \mapsto e^{2\pi it}$. Then $f^{-1}E$ is a vector bundle over $[0, 1]$. Since $[0, 1]$ is contractible, by Corollary 6.9 we have that $f^{-1}E$ is the trivial bundle $[0, 1] \times \mathbb{R}^k$. Now consider all smooth maps $[0, 1] \to S^1$. There are exactly two homotopy classes of such maps, corresponding to $[f]$ and $[-f] = [e^{-2\pi it}]$, whence by Theorem 6.8 we conclude that for each $k \in \mathbb{N}$, there are two isomorphism classes in $\text{Vect}_k(S^1)$, corresponding to $[f^{-1}E]$ and $([-f]^{-1}E)$. \hfill \square

Show that if $E$ is an oriented vector bundle, then $\pi_*\omega_\alpha = \pi_*\omega_\beta$ on $U_\alpha \cap U_\beta$. Hence \{\pi_*\omega_\alpha\}_{\alpha \in I}$ piece together to give a global form $\pi_*\omega$ on $M$. Furthermore, this definition is independent of the choice of the oriented trivialization for $E$.

Solution. Let $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m)$ be the coordinate functions on $U_\alpha$ and $U_\beta$, and $t = (t_1, \ldots, t_n), u = (u_1, \ldots, u_n)$ the fiber coordinates on $\pi^{-1}(U_\alpha)$ and $\pi^{-1}(U_\beta)$ respectively. Fix $\omega \in \Omega^*_c(E)$ and recall that $\omega_\alpha := \omega|_{\pi^{-1}(U_\alpha)}$. By chasing the inclusion maps $j_\alpha : U_\alpha \cap U_\beta \hookrightarrow U_\alpha, j_\beta : U_\alpha \cap U_\beta \hookrightarrow U_\beta, i_\beta : U_\beta \hookrightarrow M, i_\alpha : U_\alpha \hookrightarrow M$ and observing that $i_\alpha j_\alpha = i_\beta j_\beta$ is the same inclusion map, we deduce that

$$\omega_\alpha|_{\pi^{-1}(U_\alpha \cap U_\beta)} = \omega_\beta|_{\pi^{-1}(U_\alpha \cap U_\beta)}.$$  

(5.1)

A form $\omega \in \Omega^*_c(E)$ is locally of type (I) or (II). If $\omega_\alpha$ is of type (I), then $\pi_*\omega_\alpha$ is the zero form, and in particular, it is identically 0 on $U_\alpha \cap U_\beta$, whence by (5.1), we have that $\pi_*\omega_\beta = \pi_*\omega_\alpha = 0$ on $U_\alpha \cap U_\beta$.

Hence, we may now assume that both $\omega_\alpha, \omega_\beta$ are of type (II). Then there exist (see p.61) forms $\psi$ and $\tau$ on $M$, and $f, g$ compactly supported functions for each fixed $\zeta \in M$, such that

$$\omega_\alpha = (\pi^*\psi)f(x, t) \, dt, \quad \omega_\beta = (\pi^*\tau)g(y, u) \, du.$$

Owing to (5.1), it follows that

$$(\pi^*\psi|_{U_\alpha \cap U_\beta})f(x(\zeta), t) \, dt = (\pi^*\tau|_{U_\alpha \cap U_\beta})g(y(\zeta), u) \, du, \quad \text{for each } \zeta \in U_\alpha \cap U_\beta.$$

Observe the calculation

$$\pi_*\omega_\alpha|_{U_\alpha \cap U_\beta} = \psi|_{U_\alpha \cap U_\beta} \int_{\mathbb{R}^n} f(x, t) \, dt = \int_{\mathbb{R}^n} (\pi^*\psi|_{U_\alpha \cap U_\beta})f(x, t) \, dt$$

$$= \int_{\mathbb{R}^n} (\pi^*\tau|_{U_\alpha \cap U_\beta})g(y, u) \, du = \tau|_{U_\alpha \cap U_\beta} \int_{\mathbb{R}^n} g(y, u) \, du$$

$$= \pi_*\omega_\beta|_{U_\alpha \cap U_\beta},$$

as desired. It is clear then that it does not matter which oriented trivialization we choose for $E$. \qed

6 6.20.

Using a Mayer-Vietoris argument as in the proof of the Thom isomorphism (Theorem 6.17), show that if $\pi : E \to M$ is an orientable rank $n$ bundle over a manifold $M$ of finite type, then

$$H^*_c(E) \simeq H^{* - n}_c(M).$$
Solution. Our program is to show that $\pi_\ast : H_c^*(E) \to H_c^{*-n}(M)$ is an isomorphism. We adapt the proof of Theorem 6.7. Let $U$ and $V$ be open subsets of $M$. Using a partition of unity from the base $M$ we see that

$$0 \to \Omega_c^*(E|_{U \cap V}) \to \Omega_c^*(E|_U) \oplus \Omega_c^*(E|_V) \to \Omega_c^*(E|_{U \cup V}) \to 0$$

is exact, as in Proposition 2.7. So we have the diagram of Mayer-Vietoris sequences

\[ \cdots \to H_c^*(E|_{U \cap V}) \to H_c^*(E|_U) \oplus H_c^*(E|_V) \to H_c^*(E|_{U \cup V}) \xrightarrow{d^r} H_{c+1}^*(E|_{U \cup V}) \to \cdots \]

\[ \cdots \to H_c^{*-n}(U \cap V) \to H_c^{*-n}(U) \oplus H_c^{*-n}(V) \to H_c^{*-n}(U \cup V) \xrightarrow{d^r} H_c^{*-n}(U \cup V) \to \cdots \]

The above diagram is clearly commutative. By Corollary 6.9, if $U$ is diffeomorphic to $\mathbb{R}^n$, then $E|_U$ is the trivial bundle, so that by the Poincaré lemma for compact support we have that $\pi_\ast : H_c^*(E|_U) \to H_c^{*-n}(U)$ is an isomorphism. By the Five Lemma, since the desired conclusion holds for $U, V,$ and $U \cap V$, then it holds for $U \cup V$. The proof now proceeds by induction on the cardinality of a good cover for the base, as in the proof of Poincaré duality. □

References