**Week 9 Webwork #1 (3)** The problem asks what strategy you would use to figure out whether the series
\[ \sum_{n=1}^{\infty} (n + 1) \left( \frac{24^n}{5^n} \right) \]
converges or diverges. (Note your individual problem may have different numbers, but the following strategy should still work).

Remember in office hours, I applied the ratio test to the above series and concluded that the series does indeed converge. However, we felt we should be able to compare our series to a geometric series with ratio 24/25. First let’s rearrange our series so that it looks nicer.

\[ \sum_{n=1}^{\infty} (n + 1) \left( \frac{24^n}{5^n} \right) = \sum_{n=1}^{\infty} n \left( \frac{24}{25} \right)^n + \left( \frac{24}{25} \right)^n \]

\[ = \sum_{n=1}^{\infty} \left( \frac{24}{25} \right)^n \]

This may not look nicer to you right away, but I will point out that the second term in the sum is clearly convergent by geometric series so let’s only worry about the first term. We have therefore reduced the problem to the question of convergence of the series

\[ \sum_{n=1}^{\infty} \left( \frac{24\sqrt[n]{n}}{25} \right)^n \]

Now we do a little trick that we have done in our workshop involving the definition of a limit. First I have the following Proposition.

**Proposition:** \( \lim_{n \to \infty} \sqrt[n]{n} = 1 \).

**Proof.**

\[ \lim_{n \to \infty} \sqrt[n]{n} = \lim_{n \to \infty} n^{1/n} = \lim_{n \to \infty} e^{\ln(n^{1/n})} = \lim_{n \to \infty} e^{1/n \ln n} = e^{\lim_{n \to \infty} \frac{\ln n}{n}} \] since \( e^x \) is continuous

\[ = e^0 \] if you don’t believe me here use L’Hospital’s Rule

\[ = 1 \]
Now that we know that, by the definition of limit, we know that there exists some number $N$ such that for all $n \geq N$, $\sqrt[n]{n} \leq \frac{24.5}{24}$. We also have $\sqrt[n]{n} \geq 0$. So we can rewrite our series

$$\sum_{n=1}^{\infty} \left( \frac{24 \sqrt[n]{n}}{25} \right)^n = \sum_{n=1}^{N} \left( \frac{24 \sqrt[n]{n}}{25} \right)^n + \sum_{n=N+1}^{\infty} \left( \frac{24 \sqrt[n]{n}}{25} \right)^n.$$  

Then the first term on the right hand side above is definitely finite (as it is a finite sum of finite numbers) and we can say that the second term converges by the direct comparison test (since the sum $\sum_{n=N}^{\infty} \frac{24.5}{25}$ converges by geometric series with $r = \frac{24.5}{25} < 1$). So we are done.