1 6.45.

On the complex projective space $\mathbb{C}P^n$ there is a tautological line bundle $S$, called the universal subbundle; it is the subbundle of the product bundle $\mathbb{C}P^n \times \mathbb{C}^{n+1}$ given by

$$S = \{ (\ell, z) : z \in \ell \}. $$

Above each point $\ell$ in $\mathbb{C}P^n$, the fiber of $S$ is the line represented by $\ell$. Find the transition functions of the universal subbundle $S$ of $\mathbb{C}P^1$ relative to the standard open cover and compute its Euler class.

Solution. The standard open cover for $\mathbb{C}P^1$ consists of two open sets: $U_0 = [1, \frac{z_1}{z_0}]$ whenever $z_0 \neq 0$, and $U_1 = [\frac{z_0}{z_1}, 1]$ whenever $z_1 \neq 0$. Now set $x = \frac{z_1}{z_0}$ and $u = \frac{z_0}{z_1}$. Then $[1, x]$ and $[1, u]$ are new coordinates for $U_0$ and $U_1$, respectively. Note that therefore we may identify each of $U_0, U_1$ with $\mathbb{C}$. Since the fibers of $S$ are the complex lines consisting of all those points in the equivalence class of a specific point in $\mathbb{C}P^n$, then the transition functions must verify

$$(u, 1) = g_{01}(1, x), \quad (1, x) = g_{10}(u, 1),$$

for each coordinate $x, u$. This is achieved if and only if $g_{01} = u = \frac{1}{x} = \frac{z_0}{z_1}$, and $g_{10} = \frac{1}{u} = x = \frac{z_1}{z_0}$.

We now compute the Euler class. As in (6.38), we have that

$$e(S) = -\frac{1}{2\pi i} d(\rho_0 d \log g_{01}), \quad \text{on } U_1,$$
where $\rho_0$ is 1 in a neighborhood of the origin, and 0 in a neighborhood of infinity in $U_0 \simeq \mathbb{C}$. In particular,

$$e(S) = -\frac{1}{2\pi i} d \left( \rho_0 d \log \frac{1}{x} \right), \quad \text{on } U_0 \cap U_1.$$ 

We now proceed to compute $\int_{\mathbb{C}P^1} e(S)$, as in [1] pages 76-77. Fix a circle $C$ in the complex plane with so large a radius that the support of $\rho_0$ is contained inside $C$. Let $A_r$ be the annulus centered at the origin whose outer circle is $c$ and whose inner circle $B_r$ has radius $r$. As the boundary of $A_r$, the circle $C$ is oriented counterclockwise while $B_r$ is oriented clockwise. Observe the computation

$$\int_{\mathbb{C}P^1} e(S) = -\frac{1}{2\pi i} \int_C d \left( \rho_0 d \log \frac{1}{x} \right) = \frac{1}{2\pi i} \int_C d \left( \rho_0 \frac{dx}{x} \right) = \frac{1}{2\pi i} \lim_{r \to 0} \int_{A_r} d \left( \rho_0 \frac{dx}{x} \right)$$

$$= \frac{1}{2\pi i} \lim_{r \to 0} \left[ \int_C \rho_0 \frac{dx}{x} + \int_{B_r} \rho_0 \frac{dx}{x} \right] = \frac{1}{2\pi i} \lim_{r \to 0} \int_{B_r} \frac{dx}{x}$$

$$= \frac{1}{2\pi i} \left( -2\pi i \right) = -1,$$

where we used Stokes Theorem and the properties of $\rho_0$. To recap,

$$\int_{\mathbb{C}P^1} e(S) = -1.$$

□

References
