

VIRTUAL COMPLETE INTERSECTIONS IN $\mathbb{P}^1 \times \mathbb{P}^1$

JIYANG GAO, YUTONG LI, MICHAEL C. LOPER, AND AMAL MATTOO

ABSTRACT. The minimal free resolution of the coordinate ring of a complete intersection in projective space is a Koszul complex on a regular sequence. In the product of projective spaces $\mathbb{P}^1 \times \mathbb{P}^1$, we investigate which sets of points have a virtual resolution that is a Koszul complex on a regular sequence. This paper provides conditions on sets of points; some of which guarantee the points have this property, and some of which guarantee the points do not have this property.

1. INTRODUCTION

Complete intersections are a fundamental objects of study in commutative algebra and algebraic geometry. In projective space $\mathbb{P}_{\mathbb{C}}^r$, a complete intersection Y of codimension t is cut out by an ideal of codimension t which can be generated by exactly t elements of the ring $\mathbb{C}[x_0, \dots, x_r]$. In this case, there are hypersurfaces H_1, \dots, H_t such that Y is the scheme-theoretic intersection of the H_i 's. Complete intersections have coordinate rings that are Cohen–Macaulay and in \mathbb{P}^3 , complete intersections of curves have a simple formula for their genus [Har77, Exercise II.8.4]. The defining ideal of a complete intersection in \mathbb{P}^r is generated by a regular sequence and so the minimal free resolution of the coordinate ring is a Koszul complex [Pee11, Theorem 14.7].

Unfortunately, in a product of projective spaces, the nice properties of complete intersections in \mathbb{P}^r are not completely captured homologically. Even in the simplest case of points in the product of projective lines $\mathbb{P}^1 \times \mathbb{P}^1$, general regular sequences in the Cox ring do not always capture the correct geometric structure. This is where the virtual resolutions of [BES17] help. By allowing some irrelevant homology in a free complex, we expand the notion of a complete intersection via a virtual resolution, while still reaping the benefits of many properties of complete intersections. The goal of this paper is to state conditions on whether a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ do or do not form a so-called virtual complete intersection (see Definition 1.2).

Notation 1.1. The Cox ring of $\mathbb{P}^1 \times \mathbb{P}^1$ is the \mathbb{Z}^2 -graded ring $S := \mathbb{C}[x_0, x_1, y_0, y_1]$ where $\deg(x_i) = (1, 0)$ and $\deg(y_i) = (0, 1)$. The irrelevant ideal of S is $B := \langle x_0, x_1 \rangle \cap \langle y_0, y_1 \rangle = \langle x_0 y_0, x_0 y_1, x_1 y_0, x_1 y_1 \rangle$. In this setting, closed subschemes are in one-to-one correspondence with B -saturated bihomogeneous ideals [CLS11, Proposition 6.A.7]. The B -saturation of an ideal I is

$$I : B^\infty = \bigcup_{k \geq 0} I : B^k = \{s \in S \mid sB^k \subset I \text{ for some } k\}.$$

If $I \subset S$ is an ideal, then $V(I)$ denotes the subscheme of X consisting of all B -saturated bihomogeneous prime ideals that contain I . On the other hand, if X is a subvariety of $\mathbb{P}^1 \times \mathbb{P}^1$, then I_X will be the B -saturated bihomogeneous ideal of polynomials in S that vanish at every point in X .

In this paper we are concerned mostly with coherent reduced zero-dimensional schemes in the product of projective spaces $\mathbb{P}^1 \times \mathbb{P}^1$ over \mathbb{C} . We will call these reduced subschemes “sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$.” The maximum number of points on a single horizontal ruling in a set of points X is denoted as m , and the maximum number of points on a single vertical ruling is denoted as n .

We will follow the convention of [GVT15] to represent points in the biprojective space $\mathbb{P}^1 \times \mathbb{P}^1$ on a grid consisting of horizontal and vertical lines, where the two families of lines correspond to the two rulings of $\mathbb{P}^1 \times \mathbb{P}^1$, i.e., the copies of \mathbb{P}^1 .

Definition 1.1 ([BES17, Definition 1.1]). A *virtual resolution* for an ideal I in the biprojective space $\mathbb{P}^1 \times \mathbb{P}^1$ is a \mathbb{Z}^2 -graded complex of free S -modules

$$0 \longleftarrow S \xleftarrow{\varphi_1} F_1 \xleftarrow{\varphi_2} F_2 \xleftarrow{\varphi_3} \cdots$$

such that $\text{ann}(H_i(F)) \supseteq B^\ell$ for some $\ell > 0$ and $\text{im}(\varphi_1) : B^\infty = I : B^\infty$.

Definition 1.2. Let X be a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ with defining ideal I_X . We say X is a *virtual complete intersection (VCI)* if S/I_X has a virtual resolution that is a Koszul complex $K(f, g)$ of bihomogeneous forms f and g . This implies there is a regular sequence (f, g) of S such that $V(I) = V(f) \cap V(g)$ scheme-theoretically [BES17, Proposition 2.5].

Summary of Main Results. It turns out that the question of being a VCI is more subtle than the question of being a complete intersection. In particular, VCIs are not solely determined by the configuration of the points, which is a characterization of where points lie in relation to each other (see Section 2). When $|X| > mn$, the actual coordinates of points can play a role in determining whether or not X is a VCI (Remark 2.3). Our main results are summarized below.

- (1) A set of points is a VCI when it has the same number of points in each vertical (or each horizontal) ruling (Theorem 4.1).
- (2) A set of points X is not a VCI when
 - (a) $|X| < mn$, and there is at least one point in X that is on a horizontal ruling with m points and a vertical ruling with n points (Theorem 4.2).
 - (b) $|X| < mn$ and $\text{gcd}(m, n)$ does not divide $|X|$ (Theorem 5.1).
 - (c) The degrees of two forms that intersect at X are known and one of the conditions in Proposition 5.4 holds.
- (3) When all points lie on two and three vertical or horizontal rulings, we provide a complete classification of VCIs (Section 6).

Example 1.3. Consider Figure 1 of the three points

$$X = \{([a_1 : a'_1], [b_1 : b'_1]), ([a_2 : a'_2], [b_1 : b'_1]), ([a_3 : a'_3], [b_2 : b'_2])\} \subset \mathbb{P}^1 \times \mathbb{P}^1.$$

Letting I_X be the B -saturated ideal of bihomogenous polynomials vanishing at X , *Macaulay2* [M2] computes the minimal free resolution of S/I_X to be

$$\begin{array}{ccccccc}
& & S(0, -2) & & & & \\
& & \oplus & & S(-1, -2) & & \\
& & S(-1, -1) & & \oplus & & \\
0 \longleftarrow S \longleftarrow & \oplus & \longleftarrow S(-2, -2) & \longleftarrow S(-3, -2) & \longleftarrow 0. & & \\
& S(-2, -1) & & \oplus & & & \\
& \oplus & & S(-3, -1)^2 & & & \\
& S(-3, 0) & & & & &
\end{array}$$

Since the minimal free resolution has length 3, the Auslander–Buchsbaum formula tells us that S/I_X is not Cohen–Macaulay. Therefore X does not form a complete intersection. As the picture indicates, however, X is a VCI as it is the intersection of the varieties of two forms

$$f = (a'_3x_1 - a_3x_0)(b'_1y_1 - b_1y_0) \quad \text{and} \quad g = (a'_1x_1 - a_1x_0)(a'_2x_1 - a_2x_0)(b'_2y_1 - b_2y_0).$$

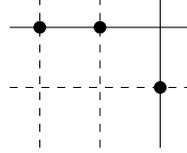


FIGURE 1. A 3-point variety that is generated by two forms geometrically but not a complete intersection.

Therefore, $K(f, g)$ is a virtual resolution of S/I_X , where

$$K(f, g): \quad 0 \longleftarrow S \xleftarrow{\begin{bmatrix} f & g \end{bmatrix}} S(-1, -1) \oplus S(-2, -1) \xleftarrow{\begin{bmatrix} -g \\ f \end{bmatrix}} S(-3, -2) \longleftarrow 0.$$

Although the points do not form a complete intersection in the original definition, they nonetheless share similar properties with complete intersections. The saturation of $\langle f, g \rangle$ by the irrelevant ideal B is equal to I_X so $V(f) \cap V(g) = X$ scheme-theoretically.

All theorems proved in this paper are from the virtual viewpoint. This differs from the viewpoint of [GVT15], which does not consider complete intersections up to irrelevance. A set of points is a complete intersection exactly when all horizontal and vertical rulings contain the same number of points [GVT15, Theorem 5.13]. That is, the set of points looks like a rectangle when put on a grid. Hence it is difficult to be a complete intersection in $\mathbb{P}^1 \times \mathbb{P}^1$ on the nose, but being a VCI is much easier, and this latter property still retains the homological benefits of the former.

Guardo and Van Tuyl not only classified when points in $\mathbb{P}^1 \times \mathbb{P}^1$ are complete intersections, they also classified when points are arithmetically Cohen–Macaulay (i.e. the coordinate ring S/I_X is Cohen–Macaulay). This occurs exactly when X can be written in a configuration that resembles a Ferrers diagram [GVT15, Theorem 4.11]. By the Auslander–Buchsbaum formula, points in $\mathbb{P}^1 \times \mathbb{P}^1$ are arithmetically Cohen–Macaulay exactly when the minimal free resolution of S/I_X is of length 2. All sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$ have virtual resolutions of length 2 [BES17, Theorem 1.5], so in this paper, we concentrate only on when sets of points are VCIs.

Outline. In Section 2, we introduce configurations, which are utilized throughout this paper to simplify many cases; unfortunately, VCIs are not always determined by their configurations, differing from [GVT15]. In Section 3, it is proved that every set of points forms a VCI when considered set-theoretically. Then, in Sections 4 and 5, we examine the scheme-theoretic case of reduced points. The majority of the proofs of our main theorems are in these sections. We name many conditions which are guaranteed to either give rise to or never give rise to VCIs. Finally, Section 6 is an application of these results, giving a complete classification of VCIs that lie on two and three horizontal (or vertical) rulings.

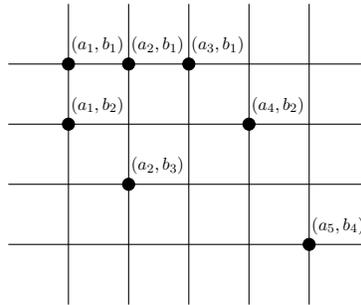


FIGURE 2. An example of a configuration

Acknowledgements. We are grateful to Christine Berkesch for her extensive guidance and feedback throughout this project, as well as Vic Reiner for many helpful discussions. We would also like to thank the University of Minnesota–Twin Cities for organizing and supporting the combinatorics REU in 2018. This work was funded by NSF RTG grant DMS-1745638.

2. CONFIGURATIONS

In this section, we introduce the notion of configurations and show that the property of being a VCI is not a combinatorial invariant. Following [GVT15, §3.2], points in $\mathbb{P}^1 \times \mathbb{P}^1$ may be placed on a grid, according to their coordinates in each copy of \mathbb{P}^1 , in the following way. There are two projections $\pi_i: \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$:

$$\pi_1(a, b) = a \quad \text{and} \quad \pi_2(a, b) = b.$$

Making a grid of horizontal and vertical lines, the vertical lines correspond to the first copy of \mathbb{P}^1 and the horizontal lines correspond to the second copy of \mathbb{P}^1 . Two points $p, q \in \mathbb{P}^1 \times \mathbb{P}^1$ lie on the same vertical line if $\pi_1(p) = \pi_1(q)$. They lie on the same horizontal line if $\pi_2(p) = \pi_2(q)$. By permuting the horizontal lines, we arrange the points so that the number on each horizontal line decreases from top to bottom, forming a *configuration*.

For example, letting a_i denote a point in the first copy of \mathbb{P}^1 and b_i denote a point in the second copy of \mathbb{P}^1 , the set of points

$$\{(a_1, b_1), (a_2, b_1), (a_3, b_1), (a_1, b_2), (a_4, b_2), (a_2, b_3), (a_5, b_4)\}$$

in $\mathbb{P}^1 \times \mathbb{P}^1$, can be represented as in Figure 2 (here the points are labeled, but in what follows they will not be labeled). We consider two sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$ to be *equivalent up to configuration* if they have the same configurations after permutation and relabeling of the rulings.

Unfortunately, the property of being a VCI depends on the coordinates of the points, not just on their configuration. This is not so surprising as the Betti numbers of points in $\mathbb{P}^1 \times \mathbb{P}^1$ also depend on more than just the configuration. To illustrate this point, we use the cross ratio.

Definition 2.1 ([Har95, Exercise 1.6]). If four points in \mathbb{P}^1 have homogeneous coordinates $[a : a']$, $[b : b']$, $[c : c']$, $[d : d']$, their *cross ratio* is:

$$\frac{(ca' - ac')(db' - bd')}{(da' - ad')(cb' - bc')}.$$

If a point is $[1 : 0]$ or $[0 : 1]$, then the terms involving this point are dropped from both the numerator and the denominator.

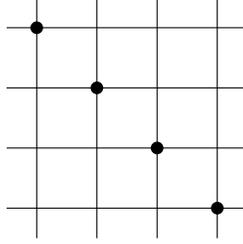


FIGURE 3. A four-point configuration whose the minimal free resolution depends on the coordinates.

In Figure 3, the total Betti numbers of the minimal free resolution depends on the cross ratio. Let I be the ideal of bihomogeneous forms vanishing at the points. When the cross ratios of the coordinates are the same after projection to each copy of \mathbb{P}^1 , *Macaulay2* [M2] shows that the minimal free resolution of S/I (omitting the twists of the free modules) is

$$S^1 \leftarrow S^6 \leftarrow S^8 \leftarrow S^3 \leftarrow 0.$$

When the cross ratios of the two copies of \mathbb{P}^1 are different, the minimal free resolution is

$$S^1 \leftarrow S^6 \leftarrow S^7 \leftarrow S^2 \leftarrow 0.$$

Moreover, for any collection of points with a subconfiguration of this kind, the minimal free resolution will depend on the value of the coordinates. By contrast, this configuration is always a VCI, regardless of the cross ratios.

Proposition 2.2. *Given the configuration of four points in Figure 3, the minimal resolution of these points depends on whether or not the cross ratios are equal after projection to each copy of \mathbb{P}^1 .*

Proof. We may change coordinates so that three of the four points are $[0 : 1], [1 : 1], [1 : 0]$ and the last point is $[1 : c]$, where c is the cross ratio [Har95, Exercise 1.19]. Now consider the form $x_0y_1 - x_1y_0$. If the cross ratios on both copies of \mathbb{P}^1 are the same, the form $x_0y_1 - x_1y_0$ vanishes, which explains the degree differences in the minimal free resolutions in the two cases - one has a $(1, 1)$ graded piece whereas the other has two $(1, 2)$ forms. Since the Hilbert function is recoverable from the minimal free resolution, the minimal free resolution changes accordingly. \square

Virtual resolutions are also not invariant under configurations when the total number of points is large relative to the number of points lying on a single horizontal ruling and a single vertical ruling, which we respectively denote by m and n .

Remark 2.3. When $|X| \geq mn$, VCIs are not always determined by configuration. That is, the same configuration may be a VCI with some coordinates, but not with others. For example, the configuration below is a VCI when either the four rightmost points or the four bottommost points lie on a conic. If these points do not lie on a conic, the configuration is not a VCI. This example is explained in more detail in Example 4.6, after the necessary machinery has been developed.

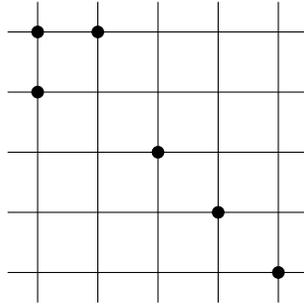


FIGURE 4

Notice that this differs from the classification of arithmetically Cohen–Macaulay points in [GVT15] which depends only on the configuration and not the actual coordinates of the points. Hence the question of when sets of points form VCIs is more subtle than the question of when sets of points form complete intersections.

3. SET-THEORETIC VCIs

In this section, we consider coherent zero-dimensional subschemes of $\mathbb{P}^1 \times \mathbb{P}^1$. We show that set-theoretically, all such subschemes (hence all corresponding configurations) are virtual complete intersections.

Theorem 3.1. *For any coherent zero-dimensional subscheme X of $\mathbb{P}^1 \times \mathbb{P}^1$, there is an ideal J so that $\sqrt{J} = \sqrt{I_X}$ and S/J has a virtual resolution that is a length two Koszul complex.*

Proof of Theorem 3.1. Let $\text{Supp}(X)$ be the underlying set of points in $\mathbb{P}^1 \times \mathbb{P}^1$. We will show that there is an ideal generated by two bihomogenous forms f and g so that $\text{Supp}(V(f, g)) = \text{Supp}(X)$. Choose f to be the product of the smallest set of $(1, 0)$ -forms needed to cover the set of points (i.e. vertical lines in every ruling containing points). Let k be the degree of f , and let n be the maximum number of points in a single vertical ruling.

Assign multiplicities to each point, so that the sum of the multiplicities of points in each vertical ruling is n . Now, construct n k -tuples each containing one point from each of the k vertical rulings. That is, each point appears as many times as its multiplicity. For each k -tuple, we can use Lagrangian interpolation to find a polynomial that will pass through those k points, and will therefore intersect f exactly at the points of the configuration. The product of these polynomials will give the desired g . \square

Example 3.2. The configuration of six points in Figure 5 is the scheme-theoretic intersection of the dotted curves,

$$V(x_0(x_0 - x_1)(x_2 - 2x_1))$$

which is the vanishing set of a $(3, 0)$ form, and the dashed curves,

$$V((y_0)(2x_1^2y_0 + x_0^2y_1 - 3x_0x_1y_1)(x_0y_1 - x_1y_0))$$

which is the vanishing set of a $(3, 3)$ form. By [Sha77, §4.2.1] (see Theorem 4.3), the intersection should have order 9, but the curves intersect at $([0 : 1], [0 : 1])$ with multiplicity 3 and at $([1 : 1], [1 : 1])$ with multiplicity 2, so the intersection set is indeed 6 points.

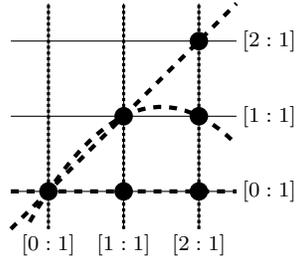


FIGURE 5

4. DETERMINATION OF VCIS

We now consider reduced zero-dimensional subschemes of $\mathbb{P}^1 \times \mathbb{P}^1$, which we refer to as “sets of points.” This requires that the homogeneous ideal generated by the two forms equal I_X after saturation by B , instead of first taking the radical and then saturating by B , which leads to a richer classification of configurations into VCIs, non-VCIs, and coordinate dependent cases.

In the previous section, we proved that set-theoretically all configurations are VCIs by assigning multiplicities so that along each ruling, there are the same total multiplicity of points. When this condition is satisfied without having to artificially “boost up” the multiplicity of any point, we have a natural environment for VCIs.

Theorem 4.1. *If X has the same number of points in each vertical (or each horizontal) ruling, it is a VCI.*

Proof. By symmetry, it is enough to prove the vertical case. Suppose X has exactly n points in every vertical ruling. We will find bihomogeneous polynomials f and g so that $K(f, g)$ is a virtual resolution for S/I_X . Let f be the polynomial such that $V(f)$ is comprised of all the vertical rulings that contain points of X . Using Lagrangian interpolation, there exists a polynomial that intersects each of these rulings once at any given point. By labeling the points in each vertical ruling from 1 to n , let g_i be the polynomial vanishing on all the points labeled i . Setting g equal to the product of the g_i gives the desired forms, so that $K(f, g)$ is a virtual resolution of S/I_X . \square

Notice that Example 1.3 illustrates this theorem. On the other hand, Theorem 4.2 below names conditions that guarantee when a set of points can never be a VCI.

Theorem 4.2. *Let X be a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$. Let m be the maximum number of points of X on a single horizontal ruling, and let n be the maximum number of points on a single vertical ruling. If $|X| < mn$, and there is at least one point in X that is on a horizontal ruling with m points and a vertical ruling with n points, then X is not a VCI.*

Before proving Theorem 4.2, we introduce the generalized Bézout’s theorem and two technical lemmas to serve as tools for providing bounds on multidegrees.

Theorem 4.3 (Generalized Bézout’s Theorem, [Sha77, §4.2.1]). *Let $f, g \in S$ be two bihomogeneous forms in $\mathbb{P}^1 \times \mathbb{P}^1$. If f and g are in general position of multidegree (a, b) and (c, d) respectively, then $|V(f) \cap V(g)| = ad + bc$, counting multiplicities.*

This theorem will be used extensively to help combinatorially determine virtual complete intersections.

Lemma 4.4. *Given a configuration of finitely many points X in $\mathbb{P}^1 \times \mathbb{P}^1$, let m be the maximum number of points on the same horizontal ruling and n be the maximum number of points on the same vertical ruling. If $K(f, g)$ is a virtual resolution of S/I_X , where polynomials f and g are of degrees (a, b) and (c, d) , respectively, then $\max(a, c) \geq m$ and $\max(b, d) \geq n$.*

Proof. Assume, for the sake of contradiction, that both a and c are less than m . Without loss of generality, we can change coordinates to assume that the m points are on the horizontal ruling with coordinates $[1 : 0]$ and assume none of the m points lie on the vertical ruling $[0 : 1]$. We can restrict f to the ruling $[1 : 0]$ by substituting $y_0 = 1, y_1 = 0, x_0 = 1$ yielding a single variable polynomial of degree a with m roots. By the assumption that $a < m$, this restriction of f must be identically 0, and so $V(f)$ contains the entire ruling $[1 : 0]$. By an identical argument on g using $c < m$, we have $V(g)$ also containing the entire ruling $[1 : 0]$. Therefore, $V(f) \cap V(g)$ contains that entire ruling, and so cannot be the original finite set of points. Thus our assumption that both a and c are less than m was false, and so $\max(a, c) \geq m$. The proof that $\max(b, d) \geq n$ is analogous. \square

Lemma 4.5. *Let X be a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$. Let m and n be as in Lemma 4.4. If $K(f, g)$ is a virtual resolution for S/I_X , where f and g have multidegrees (a, b) and (c, d) respectively, and $|X| < mn$, then,*

- (1) *Either (i) $a \geq m$ and $b \geq n$, or (ii) $c \geq m$ and $d \geq n$.*
- (2) *In case (i), g has horizontal components on the lines containing the m points, and vertical components on the lines containing the n points. In case (ii), the same is true of f .*

Proof. By Lemma 4.4, we have

$$\max(a, c) \geq m \quad \text{and} \quad \max(b, d) \geq n.$$

Without loss of generality, suppose $a \geq c$ and $d \geq b$. Then, $a \geq m$ and $d \geq n$. However, in this case $ad \geq mn$, so $ad + bc \geq mn$, which contradicts $|X| < mn$. Therefore, we must have $a \geq c$ and $b \geq d$, so $a \geq m$ and $b \geq n$. This proves (1).

If g does not contain the entire line of the m collinear points, then g restricted to that line is a nonzero polynomial with m roots, and so has degree at least m . This means that $c \geq m$, which gives the contradiction $|X| = ad + bc \geq bc \geq mn$. Similarly, if g does not contain the ruling with n points, then its restriction to that line must have degree at least n giving the contradiction $|X| = ad + bc \geq ad \geq mn$. This completes the proof. \square

Proof of Theorem 4.2. Assume that $|X| < mn$, and there is at least one point in X that lies both on a horizontal ruling with m points and a vertical ruling with n points. If $K(f, g)$ is a virtual resolution for S/I_X , where f has multidegree (a, b) and g has multidegree (c, d) , then by the first part of Lemma 4.5, we may assume $a \geq m$ and $b \geq n$. Suppose $V(g)$ includes s horizontal lines and t vertical lines. Now using the second part of Lemma 4.4, s and t are at least one, and by assumption, the intersection of these $s + t$ lines contains at least one point of X . Factoring g , and changing coordinates if necessary so that no points have coordinate $[0 : 1]$, yields

$$g = \lambda(x_1 - \alpha_1 x_0)(x_1 - \alpha_2 x_0) \cdots (x_1 - \alpha_s x_0)(y_1 - \beta_1 y_0) \cdots (y_1 - \beta_t y_0) \cdot g_0,$$

where g_0 is a bihomogeneous polynomial of multidegree (p, q) . Let $Y \subseteq X$ be the points covered by the $s + t$ components of g . We have $|Y| \leq ms + nt - 1$, because we are certainly double counting the point lying on the intersection of the vertical and horizontal rulings. The remaining set of points,

$X \setminus Y$, must be precisely the intersection of f and g_0 , whose cardinality is $aq + bp$ according to Theorem 4.3.

Applying Theorem 4.3 again to f and g , it follows that

$$a(s + q) + b(t + p) = |X| \leq ms + nt - 1 + aq + bp.$$

Simplifying the inequality above yields

$$as + bt \leq ms + nt - 1.$$

Since $a \geq m$, $b \geq n$, and both s and t are at least one, we have a contradiction. Thus, X cannot be a VCI. \square

Using Lemma 4.4, we can now illustrate Remark 2.3 with an example.

Example 4.6. Consider the configuration of six points in Figure 4, and suppose it is the VCI of f and g with multidegree (a, b) and (c, d) respectively. If any of the degrees were 0, say a , then $V(f)$ would be parallel lines, and since there are five distinct coordinates, f would have degree $(0, 5)$. There is no choice of c and d that satisfies $ad + bc = 6$, so none of the degrees are 0.

Furthermore, by Lemma 4.4 and Theorem 4.3, we find that the only possible multidegrees (up to permutation) are $(2, 1)$ for f and $(2, 2)$ for g . Since there are two points sharing a ruling (both vertically and horizontally), f must have a degree $(0, 1)$ component passing through the vertical one, and therefore must have a degree $(2, 0)$ component passing through the remaining four points. Conics are determined by three points, though, so this is impossible in most cases. Thus, in these cases the set of points could not be a VCI.

However, in the cases where the remaining four points do lie on a conic, the points may be a VCI. For instance, if the points have coordinates:

$$X = \left\{ ([1 : 1], [1 : 1]), ([2 : 1], [1 : 2]), ([3 : 1], [1 : 3]), \right. \\ \left. ([4 : 1], [1 : 4]), ([1 : 0], [1 : 1]), ([1 : 0], [1 : 0]) \right\},$$

then $K(f, g)$ is a virtual resolution of S/I_X , where

$$f = x_0x_1y_0 - x_1^2y_1 \quad \text{and} \quad g = 24x_1^2y_0^2 - x_0^2y_0y_1 - 50x_1^2y_0y_1 + x_0^2y_1^2 - 9x_0x_1y_1^2 + 35x_1^2y_1^2.$$

Theorem 4.2 enables us to give a complete classification of VCI points in $\mathbb{P}^1 \times \mathbb{P}^1$ that lie in a configuration that forms a Ferrers diagram. A Ferrers diagram of points in an m by n grid will be called a **rectangle**. As mentioned in Section 1, [GVT15] show a set of points is arithmetically Cohen–Macaulay exactly when it forms a Ferrers diagram. The corollary below states that if X is arithmetically Cohen–Macaulay, then it is a VCI if and only if it is a complete intersection.

Corollary 4.7. *If X is a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ forming a Ferrers diagram, then X is a virtual complete intersection if and only if it is a rectangle.*

Proof. Defining m and n as before, if X forms a Ferrers diagram that is not a rectangle, then the number of points is strictly lower than mn . Further, the corner of the diagram is one of m points on its horizontal ruling and n points on its vertical ruling, so applying Theorem 4.2 proves that X is not a VCI.

Conversely, if the configuration is a rectangle, then let f denote the $(m, 0)$ -form whose vanishing set consists of the m vertical rulings the points lie on, and let g denote the $(0, n)$ -form whose vanishing

set consists of the n horizontal rulings the points lie on. Then (f, g) is a regular sequence, indicating X is a complete intersection. \square

5. BOUNDS ON MULTIDEGREES AND SIZE OF A CONFIGURATION

In some cases, the property of being a VCI cannot be directly determined based on the maximum number of points on a single vertical/horizontal ruling. In this section, we provide characterizations for VCIs by more closely examining the relationship between the multidegrees of f and g and the total number of points in the configuration.

Theorem 5.1. *If $|X| < mn$ and $\gcd(m, n)$ does not divide $|X|$, then X is not a VCI.*

Before proving Theorem 5.1, we first prove the following lemma.

Lemma 5.2. *Let $K(f, g)$ be a virtual resolution for S/I_X with $|X| < mn$. Assume, without loss of generality, that $a \geq m$ and $b \geq n$. Then,*

- (1) $a = m$, $b = n$, and
- (2) $V(g)$ has vertical components exactly on rulings with n points of X and has horizontal components exactly on rulings with m points of X .

Proof. As in the proof of Theorem 4.2, let s be the number of horizontal lines and t be the number of vertical lines of $V(g)$. Set Y to be the points of g covered by those lines, and let (p, q) be the multidegree of the remaining components of g . By Lemma 4.5, we have $s, t \geq 1$ and

$$as + bt = |Y| \leq ms + nt.$$

By hypothesis, $a \geq m$ and $b \geq n$. Either of these being strict would contradict the above, so $a = m$ and $b = n$. This implies

$$ms + nt = |Y|.$$

Thus, each vertical component of $V(g)$ must contain n points of X and each horizontal component must contain m points of X , because Theorem 4.2 guarantees no point can lie on both a horizontal and vertical component. \square

Note that when $|X| < mn$, the values of s and t are intrinsically determined by the configuration of X : they are the maximum number of points on any horizontal and vertical ruling respectively.

Proof of Theorem 5.1. Suppose $|X| < mn$ with $\gcd(m, n)$ not dividing $|X|$. Assume $K(f, g)$ is a virtual resolution for S/I_X . From Lemma 5.2, it can be assumed f has multidegree (m, n) . Letting g have multidegree (c, d) , Theorem 4.3 implies $|X| = dm + cn$. This is divisible by $\gcd(m, n)$, which is a contradiction. \square

Proposition 5.3. *If $K(f, g)$ is a virtual resolution for S/I_X with $|X| < mn$, $\gcd(m, n) = 1$, and (m, n) the multidegree of f , then the multidegree of g is (c, d) where:*

$$c = n^{-1}|X| \pmod{m} \quad \text{and} \quad d = m^{-1}|X| \pmod{n}$$

(with $0 \leq c < m$ and $0 \leq d < n$).

Proof. By Theorem 4.3,

$$dm + cn = |X|.$$

Considering modulo m and n , we have:

$$c \equiv n^{-1}|X| \pmod{m} \quad \text{and} \quad d \equiv m^{-1}|X| \pmod{n}.$$

Since both cn and dm are less than mn we must have $c < m$ and $d < n$. Thus c and d must have the desired values. \square

Proposition 5.4. *Assume X is a finite set of points with $|X| < mn$, $\gcd(m, n) = 1$, and let*

$$c \equiv n^{-1}|X| \pmod{m} \quad \text{and} \quad d \equiv m^{-1}|X| \pmod{n}.$$

Let s and t be defined as in the proof of Theorem 4.2, and set $p := d - s$ and $q := c - t$. If any of the following are true, X will not be a VCI.

- (1) $dm + cn \neq |X|$
- (2) $d < s$ or $c < t$
- (3) *There is a horizontal ruling with strictly between q and m points of X , or a vertical ruling with strictly between p and n points of X .*

Proof. We will prove the contrapositive of (1). Assume that $K(f, g)$ is a virtual resolution for S/I_X . Then by Lemma 5.2, $\deg(f) = (m, n)$ and by Proposition 5.3, $\deg(g) = (c, d)$. Thus Theorem 4.3 guarantees $|X| = dm + cn$.

If X is a VCI of f and g , then g has s horizontal line components, so $d \geq s$. Similarly since it has t vertical line components, then $c \geq t$. Thus (2) is proved.

By Lemma 5.2, any horizontal ruling with fewer than m points of X cannot be contained in $V(g)$. However, a polynomial of multidegree (p, q) cannot vanish on more than p points of a horizontal ruling without containing the entire ruling. Analogously, g cannot vanish on between q and n points of a vertical ruling, completing (3). \square

Note that when X is a VCI of f and g with $|X| < mn$, we have determined not only the multidegree of g but also the multidegree of components that are not degree 1 lines, that is, p and q are intrinsically determined.

Using these restrictions on VCIs, it is possible to classify all possible configurations of VCIs when the parameters are small and eliminate a sizable number of cases in general. Nevertheless, it is important to keep in mind Remark 2.3, which illustrates the limit of the applicability of combinatorics of configurations to determine VCIs when the size of the configuration gets sufficiently large (with respect to m and n).

6. COMPLETE CLASSIFICATIONS OF POINTS ON 2 OR 3 RULINGS

In this section, we give a complete classification of VCIs when all points lie on 2 or 3 rulings.

Theorem 6.1. *If all points lie on two horizontal rulings, they form a VCI if and only if either:*

- (1) *no two of them lie on the same vertical ruling, or*
- (2) *both horizontal rulings contain the same number points.*

Proof. If no two points lie on the same vertical ruling, then each vertical ruling contains exactly one point, so the configuration is a VCI by Theorem 4.1. If both horizontal rulings contain the same number of points, we can match point p_i in one ruling with point p'_i in the other; then the configuration is the VCI of two horizontal lines through the two rulings and the product of $(1, 1)$ forms each passing through one pair of p_i and p'_i .

Inversely, suppose two points lie on the same vertical ruling, and the two horizontal rulings have different numbers of points. Then the maximum number of points on the same horizontal ruling times 2 (the maximum number of points on the same vertical ruling) is greater than the total number of points. So by Theorem 4.2, this configuration is not a VCI. \square

Theorem 6.2. *If all points lie on three horizontal rulings, the configuration is a VCI if and only if it satisfies one of the following conditions:*

- (1) *All horizontal rulings contain the same number of points.*
- (2) *All vertical rulings contain the same number of points.*
- (3) *On two of the horizontal rulings all points are in pairs on the same vertical ruling, and no vertical rulings contain 3 points.*
- (4) *Two of the horizontal rulings contain the same number of points, k , and all points lie on a $(k, 1)$ -curve.*

Proof. All four conditions are sufficient. Conditions (1) and (2) follow from Theorem 4.1. Configurations satisfying condition (3) can always be decomposed into two rectangular blocks and be seen as an intersection of two forms that resemble the structure in Figure 6. That is, $V(f)$ consists of the two paired horizontal rulings and vertical rulings through each remaining point, and $V(g)$ consists of vertical rulings through each pair of points and a horizontal ruling through the remaining points.

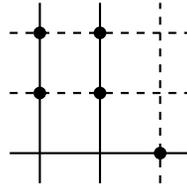


FIGURE 6. f is the product of solid lines; g of dashed lines.

Condition (4) can always be decomposed into the intersection of a $(k, 1)$ -curve, and the union of two horizontal lines and one vertical line through each point of X on the remaining ruling, as demonstrated in Figure 7. By Theorem 4.3, the $(k, 1)$ -curve will intersect the two horizontal rulings in exactly the k points each, and will intersect each vertical ruling in exactly one point, as desired. The vertical rulings need to be carefully chosen so that their intersections with the curve lie on the same horizontal ruling.

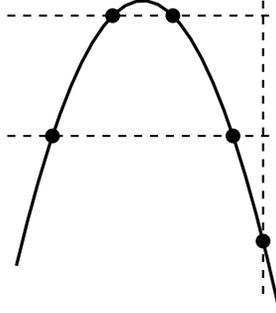


FIGURE 7. f is the solid parabola; g is the product of dashed lines.

We now show that these are the only cases. Assume X is a VCI that satisfies none of the conditions above. Let n be the maximum number of points on a single vertical ruling so n may be 1, 2 or 3. We will show that in each case X must satisfy one of the conditions above.

Notice that if $n = 1$, then condition (2) is satisfied. If $n = 3$, one of the 3 points on such a ruling will also be on the horizontal ruling with the maximal number m of points. Theorem 4.2 implies X will not be a VCI unless all three horizontal rulings contain m points, in which case (1) holds.

We now consider the the most difficult case of $n = 2$. Assume the three horizontal rulings contain $\alpha \geq \beta \geq \gamma$ points, respectively. Notice that α, β, γ are not all equal since otherwise X satisfies condition (1). Since X is a VCI, $K(f, g)$ is a virtual resolution of S/I_X for some f, g of multidegrees (a, b) and (c, d) . According to Lemma 4.4, $\max\{a, c\} \geq \alpha$ and $\max\{b, d\} \geq 2$. Without loss of generality, suppose $a \geq c$. Then $a \geq \alpha$. This means $d < 3$ since $3\alpha > |X| = ad + bc \geq d\alpha$.

There are two cases:

Case 1: $b \geq 2$. If $d = 0$, then $V(g)$ is the union of vertical rulings. By Bézout's Theorem, the number of intersection points of $V(f)$ and each vertical ruling in $V(g)$ is b , so the configuration must satisfy (2).

Therefore, $d > 0$. Notice that g having a high degree in y -dimension implies a low degree in x -dimension. In particular, degree $c \leq \beta$, because

$$\alpha + 2\beta \geq \alpha + \beta + \gamma = |X| = ad + bc \geq \alpha + 2c \quad (6.1)$$

(using $a \geq \alpha$ and $b \geq 2$ in the last step).

If $c < \beta$, then $V(g)$ must contain the entirety of the horizontal rulings of α points and β points. This is because when restricting to those rulings g would be a polynomial of degree c , and so could vanish at β points only if it vanished on the whole ruling. Thus, we have $d \geq 2$, and so $\alpha + \beta + \gamma = ad + bc \geq 2\alpha + 2c$. From this we find $c < \gamma$, and again using the same argument, $V(g)$ vanishes on all 3 horizontal rulings, which violates the fact that $d < 3$.

On the other hand, if $c = \beta$, then using the inequality (6.1) and recalling $b \geq 2, d > 0$, we have

$$c = \beta = \gamma, \quad a = \alpha, \quad b = 2, \quad \text{and} \quad d = 1.$$

In this case, $\alpha > \beta = c$, so $V(g)$ must contain the entire horizontal ruling of α points. Since $d = 1$, the rest of $V(g)$ can only consist of vertical rulings, and there must be β of them. Each vertical ruling intersects $V(f)$ at 2 points by Theorem 4.3, and so the configuration satisfies (3).

Case 2: $b < 2$. Since $\max\{b, d\} \geq 2$, d must equal 2. If $b = 0$, then $V(f)$ is the union of multiple vertical rulings. Since the number of intersections of $V(g)$ and each vertical ruling is always c , the configuration must satisfy (2). Thus, $b = 1$. As before, notice that

$$\alpha + \beta + \gamma = |X| = ad + bc \geq 2\alpha + c.$$

Then, $c \leq \gamma$. If $c < \gamma$, then $V(g)$ has to contain all three horizontal rulings, which contradicts $d = 2$. Therefore, $c = \gamma$, and $\alpha = \beta > \gamma$. In this case, $V(g)$ must contain the entire horizontal rulings of α points and β points. Since $d = 2$, the rest of $V(g)$ can only be vertical rulings, and there must be γ of them, and recalling $b = 1$, the configuration satisfies (4). \square

REFERENCES

- [BES17] Christine Berkesch, Daniel Erman, and Gregory G. Smith, *Virtual Resolutions for a Product of Projective Spaces* (2017), available at [arXiv:1703.07631](https://arxiv.org/abs/1703.07631).
- [CLO05] David A. Cox, John B. Little, and Donal O’Shea, *Using algebraic geometry*, Graduate Studies in Mathematics 124, vol. 185, Springer, New York, 2005.
- [CLS11] David A. Cox, John B. Little, and Henry K. Schenck, *Toric varieties*, Graduate Studies in Mathematics 124, Amer. Math. Soc., Providence, RI, 2011.
- [Eis05] David Eisenbud, *The geometry of syzygies: a second course in commutative algebra and algebraic geometry*, Graduate Texts in Mathematics 229, Springer Science+Business Media, Inc, New York, 2005.
- [GVT15] Elena Guardo and Adam Van Tuyl, *Arithmetically Cohen–Macaulay sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$* , SpringerBriefs in Mathematics, Springer, Cham, 2015.
- [Har77] Robin Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics 52, Springer-Verlag, New York-Heidelberg, 1977.
- [Har95] Joe Harris, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 133, Springer-Verlag, New York, 1995. A first course, Corrected reprint of the 1992 original.
- [M2] Daniel R. Grayson and Michael E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, available at <http://www.math.uiuc.edu/Macaulay2/>.
- [Pee11] Irena Peeva, *Graded syzygies*, Algebra and Applications, vol. 14, Springer-Verlag London, Ltd., London, 2011.
- [Sha77] Igor R. Shafarevich, *Basic Algebraic Geometry I: Varieties in Projective Space*, 3rd, Springer-Verlag, New York-Heidelberg, 1977.

Jiyang Gao, DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MA 02139

E-mail address: gaojy98@mit.edu

Yutong Li, DEPARTMENT OF MATHEMATICS AND STATISTICS, HAVERFORD COLLEGE, HAVERFORD, PA 19041

E-mail address: yli11@haverford.edu

Michael C. Loper, SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MN 55455

E-mail address: loper012@umn.edu

Amal Mattoo, DEPARTMENT OF MATHEMATICS, HARVARD COLLEGE, CAMBRIDGE, MA 02138

E-mail address: amattoo@college.harvard.edu