(1) Recall that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

(a) Estimate $e$ with error less than 0.001. [If $f(x) = e^x$ and $n$ is large enough so that $|R_n(1; 0)| < 0.001$, then $P_n(1)$ is an estimate of $e$ with error less than 0.001]

$$e = e^1 = \sum_{n=0}^{\infty} \frac{1}{n!},$$

By Taylor’s Theorem:

$$|R_n(1; 0)| = \left| \frac{1}{n!} \int_0^1 f^{(n+1)}(t)(x-t)^n dt \right|$$

$$\leq \frac{1}{n!} \int_0^1 |f^{(n+1)}(t)||x-t|^n dt$$

$$\leq \frac{1}{n!} \int_0^1 e|x-t|^n dt$$

$$= \frac{e}{n!} \cdot \frac{1}{n+1}$$

$$= \frac{e}{(n+1)!}$$

So we are looking for $n$ such that $\frac{e}{n+1} \leq 0.001$. We see $\frac{e}{6!} \approx 0.003$ and $\frac{e}{7!} \approx 0.0005$.

Therefore our estimate for $e$ is $\sum_{n=0}^{6} \frac{1}{n!}$.

(b) Use the Alternating Series Test to estimate $\cos 1$ with error less than 0.001. Make sure you satisfy the hypotheses of the AST.

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

So

$$\cos 1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}$$

Notice that this is an alternating series. We wish to use the Alternating Series Test, but before we do we must check that this series satisfies the conditions on the theorem. We see that $\frac{1}{(2n)!}$ is positive and decreasing and $\lim \frac{1}{(2n)!} = 0$. Therefore,

$$\left| \cos 1 - \sum_{k=0}^{n} \frac{(-1)^n}{(2n)!} \right| \leq \frac{1}{(2n+2)!}.$$

We see $\frac{1}{6!} \approx 0.0013$ and $\frac{1}{8!} \approx 0.000025$ so our estimate is $\left[ 1 - \frac{1}{2} + \frac{1}{24} - \frac{1}{720} \right]$.

(2) Show that for $0 \leq x \leq 1$,

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1} \frac{x^n}{n} + \cdots$$

by going through the following steps:
(a) Let \( f(x) = \ln(1 + x) \). Find an expression for \( f^{(n)}(x) \).

\[
f^{(n)}(x) = (-1)^{n+1} \frac{(n-1)!}{(1+x)^n}
\]

(b) Find and simplify the integral expression for \( R_n(x) \).

\[
R_n(x; 0) = \frac{1}{n!} \int_0^x f^{(n+1)}(t)(x-t)^n dt
= \frac{1}{n!} \int_0^x (-1)^{n+2} \frac{n!}{(1+t)^n} (x-t)^n dt
= \int_0^x (-1)^{n+2} \frac{(x-t)^n}{(1+t)^n} dt
\]

(c) Use that fact that for \( 0 \leq t \),

\[
\frac{1}{(1+t)^n} \leq 1,
\]

to show that for \( 0 \leq x \leq 1 \),

\[
|R_n(x)| \leq \frac{x^{n+1}}{n+1} \rightarrow 0.
\]

For all \( x \) such that \( 0 \leq x \),

\[
|R_n(x; 0)| = \left| \int_0^x (-1)^{n+2} \frac{(x-t)^n}{(1+t)^n} dt \right|
\leq \int_0^x \frac{(x-t)^n}{(1+t)^n} dt
\leq \int_0^x (x-t)^n dt
= \frac{x^{n+1}}{n+1}
\]

which converges to 0 for all \( x < 1 \).

(3) (Only if you have time): In this problem we will prove what is widely considered the most beautiful mathematical equation: \( e^{ix} + 1 = 0 \). [Here \( i = \sqrt{-1} \].

(a) Let \( f(x) = e^{ix} \). This function has a power series representation (centered about 0) which converges for all \( x \in \mathbb{R} \). Find it. (Hint: Either find derivatives of \( f \) or know the power series of \( e^x \) and substitute \( ix \).]

\[
e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!}
\]

(b) Find the power series representation (centered about 0) for \( g(x) = \cos x \).

\[
\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}
\]
(c) Find the power series representation (centered about 0) for \( h(x) = i \sin x \).

\[
i \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n i x^{2n+1}}{(2n+1)}
\]

(d) Given that all of the above power series have radius of convergence \( \infty \), show that 
\( e^{ix} = \cos x + i \sin x \) (This formula is known as Euler’s identity). [Hint: Start with the RHS and use that \( i^2 = -1 \).]

\[
\cos x + i \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(-1)^n i x^{2n+1}}{(2n+1)}
\]

\[
= \sum_{n=0}^{\infty} \frac{i^2 x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{i^{2n+1} x^{2n+1}}{(2n+1)}
\]

\[
= \sum_{n=0}^{\infty} \frac{(ix)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(ix)^{2n+1}}{(2n+1)} \quad \text{because both series converge for every} \ x \in \mathbb{R}
\]

\[
= \sum_{n=0}^{\infty} \frac{(ix)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(ix)^{2n+1}}{(2n+1)}
\]

\[
= \sum_{n=0}^{\infty} \frac{(ix)^n}{n!}
\]

\[
= e^{ix}
\]

(e) Compute \( e^{i\pi} + 1 \).

\[
e^{i\pi} + 1 = \cos(\pi) + i \sin(\pi) + 1 = -1 + 0 + 1 = 0
\]