(1) The fibonacci sequence \( \{f_n\} \) is defined by
\[
\begin{align*}
    f_1 &= 1 \\
    f_2 &= 1 \\
    f_n &= a_{n-1} + f_{n-2}, \quad \text{for } n \geq 3.
\end{align*}
\]

Define a new sequence \( x_n = \frac{f_{n+1}}{f_n} \). Given that \( \{x_n\} \) converges, find the limit.
[This limit is called the Golden Ratio and shows up in the most unexpected places.]

First let’s find a recurrence relation for \( x_n \).
\[
x_{n+1} = \frac{f_{n+2}}{f_{n+1}} = \frac{f_{n+1} + f_n}{f_{n+1}} = \frac{f_{n+1} + f_n}{f_{n+1}} = x_n + \frac{1}{x_n}
\]

Now suppose \( \lim x_n = L \). then
\[
\lim x_{n+1} = \lim \left( 1 + \frac{1}{x_n} \right)
\]
\[
\Rightarrow L = 1 + \frac{1}{L}
\]
\[
\Rightarrow L^2 = L + 1
\]
\[
\Rightarrow L^2 - L - 1 = 0
\]
\[
\Rightarrow L = \frac{1 \pm \sqrt{5}}{2}
\]

Now since \( x_n > 0 \) for all \( n \), we must choose the positive root. So
\[
x_n = \frac{1 + \sqrt{5}}{2}
\]

(2) Let \( \{a_n\} \) be a Cauchy sequence such that \( a_n \) is an integer for all \( n \in \mathbb{N} \). Show that there is a positive integer \( n_0 \) such that \( a_n = C \) for all \( n \geq n_0 \), where \( C \) is a constant.

Proof. Since \( a_n \) is Cauchy, there exists some \( n_0 \) such that for all \( n, m \geq n_0 \), we have \( |a_m - a_n| < 1/2 \). But \( a_m \) and \( a_n \) are integers and so their difference is an integer. But the only integer whose absolute value is less than \( 1/2 \) is 0. Therefore \( |a_m - a_n| = 0 \) so \( a_m = a_n \) for all \( m, n \geq n_0 \). So \( C = a_{n_0} \). Then for all \( n \geq n_0 \), \( a_n = C \) as desired. \( \Box \)
(3) (a) Show that if \(\{a_n\}\) is Cauchy then \(\{a_n^2\}\) is also Cauchy.

(b) Is the converse of part (a) true? If so, prove it. If not, provide a counterexample.

(a) Proof. Since \(\{a_n\}\) is Cauchy, it is convergent. Since \(\{a_n\}\) is convergent, the limit laws tell us \(\{a_n^2\}\) is convergent (product of two convergent sequences is convergent). Since \(\{a_n^2\}\) is convergent, it is Cauchy.

(b) The converse of part (a) states: If \(\{a_n^2\}\) is Cauchy, then \(\{a_n\}\) is Cauchy. This is false as evidenced by the counterexample \(a_n = (-1)^n\). Then \(a_n^2\) is the constant sequence \(\{1\}\) and is trivially convergent so Cauchy, but \(a_n\) is not convergent so cannot be Cauchy.

(4) Let \(\{a_n\}\) be a sequence that satisfies

\[|a_{n+2} - a_{n+1}| < c|a_{n+1} - a_n| \text{ for all } n \in \mathbb{N}\]

where \(0 < c < 1\).

(a) Show that \(|a_{n+1} - a_n| < c^n|a_2 - a_1|\) for all \(n \geq 2\).

(b) Show that \(\{a_n\}\) is a Cauchy sequence.

(a) Proof. We will prove this by induction. Notice that the base case is true by assumption. Now assume \(|a_{n+1} - a_n| < c^n|a_2 - a_1|\). Then

\[|a_{n+2} - a_{n+1}| < c|a_{n+1} - a_n| < c \cdot c^n|a_2 - a_1| = c^{n+1}|a_2 - a_1|.

So \(|a_{n+1} - a_n| < c^n|a_2 - a_1|\) for all \(n \geq 2\).

(b) Proof. Let \(\varepsilon > 0\) be arbitrary. Then by the same steps as the example done in class we have for all \(n < m\) that \(|a_m - a_n| < \frac{c^n}{1-c}|a_2 - a_1|\). (Go through this part yourself to double check). Choose \(n_0 > \frac{\ln(\frac{\varepsilon(1-c)}{|a_2 - a_1|})}{\ln c}\). This way \(c^{n_0} < \frac{\varepsilon(1-c)}{|a_2 - a_1|}\). Then for all \(m > n \geq n_0\), we have

\[|a_m - a_n| < \frac{c^n}{1-c}|a_2 - a_1| \leq \frac{c^{n_0}}{1-c}|a_2 - a_1| < \frac{\varepsilon(1-c)}{|a_2 - a_1|} \cdot \frac{|a_2 - a_1|}{1-c} = \varepsilon.

Thus \(a_n\) is a Cauchy sequence.