MATH 1271
Example Optimization Problem

We want to construct a box whose base length is three times the base width. The material used to build the top and bottom cost $10/ft^2$ and the material to build the sides cost $6/ft^2$. If the box must have volume 50 ft$^3$, what is the minimum cost of the box?

We start by drawing a picture of the box.

![Diagram of a box with base length three times the base width]

We have the following formula for the volume of the box

\[ V = lwh \]

\[ 50 = 3w^2h \]  \hspace{1cm} (1)

We also have the following formula for the cost of the box

\[ C = 2(\text{area of top/bottom})(10) + 2(\text{area of front/back})(6) + 2(\text{area of sides})(6) \]

\[ = 2(3w^2)(10) + 2(3wh)6 + 2(wh)(6) \]

\[ = 60w^2 + 36wh + 12wh \]

\[ = 60w^2 + 48wh \]  \hspace{1cm} (2)

So we have written the cost as a function of two variable, height and width. But we would like to rewrite the cost as the function of only one variable (probably width). This is where we look back at equation (1) and solve for $h$ in terms of $w$. That is

\[ h = \frac{50}{3w^2} \]  \hspace{1cm} (3)

Plugging the value for $h$ from (3) above into equation (2) yields

\[ C = 60w^2 + 48w \left( \frac{50}{3w^2} \right) \]

\[ = 60w^2 + \frac{800}{w} \]  \hspace{1cm} (4)
Now we want to minimize cost. To do this we want to find the critical points of $C$ and hope that one is our absolute minimum. We have

$$\frac{dC}{dw} = 120w - \frac{800}{w^2} = \frac{120w^3 - 800}{w^2} \quad (5)$$

Then the critical points of $C$ are $w = 0$ and $w = \sqrt[3]{\frac{800}{12}} \approx 1.8821$. Clearly $w = 0$ does not make any sense so we throw out that point. We then notice that when $w < 1.88, C'(w) < 0$ and when $w > 1.88, C'(w) > 0$ so we can conclude that $w = 1.88$ is an absolute minimum.

Our final step is to find cost of the box at this point. We get

$$C(1.88) = \boxed{637.60}$$