(1) Estimate the value of \( \int_0^1 \sin(t^2) dt \) with an error of less than 0.001 using the following steps:

(a) Write out the power series expansion for \( \sin t \).

(b) Write out the power series expansion for \( \sin(t^2) \).

(c) Integrate the power series term by term. Notice the result is an alternating series which makes it easy to put a bound on the error.

(a) \( \sin t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} \)

(b) \( \sin(t^2) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{4n+2}}{(2n+1)!} \)

(c) \[
\int_0^1 \sin(t^2) dt = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n t^{4n+2}}{(2n+1)!} dt \\
= \sum_{n=0}^{\infty} \int_0^1 \frac{(-1)^n t^{4n+2}}{(2n+1)!} dt \\
= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int_0^1 t^{4n+2} dt \\
= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left[ \frac{t^{4n+3}}{4n+3} \right]_0^1 \\
= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{1}{4n+3} \\
= \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+3)(2n+1)!} \\
\]

Notice that \( \frac{1}{(4n+3)(2n+1)!} > 0 \), is obviously decreasing, and \( \lim_{n \to \infty} \frac{1}{(4n+3)(2n+1)!} = 0 \) so the hypotheses of the Alternating Series Test are satisfied. So we need to find \( n \) so that \( 0.001 > \frac{1}{(4(n+1)+3)(2(n+1)+1)!} = \frac{1}{(4n+7)(2n+3)!} \)

If we check we see that \( \frac{1}{(7)(3)!} > 0.001 \) but \( \frac{1}{(11)(5)!} < 0.001 \). So \( n = 1 \). Our estimate is \( \frac{1}{3} - \frac{1}{42} \).
(2) Estimate the value of $\int_0^1 e^{-t^2} \, dt$ with error less than 0.001.

We go through the same steps that we did above (starting with (b)). We have

$$e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}.$$

So

$$\int_0^1 e^{-t^2} \, dt = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!} \, dt$$

$$= \sum_{n=0}^{\infty} \int_0^1 \frac{(-1)^n t^{2n}}{n!} \, dt$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{3}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!3}.$$

Also, $\frac{1}{n!3} > 0$, is decreasing, and $\lim \frac{1}{n!3} = 0$. So we may use the Alternating Series Test. We are looking for $n$ such that $\frac{1}{(n+1)!3} < 0.001$. It turns out that $n = 5$ is the first that works. So our estimate is

$$\sum_{n=0}^{5} \frac{(-1)^n t^{2n}}{n!).$$
(3) Prove Taylor’s Theorem: Assume that \( f \) has continuous derivatives of order \( n + 1 \) on the closed interval between 0 and \( x \). Then

\[
R_n(x) = \frac{1}{n!} \int_0^x f^{(n+1)}(t)(x - t)^n dt.
\]

[Hint: Use induction. Start with \( n = 0 \) and use the Fundamental Theorem of Calculus (regular integration). Then for the induction, start with your inductive hypothesis and use integration by parts with \( u = f^{(n+1)}(t) \) and \( dv = (x - t)^n dt \).]

We prove this by induction. Our base case is \( n = 0 \).

\[
\frac{1}{0!} \int_0^x f'(t)(x - t)^0 dt = \int_0^x f'(t) dt = f(x) - f(0) = f(x) - P_0(x; 0) = R_0(x; 0).
\]

Now assume that

\[
R_n(x; 0) = \frac{1}{n!} \int_0^x f^{(n+1)}(t)(x - t)^n dt.
\]

We integrate the right hand side above by parts using \( u = f^{(n+1)}(t) \) so \( du = f^{(n+2)}(t)dt \) and \( dv = (x - t)^n dt \) so \( v = -\frac{(x - t)^{n+1}}{(n + 1)} \). This gives:

\[
R_n(x; 0) = \frac{1}{n!} \left[ \frac{f^{(n+1)}(t)(x - t)^{n+1}}{(n + 1)} \right]_0^x + \frac{1}{n!} \int_0^x \frac{f^{(n+2)}(t)(x - t)^{n+1}}{n + 1} dt
\]

\[
= 0 + \frac{f^{(n+1)}(0)x^{n+1}}{(n + 1)!} + \frac{1}{(n + 1)!} \int_0^x f^{(n+2)}(t)(x - t)^{n+1} dt
\]

Subtracting \( \frac{f^{(n+1)}(0)x^{n+1}}{(n + 1)!} \) from both sides yields:

\[
\frac{1}{(n + 1)!} \int_0^x f^{(n+2)}(t)(x - t)^{n+1} dt = R_n(x; 0) - \frac{f^{(n+1)}(0)x^{n+1}}{(n + 1)!}
\]

\[
= f(x) - P_n(x; 0) - \frac{f^{(n+1)}(0)x^{n+1}}{(n + 1)!}
\]

\[
= f(x) - \sum_{k=0}^{n} \frac{f^{(n)}(0)x^n}{n!} - \frac{f^{(n+1)}(0)x^{n+1}}{(n + 1)!}
\]

\[
= f(x) - P_{n+1}(x; 0)
\]

\[
= R_{n+1}(x; 0)
\]

So by Mathematical Induction we are done.

(4) Find the power series representation for \( f(x) = \sqrt{x} \) centered about \( a = 1 \) (notice that we cannot center about 0 because \( \sqrt{x} \) is not differentiable at \( x = 0 \)).

This will not be on the exam as the final will cover sections 1.1 to 4.9 (this is in section 4.10). But the answer is below.

To make the notation in this problem a little nicer we will introduce some new notation called the double factorial or semifactorial (for more information on this, check out the wikipedia page on double factorial. The double factorial is defined as

\[
n!! := \begin{cases} \frac{n(n-2)(n-4) \cdots 1}{2} & \text{if } n \text{ is odd,} \\
\frac{n(n-2)(n-4) \cdots 2}{2} & \text{if } n \text{ is even.} \end{cases}
\]
Essentially, the double factorial is the product of every other natural number below \( n \) (starting with \( n \)). Now that that is out of the way, we will get back to solving the problem. We make the following table:

<table>
<thead>
<tr>
<th>( n )</th>
<th>( f^{(n)}(x) )</th>
<th>( f^{(n)}(1) )</th>
<th>( \frac{f^{(n)}(4)}{n!} (x - 1)^n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \sqrt{x} )</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>( \frac{1}{2x^{1/2}} )</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{x - 1}{2} )</td>
</tr>
<tr>
<td>2</td>
<td>( -\frac{1}{2^2x^{3/2}} )</td>
<td>( -\frac{1}{2^2} )</td>
<td>( -\frac{(x - 1)^2}{2^2 \cdot 2!} )</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{3 \cdot 1}{2^3x^{5/2}} )</td>
<td>( \frac{3 \cdot 1}{2^3} )</td>
<td>( \frac{3 \cdot 1(x - 1)^3}{2^3 \cdot 3!} )</td>
</tr>
<tr>
<td>4</td>
<td>( -\frac{5!!}{2^4x^{7/2}} )</td>
<td>( -\frac{5!!}{2^4} )</td>
<td>( -\frac{5!!(x - 1)^4}{2^4 \cdot 4!} )</td>
</tr>
<tr>
<td>5</td>
<td>( \frac{7!!}{2^5x^{9/2}} )</td>
<td>( \frac{7!!}{2^5} )</td>
<td>( \frac{7!!(x - 1)^5}{2^5 \cdot 5!} )</td>
</tr>
<tr>
<td>( n )</td>
<td>( \frac{(-1)^{n+1}(2n - 3)!!}{2^n x^{(2n-1)/2}} )</td>
<td>( \frac{(-1)^{n+1}(2n - 3)!!}{2^n} )</td>
<td>( \frac{(-1)^{n+1}(2n - 3)!!(x - 1)^n}{2^n \cdot n!} )</td>
</tr>
</tbody>
</table>

So we conclude that if \( \sqrt{x} \) can be written as a power series centered about 1, then

\[
\sqrt{x} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n - 3)!!(x - 1)^n}{2^n \cdot n!}.
\]