WHAT MAKES A COMPLEX VIRTUAL

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Abstract. Virtual resolutions are homological representations of finitely generated Pic(X)-graded modules over the Cox ring of a smooth projective toric variety. In this paper, we identify two algebraic conditions that characterize when a chain complex is virtual. We then turn our attention to the saturation of Fitting ideals by the irrelevant ideal of the Cox ring and prove some results that mirror the classical theory of Fitting ideals for Noetherian rings.

1. Introduction

In a famous paper, Buchsbaum and Eisenbud present two criteria that completely determine whether or not a chain complex is exact over a Noetherian ring [BE73]. This is done without examining the homology of the complex. These criteria are useful in investigating a module by examining the minimal free resolution. Coherent sheaves over projective space correspond to finitely generated graded modules over a standard-graded polynomial ring. Properties of this module and the sheaf associated to the graded module such as degree, dimension, and Hilbert polynomial, are encoded in the minimal free resolution of the graded module. Below is the main theorem from [BE73].

Theorem 1.1 ([BE73])**.** *Let R be a Noetherian ring. Suppose*

$$F\colon 0 \to F_n \xrightarrow{\varphi_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0$$

is a chain complex of free R-modules. Then F is exact if and only if both of the following conditions are satisfied:

(a) $\operatorname{rank}(\varphi_i) + \operatorname{rank}(\varphi_{i+1}) = \operatorname{rank}(F_i)$ *for each* $i = 1, 2, ..., n$ *(taking* $\varphi_{n+1} = 0$*),*

(b) $\operatorname{depth}(I(\varphi_i)) \geq i$*.*

Unfortunately when studying coherent sheaves over more general smooth projective toric varieties, the situation is not as well-behaved. Locally free resolutions of a coherent sheaf are often shorter and thinner than the corresponding minimal free resolutions of the modules. Tying these concepts more closely together, Berkesch, Erman, and Smith introduced the notion of virtual resolutions in [BES17]. The main theorem in the present paper (Theorem 1.3) is the virtual analogue to the main theorem of [BE73] (Theorem 1.1).

Notation. Throughout this paper, X will be a smooth projective toric variety and S will denote the Cox ring of X over an algebraically closed field $\Bbbk$. The Cox ring S is graded by the Picard group of X, which we denote by Pic(X) [Cox95, §1]. In particular, S is a polynomial ring with a multigrading. Denote the irrelevant ideal of S by B; M will be a finitely generated $\operatorname{Pic}(X)$-graded module over S and $\widetilde{M}$ denotes the sheaf of M over X, as constructed in [Cox95, §3]. Given an ideal I of S, we denote the set of all homogenous prime ideals containing I by $V(I)$.

Definition 1.2. A graded free complex $F\colon \cdots \to F_n \xrightarrow{\varphi_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0$ of S is called a **virtual resolution** of M if the corresponding complex of vector bundles $\widetilde{F}$ is a locally free resolution of the sheaf $\widetilde{M}$.

We will also call a complex **virtual** if it is a virtual resolution of some $\operatorname{Pic}(X)$-graded S-module. The above definition uses the geometric language, but virtual resolutions can be equivalently defined algebraically. The $\operatorname{Pic}(X)$-graded S-module associated to a sheaf $\mathcal{F}$ is defined to be $\Gamma_*(\mathcal{F}) = \bigoplus_{\alpha\in\operatorname{Pic}(X)} \Gamma(X, \mathcal{F}(\alpha))$ (see [Cox95, Theorem 3.2]). The complex F is a virtual resolution of M if $\Gamma_*\big(\widetilde{M}\big) \cong \Gamma_*(\widetilde{\operatorname{coker}(\varphi_1)})$ and for every $i \geq 1$, there is an n such that $B^n H_i(F) = 0$.

A map of free S-modules $\varphi\colon F \to G$ can be expressed as a matrix with entries in S by choosing bases of F and G. Denote by $I_r(\varphi)$ the ideal generated by the $r \times r$ minors of φ. Then $\operatorname{rank}(\varphi)$ will be the largest r such that $I_r(\varphi) \neq 0$. The ideal $I_{\operatorname{rank}(\varphi)}(\varphi)$ will be the most important of these ideals of minors, and we set $I(\varphi) := I_{\operatorname{rank}(\varphi)}(\varphi)$. By convention, we define $I_k(\varphi) = S$ for every integer $k \leq 0$.

In fact, these ideals of minors can be extended to projective modules (that may not be finitely generated). Indeed φ gives rise to a map $\bigwedge^r \varphi\colon \bigwedge^r F \to \bigwedge^r G$, and the rank of φ is the largest r such that $\bigwedge^r \varphi \neq 0$. In this context, $I_r(\varphi)$ is the image of $\bigwedge^r F \otimes (\bigwedge^r G)^* \to S$.

Our main result is:

Theorem 1.3. *Let X be a smooth projective toric variety with $S = \operatorname{Cox}(X)$. Suppose*

$$F\colon 0 \to F_n \xrightarrow{\varphi_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0$$

is a $\operatorname{Pic}(X)$-graded complex of free S-modules. Then F is virtual if and only if both of the following conditions are satisfied:

(a) $\operatorname{rank}(\varphi_i) + \operatorname{rank}(\varphi_{i+1}) = \operatorname{rank}(F_i)$ *for each $i = 1, 2, ..., n$ (with $\varphi_{n+1} = 0$),*
(b) $\operatorname{depth}(I(\varphi_i) : B^\infty) \geq i$.

As in [BE73], we assign the unit ideal infinite depth, so that condition (b) holds if $I(\varphi_i) : B = S$. The difference between Theorem 1.1 and Theorem 1.3 is the replacement of exactness with virtuality and the addition of the saturation of ideals of minors by the irrelevant ideal B. Of course, any complex that is exact will also be a virtual complex. Further, if a complex is exact, then the conditions of Theorem 1.3 will be satisfied by Theorem 1.1. On the other hand, below is an example of a complex that is virtual but not exact.

Example 1.4. Let $X = \mathbb{P}^1 \times \mathbb{P}^2$ so that $S = \Bbbk[x_0, x_1, y_0, y_1, y_2]$ with $\deg(x_i) = (1, 0)$ and $\deg(y_i) = (0, 1)$. Then the irrelevant ideal B is $\langle x_0, x_1\rangle \cap \langle y_0, y_1, y_2\rangle$. Let I be the following

B-saturated ideal of 4 points:

$$\begin{gathered}\langle y_0y_1 - 22y_1^2 - 20y_0y_2 - 26y_1y_2 - 6y_2^2, y_0^2 - 20y_1^2 - 14y_0y_2 - 22y_1y_2 + 33y_2^2,\\ x_0y_1 + 10x_0y_2 - x_1y_0 - 19x_1y_1 + 47x_1y_2, x_0y_0 + 43x_0y_2 - 13x_1y_0 + 50x_1y_1 + 20x_1y_2,\\ x_0^2y_2 - 42x_0x_1y_2 + 42x_1^2y_0 - 48x_1^2y_1 + 13x_1^2y_2, x_0^4 + 27x_0^3x_1 + 25x_0^2x_1^2 - 29x_0x_1^3 - 40x_1^4\rangle.\end{gathered}$$

The minimal free resolution of S/I is included below, calculated using `Macaulay2` [M2].

$$S \longleftarrow \begin{array}{c} S(-1,-1)^2\\ \oplus\\ S(0,-2)^2\\ \oplus\\ S(-2,-1)\\ \oplus\\ S(-4,0)\end{array} \longleftarrow \begin{array}{c} S(-1,-2)^2\\ \oplus\\ S(-2,-2)^3\\ \oplus\\ S(-1,-3)^2\\ \oplus\\ S(0,-4)\\ \oplus\\ S(-1,-4)^3\end{array} \longleftarrow \begin{array}{c} S(-2,-3)^3\\ \oplus\\ S(-1,-4)^2\\ \oplus\\ S(-4,-2)^3\end{array} \longleftarrow \begin{array}{c} S(-2,-4)\\ \oplus\\ S(-4,-3)\end{array} \longleftarrow 0.$$

The virtual resolution of the pair $(S/I, (1,1))$ [BES17, Theorem 1.3] is

$$S \xleftarrow{\varphi_1} \begin{array}{c} S(-1,-1)^2\\ \oplus\\ S(0,-2)^2\\ \oplus\\ S(-2,-1)\end{array} \xleftarrow{\varphi_2} \begin{array}{c} S(-1,-2)^2\\ \oplus\\ S(-2,-2)^3\\ \oplus\\ S(-1,-3)^2\end{array} \xleftarrow{\varphi_3} S(-2,-3)^3 \longleftarrow 0.$$

Notice that this virtual resolution is both shorter and thinner than the minimal free resolution, as is often the case. `Macaulay2` [M2] calculates that this virtual resolution has nonzero first homology module $S/\langle x_0, x_1\rangle$, which is annhilated by B. Consequently, this virtual complex is not exact. Calculating the depth of the (non-B-saturated) ideal of minors of each differential yields

$$\operatorname{depth}(I(\varphi_1)) = 3, \qquad \operatorname{depth}(I(\varphi_2)) = 2, \qquad \operatorname{depth}(I(\varphi_3)) = 2,$$

indicating again that the virtual complex is not exact, because the depth of $I(\varphi_3)$ is not large enough. Though the complex is not exact, $\operatorname{depth}(I(\varphi_3) : B^\infty) = 3$, so it is indeed virtual. ⋄

Remark 1.5. In the special case of the smooth projective toric variety $\mathbb{P}^n_{\Bbbk}$, the Cox ring is $S = \Bbbk[x_0, x_1, ..., x_n]$, and the irrelevant ideal is the maximal homogeneous ideal $B = \mathfrak{m} = \langle x_0, x_1, ..., x_n\rangle$. If the length of the complex F is not longer than $n+1$ (the bound from the Hilbert Syzygy Theorem), then Theorem 1.3 recovers Theorem 1.1. This is because each $I(\varphi_i)$ is homogeneous (in fact, every minor of φ_i is homogeneous), and for any homogeneous ideal $I \subsetneq \mathfrak{m}$,

$$\operatorname{depth}(I) \leq \operatorname{depth}(I : \mathfrak{m}^\infty) \leq \operatorname{depth}(\sqrt{I} : \mathfrak{m}^\infty) = \operatorname{depth}(\sqrt{I}) = \operatorname{depth}(I),$$

where the second to last equality follows from the fact that any homogeneous radical ideal properly contained in $\mathfrak{m}$ is $\mathfrak{m}$-saturated. Therefore, condition (b) in Theorem 1.3 becomes

$$i \leq \operatorname{depth}(I(\varphi_i) : B^\infty) = \operatorname{depth}(I(\varphi_i)),$$

which exactly matches condition (b) in Theorem 1.1.

As condition (a) is the same in the two theorems, we have thus proved the following proposition. A slightly different proof is offered below.

Proposition 1.6. *Let* $S = \mathrm{Cox}(\mathbb{P}^n_{\Bbbk}) = \Bbbk[x_0, x_1, .., x_n]$. *If*

$$F : 0 \longrightarrow F_m \longrightarrow F_{m-1} \longrightarrow \cdots \longrightarrow F_0$$

is a virtual resolution with $m \leq n+1$, *then* F *is exact.*

Proof. The irrelevant ideal is $B = \langle x_0, x_1, ..., x_n \rangle$. Since F is a virtual resolution, all of its homology modules are annihilated by a power of B. Assume F has a nonzero homology module and let $H := H_i(F)$ where i is the largest index with $H_i(F)$ nonzero. The Peskine–Szpiro Acyclicity Lemma [PS73, Lemma 1.8] guarantees $\mathrm{depth}(H) \geq 1$ and hence B has a nonzerodivisor on H, a contradiction to the assumption that F is a virtual resolution. Therefore, in this setting, if F is a virtual resolution, it must be exact. ☐

Outline. Section 2 lays the groundwork for the rest of the paper. This includes notation, relevant definitions, and some preliminary facts about B-saturated prime ideals. These primes are important, because localizing a module at a B-saturated homogeneous prime corresponds to taking the stalk of a sheaf over the toric variety X. Section 3 contains the proof of the forward direction of Theorem 1.3, while Section 4 contains the proof of the converse. In Section 5, the invariance of saturated Fitting ideals of virtual presentations is presented along with an obstruction to the number of generators of an ideal up to saturation. Finally, Section 6 contains a connection between saturated Fitting ideals and locally free sheaves. It ends with a useful result concerning unbounded virtual resolutions. The results in Sections 5 and 6 can be used to prove Theorem 1.3 in the same way that results of Fitting ideals can be used to prove Theorem 1.1.

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2. B-Saturated Prime Ideals

In this section the structure of B-saturated prime ideals in the Cox ring S is discussed, which will aid in later proofs. Indeed, in order to show that a complex is virtual, we will show that after sheafifying, the complex of vector bundles is acyclic by showing it is exact in each place. The latter will be true if and only if the stalk at each B-saturated homogeneous prime is exact.

Recall the **saturation of an ideal** I **by** B is

$$I : B^\infty = \bigcup_{n \geq 0} (I : B^n) = \{s \in S \mid sB^n \subset I \text{ for some } n\}.$$

There is a correspondence between B-saturated ideals of S and closed subschemes of X [CLS11, Proposition 6.A.7].

Throughout the paper we will often be concerned with the structure of the homogeneous B-saturated prime ideals of S, which form a proper subset of the homogeneous prime ideals of S. It will be important to see which prime ideals are B-saturated.

Proposition 2.1. *Suppose P is a homogenous prime ideal of S. Then either P is B-saturated or a prime component of B is contained in P, in which case $P : B^\infty = S$.*

Proof. We first prove the second statement. Let $B = \bigcap_{i=1}^m Q_i$ with each Q_i a homogeneous prime ideal. Then

$$P : B^\infty = (\cdots((P : Q_1^\infty) : Q_2^\infty) : \cdots : Q_m^\infty).$$

Now if $P \supset Q_i$, then $P : Q_i^\infty = P : Q_i = S$. This proves the second part of the proposition.

In order to show the first part, it suffices to show that if P and Q are prime ideals so that P does not contain Q, then $P : Q^\infty = P$. Suppose $s \in P : Q^\infty$ and let $\ell \in \mathbb{Z}_{\geq 0}$ be such that $sQ^\ell \subset P$. Since $Q \not\subset P$, there is a $q \in Q - P$. Thus $q^\ell \in Q^\ell - P$, but $sq^\ell \in P$ so $s \in P$. □

This first proposition says that every homogenous prime ideal of S is either B-saturated or irrelevant. The lemma below implies we need only consider prime ideals of small enough height.

Lemma 2.2. *If I is a homogeneous ideal of codimension greater than $\dim X$, then $I : B^\infty = S$.*

Proof. Homogeneous ideals that saturate to all of S are ideals that correspond to the empty subvariety of X. It is enough to show that if I corresponds to a nonempty subvariety of X, then the height of I is less than or equal to the dimension of X. Let $d := \dim X$, $c := \operatorname{codim} I$, and n be the number of variables of S. Suppose x is in the subvariety of X corresponding to I. Considering the quotient construction of X, there is a torus G that acts on n-dimensional affine space $\mathbb{A}^n$. As G acts freely on $\mathbb{A}^n \backslash V(B)$ [CLS11, Exercise 5.1.11], the dimension of both G and the orbit Gx in $\mathbb{A}^n$ is $n - d$. Then $Gx \subset V(I) \subset \mathbb{A}^n$ implies that $c \leq d$, since

$$n - d = \dim(Gx) \leq \dim V(I) = n - c.$$ □

Therefore, when considering homogeneous B-saturated primes of the Cox ring S of X, we need only consider the homogeneous primes of codimension at most the dimension of X that do not contain any prime components of the irrelevant ideal B.

3. Sufficiency of Criteria

Here we prove the forward direction of Theorem 1.3.

Proof that conditions (a) and (b) imply the complex is virtual. It is enough to show for each B-saturated homogeneous prime ideal P of S, the complex $\widetilde{F}_P$ is exact. Since S is a polynomial ring, it is an integral domain, so each $I(\varphi_i)$ contains a nonzerodivisor. Therefore, as localization commutes with taking Fitting ideals [Eis95, Corollary 20.5], the hypotheses are not weakened. We will show that $\widetilde{F}_P$ is exact by induction on the codimension of P. The unique minimal prime ideal of S is $\langle 0\rangle$, and localizing at this ideal gives a complex of vector spaces. The complex $F_{\langle 0\rangle}$ therefore becomes exact by assumption (a).

Now suppose P is a B-saturated prime of codimension $c > 0$. By assumption (b), $\operatorname{depth}(I(\varphi_{c+1}) : B^\infty) \geq c+1$, so $I(\varphi_{c+1}) : B^\infty$ is not contained in P. Therefore $I(\varphi_{c+1})$ is also not contained in P. Indeed, if $I(\varphi_{c+1}) \subset P$, then $I(\varphi_{c+1}) : B^\infty \subset P : B^\infty = P$. Hence $I(\varphi_{c+1})_P = S_P$, so F_P can be broken into two complexes:

$$G_P\colon \quad 0 \longrightarrow (F_n)_P \longrightarrow \cdots (F_{c+1})_P \xrightarrow{(\varphi_{c+1})_P} (F_c)_P \longrightarrow (\operatorname{coker}\varphi_{c+1})_P \longrightarrow 0,$$

$$H_P\colon \quad 0 \longrightarrow (\operatorname{coker}\varphi_{c+1})_P \xrightarrow{(\varphi'_c)_P} (F_{c-1})_P \xrightarrow{(\varphi_{c-1})_P} (F_{c-2})_P \longrightarrow \cdots \longrightarrow (F_0)_P,$$

where φ'_c is induced from φ_c. Now since $I((\varphi_i)_P) = S_P$ for every $i \geq c+1$, the complex G_P is exact. All that is left to show is that H_P is exact. Since $I((\varphi_{c+1})_P) = S_P$, the S_P-module $(\operatorname{coker}\varphi_{c+1})_P$ is projective and hence free. Also notice that $I((\varphi'_c)_P) = I((\varphi_c)_P)$.

By induction, H_P becomes exact when localizing at any homogeneous B-saturated prime ideal Q properly contained in P. So by Proposition 2.1, it becomes exact when localizing at any prime ideal contained in P (if a smaller prime contains a prime component of B, then P does as well, which means that P would not be B-saturated). Therefore the homology modules are only supported on the maximal ideal P_P and thus have depth 0. The depth of each free S_P-module is equal to the codimension of P (since S is Cohen–Macaulay), which is strictly positive. Therefore, applying the Peskine–Szpiro Acyclicity Lemma [PS73, Lemma 1.8] completes the proof. □

4. Necessity of the Conditions

In order to prove the reverse direction of Theorem 1.3, the following lemma is needed.

Lemma 4.1. *If I is a homogeneous ideal of S with $I : B^\infty \neq S$ and* $\operatorname{depth}(I : B^\infty) = k$*, then there is a B-saturated homogeneous prime P such that* $\operatorname{depth}(I_P) \leq k$*.*

Proof. Let $x_1, x_2, ..., x_k$ be a maximal regular sequence in $I : B^\infty$. Set $J = \langle x_1, x_2, ..., x_k\rangle$, so that $I : B^\infty$ consists of zero divisors on S/J. All associated primes of $S/(I : B^\infty)$ are homogeneous. These associated primes are also B-saturated, because saturation removes the Q-primary components of I with $Q \subset B$. Let $P \in \operatorname{Ass}(S/(I : B^\infty))$. Since $J \subset I : B^\infty \subset P$, all elements of $(I : B^\infty)_P$ are zero divisors on $(S/J)_P$. Thus $x_1, ..., x_k$ is a maximal sequence of $(I : B^\infty)_P$. Therefore,

$$\operatorname{depth}(I_P) \leq \operatorname{depth}((I : B^\infty)_P) = k.$$ □

Proof that virtuality of F implies (a) and (b). As F is virtual, F becomes exact after localizing at any B-saturated prime ideal of S. The ideal $\langle 0\rangle$ is a B-saturated prime and after localizing, $F_{\langle 0\rangle}$ becomes an exact sequence of vector spaces. This proves (a).

To prove (b), suppose that there is some i such that $I(\varphi_i) : B^\infty \neq S$, and $\operatorname{depth}(I(\varphi_i) : B^\infty) < i$. Since S is an integral domain, $I(\varphi_i)$ contains a nonzerodivisor and since taking Fitting ideals commutes with localization, $I((\varphi_i)_P) = I(\varphi_i)_P$. Using Lemma 4.1, there is a

B-saturated prime ideal P such that $\operatorname{depth}(I(\varphi_i)_P) \leq \operatorname{depth}(I(\varphi_i) : B^\infty) < i$. Therefore by Theorem 1.1, F_P is not exact, a contradiction to the assumption that F is virtual. Hence (b) holds. □

5. Invariance of Saturated Fitting Ideals

Between this section and Section 6, we record the groundwork of tools for an alternative proof of Theorem 1.3, using B-saturated Fitting ideals in the Cox ring S of a smooth projective toric variety X. Here, we present some facts about the B-saturation of Fitting ideals of an S-module. The Fitting ideals of a finitely-generated module M over a Noetherian ring can be calculated by a free presentation $F \xrightarrow{\varphi} G \to M \to 0$. Let r denote the rank of G. The ***i*th Fitting ideal of** M, $\operatorname{Fitt}_i(M)$, is defined to be $I_{r-i}(\varphi)$. This ideal is independent of the free presentation [Fit36]. We now adapt this idea of Fitting ideals to virtual presentations of a $\operatorname{Pic}(X)$-graded S-module M, beginning with the invariance of the B-saturated Fitting ideals. We then produce further facts about saturated Fitting ideals that mirror the classical theory of Fitting ideals.

Lemma 5.1 (Saturated Fitting's Lemma)**.** *Suppose X is a smooth projective toric variety and S is the Cox ring of X with irrelevant ideal B. Let*

$$F \xrightarrow{\varphi} G \longrightarrow M \longrightarrow 0 \qquad \text{and} \qquad F' \xrightarrow{\varphi'} G' \longrightarrow M' \longrightarrow 0$$

be finite virtual presentations of M and M', respectively, with G and G' of ranks r and r'. If $\widetilde{M} \cong \widetilde{M'}$, then $I_{r-i}(\varphi) : B^\infty = I_{r'-i}(\varphi') : B^\infty$ for every $i \in \mathbb{Z}_{\geq 0}$.

Proof. We may harmlessly assume that M is already B-saturated and that $F \longrightarrow G \longrightarrow M \longrightarrow 0$ is the truncation of the minimal free resolution of M. By replacing M' with $\operatorname{coker} \varphi'$ if necessary, we may also assume that $F' \longrightarrow G' \longrightarrow M' \longrightarrow 0$ is a free presentation of M'.

Now $I_{r-i}(\varphi) : B^\infty = I_{r'-i}(\varphi') : B^\infty$ if and only if the sheaves $\widetilde{I_{r-i}(\varphi)}$ and $\widetilde{I_{r'-i}(\varphi')}$ are equal. We will show the equality of the sheaves by showing that they agree on every distinguished open affine $D(f)$ of X. Given $f \in S$, $D(f)$ can be written (by abusing notation) as $\{p \in X | f(p) \neq 0\}$. In this case,

$$\widetilde{I_{r-i}(\varphi)}(D(f)) \cong I_{r-i}(\varphi)_f = I_{r-i}(\varphi_f),$$

where the last equality holds because taking Fitting ideals commutes with base change. Because localization is exact, $F_f \xrightarrow{\varphi_f} G_f \longrightarrow M_f \longrightarrow 0$ is a free presentation of M_f. Furthermore, since $\widetilde{M} = \widetilde{M'}$, we have $M_f \cong \widetilde{M}(D(f)) = \widetilde{M'}(D(f)) \cong M'_f$. Fitting's Lemma [Fit36] implies $I_{r-i}(\varphi)_f = I_{r'-i}(\varphi'_f)$.

Thus $\widetilde{I_{r-i}(\varphi)}$ and $\widetilde{I_{r'-i}(\varphi')}$ agree on every distinguished open affine $D(f)$ of X, so the sheaves are equal. □

We will call $I_{r-i}(\varphi) : B^\infty$ the ***i*th saturated Fitting ideal of** M. Lemma 5.1 allows us to prove that the jth saturated Fitting ideal not equaling S is an obstruction to generating a module up to saturation by j elements. It is analogous to the fact that the jth Fitting ideal not equaling S is an obstruction to generating a module by j elements in the classical theory of Fitting ideals of Noetherian rings (see [Eis95, Proposition 20.6]).

Define $\mathbb{V}(I)$ to be the set of homogeneous B-saturated prime ideals P of S such that $P \supset I$. In particular,

$$\mathbb{V}(I) := \{P \in V(I) \mid P \text{ is } B\text{-saturated}\}.$$

Proposition 5.2. *The set $\mathbb{V}(\mathrm{Fitt}_j(M) : B^\infty)$ consists of exactly the homogeneous B-saturated primes P such that there does not exist an S-module N where N_P can be generated by j elements and M and N have the same saturation.*

Proof. Suppose $P \in \mathbb{V}(\mathrm{Fitt}_j(M) : B^\infty)$ and N is an ideal such that the sheaves $\widetilde{M}$ and $\widetilde{N}$ are isomorphic. Then by Lemma 5.1, for every $j \in \mathbb{Z}_{\geq 0}$, $\mathrm{Fitt}_j(M) : B^\infty = \mathrm{Fitt}_j(N) : B^\infty$. So

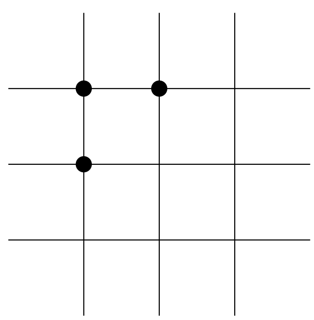
$$\mathrm{Fitt}_j(N) \subset \mathrm{Fitt}_j(N) : B^\infty \subset P,$$

and therefore N_P cannot be generated by j elements [Eis95, Proposition 20.6].

On the other hand, suppose P is a homogeneous B-saturated prime not belonging to $\mathbb{V}(\mathrm{Fitt}_j(M) : B^\infty)$. In this case, $\mathrm{Fitt}_j(M)$ cannot be contained in P. For if $\mathrm{Fitt}_j(M) \subset P$, then $\mathrm{Fitt}_j(M) : B^\infty \subset P : B^\infty = P$. Thus M_P can be generated by j elements. □

Notice that the previous proposition does not say that if $\mathrm{Fitt}_j(M) : B^\infty = S$, then there is a module N generated by j elements such that $\widetilde{M} \cong \widetilde{N}$. Indeed, this fails to be true as shown by the following example.

Example 5.3. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ so that $S = \Bbbk[x_0, x_1, y_0, y_1]$ with $\deg(x_i) = (1,0)$ and $\deg(y_i) = (0,1)$. Consider the B-saturated ideal $I = \langle y_0y_1, x_0y_0, x_0x_1 \rangle$ of three points lying in the following way on a ruling of $\mathbb{P}^1 \times \mathbb{P}^1$.



Then S/I has minimal resolution

$$S \longleftarrow \begin{matrix} S(-2,0) \\ \oplus \\ S(-1,-1) \\ \oplus \\ S(0,-2) \end{matrix} \xleftarrow{\begin{bmatrix} -y_0 & 0 \\ x_1 & -y_1 \\ 0 & x_0 \end{bmatrix}} \begin{matrix} S(-2,-1) \\ \oplus \\ S(-1,-2) \end{matrix} \longleftarrow 0.$$

The ideal $\mathrm{Fitt}_2(S/I)$ is generated by the entries of the matrix above, so $\mathrm{Fitt}_2(S/I) : B^\infty = S$.

However, there cannot be two homogenous polynomials of S whose intersection vanishes exactly at the above three points. For if f and g are homogeneous forms of degree (a,b) and (c,d) respectively, the multigraded version of Bézout's Theorem (see example 4.9 in [Sha77]) says that the intersection multiplicity of f and g is $ad+bc$. For concreteness, the leftmost vertical line is $V(x_0)$, the middle vertical line is $V(x_1)$, the topmost horizontal line is $V(y_0)$ and the middle horizontal line is $V(y_1)$. Then there are two cases for which $ad+bc=3$.

Case 1: One of the terms (ad or bc) is 3 and one is 0. Without loss of generality, assume $a=3$, $d=1$. If $c=0$, then $\deg(g)=(0,1)$, which cannot vanish at all three points. Hence, $b=0$ and $\deg(f)=(3,0)$. Plugging in $x_0=0$, $x_1=1$ gives a degree 0 form that must vanish at $[0:1]$ and $[1:0]$ which means f must vanish on all of $V(x_0)$. Similarly, plugging in $x_0=0$, $x_1=1$ to g yields a degree 1 form that also vanishes at these two points, implying g vanishes on all of $V(x_0)$. Therefore $V(f)\cap V(g)\neq X$.

Case 2: One of the terms is 2, and the other is equal to 1. It suffices to assume $a=2$ and $b=c=d=1$. Then both f and g restricted to $V(x_0)$ are degree one polynomials in y_0 and y_1 that vanish at two points so again $V(x_0)\subset V(f)\cap V(g)$. Therefore, no ideal generated by two elements will saturate to the ideal I. ◇

6. Saturated Fitting Ideals and Locally Free Sheaves

In the classical theory of Fitting ideals, a module is projective if and only if its first nonzero Fitting ideal is the whole ring. Sheafifying a projective module yields a locally free sheaf. The situation is similar for modules over the Cox ring and saturated Fitting ideals.

Proposition 6.1. *If M is a $\operatorname{Pic}(X)$-graded S-module, then $\operatorname{Fitt}_r(M):B^\infty=S$ and $\operatorname{Fitt}_{r-1}(M):B^\infty=0$ if and only if the sheaf $\widetilde{M}$ is locally free of constant rank r.*

Proof. First, if $\widetilde{M}$ is locally free of rank r, then for any B-saturated prime P, $M_P\cong\widetilde{M}_P$ is a free S_P-module of rank r. Therefore, M_P has the free presentation $0\longrightarrow S_P^r$, so $\operatorname{Fitt}_r(M_P)=S_P$ and $\operatorname{Fitt}_{r-1}(M_P)=0$. Since taking Fitting ideals commutes with localization, $\operatorname{Fitt}_{r-1}(M)=0$ and for every B-saturated prime P, $\operatorname{Fitt}_r(M)\not\subset P$. By definition,

$$\operatorname{codim}\operatorname{Fitt}_r(M)=\sup_{P\in\operatorname{Spec}S}\{\operatorname{codim}P:\operatorname{Fitt}_r(M)\subset P\}$$

is strictly greater than the codimension of any B-saturated prime (see Section 2). The maximal codimension of any B-saturated prime of S is $\dim X$ by Lemma 2.2. So again by Lemma 2.2, $\operatorname{Fitt}_r(M):B^\infty=S$ as desired.

Now suppose $\operatorname{Fitt}_r(M):B^\infty=S$ and $\operatorname{Fitt}_{r-1}(M):B^\infty=0$, and let P be a B-saturated prime ideal of S. By Proposition 5.2, there is an S-module N with $\widetilde{N}\cong\widetilde{M}$ such that N_P can be generated by r elements over S_P. Let $\varphi_P\colon G_P\to S_P^r$ be a free presentation of N_P. Then

$$I_1(\varphi_P)=\operatorname{Fitt}_{r-1}(N_P)\subset\operatorname{Fitt}_{r-1}(N_P):B^\infty=\operatorname{Fitt}_{r-1}(M_P):B^\infty=0.$$

Thus $\varphi_P=0$, which implies $N_P\cong S_P^r$. Since each the stalk at each point of the sheaf $\widetilde{N}$ is free of rank r, this means $\widetilde{N}$ is locally free of rank r, as desired. □

Notice that in everything up to this point, we have required that the complex be bounded. This may seem unsatisfying as Definition 1.2 does not require the length of a virtual resolution to be finite. The following result does not require this hypothesis, again mirroring the classical theory of Fitting ideals and exactness.

Proposition 6.2. *A complex of free S-modules $F \xrightarrow{\varphi} G \xrightarrow{\psi} H$ with $I(\varphi) : B^\infty = S = I(\psi) : B^\infty$ is virtual if and only if* $\operatorname{rank}(\varphi) + \operatorname{rank}(\psi) = \operatorname{rank}(G)$.

Proof. We have the right exact sequence

$$F \xrightarrow{\varphi} G \to \operatorname{coker}\varphi \to 0.$$

This is a free presentation of $\operatorname{coker}\varphi$. Letting $\operatorname{rank}(G) = n$ and $\operatorname{rank}(\varphi) = r$,

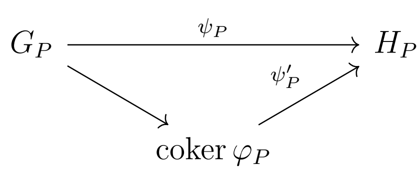
$$\operatorname{Fitt}_{n-r}(\operatorname{coker}\varphi) : B^\infty = I_r(\varphi) : B^\infty = S, \text{ and}$$

$$\operatorname{Fitt}_{n-r-1}(\operatorname{coker}\varphi) : B^\infty = I_{r+1}(\varphi) : B^\infty = 0.$$

By Proposition 6.1, $\widetilde{\operatorname{coker}\varphi}$ is a locally free sheaf of constant rank $n - r$. As localization is exact, localizing at a B-saturated prime P gives

$$F_P \to G_P \to \operatorname{coker}(\varphi_P) \to 0.$$

The rank of $\operatorname{coker}\varphi_P$ is equal to $\operatorname{rank}(G_P) - \operatorname{rank}(\varphi_P)$. Notice that $\operatorname{rank}(\varphi_P) = \operatorname{rank}(\varphi)$ since $I(\varphi)$ contains a nonzerodivisor. Now since $F_P \xrightarrow{\varphi_P} G_P \xrightarrow{\psi_P} H_P$ is a complex, ψ_P factors through $\operatorname{coker}\varphi_P$:



As $\operatorname{coker}\varphi_P$ is free of rank $n-r$, $\operatorname{rank}(\psi_P) = \operatorname{rank}(\psi'_P)$ and $I_j(\psi_P) = I_j(\psi'_P)$ for every $j \in \mathbb{Z}_{\geq 0}$.

Now $F_P \to G_P \to H_P$ is exact if and only $G_P \xrightarrow{0} \operatorname{coker}\varphi_P \xrightarrow{\psi'_P} H_P$ is exact in which case $\operatorname{rank}(\psi'_P) = \operatorname{rank}(\operatorname{coker}\varphi_P)$. Assembling the equalities shows

$$\operatorname{rank}(\varphi)+\operatorname{rank}(\psi) = \operatorname{rank}(\varphi_P)+\operatorname{rank}(\psi'_P) = \operatorname{rank}(\varphi_P)+\operatorname{rank}(G_P)-\operatorname{rank}(\varphi_P) = \operatorname{rank}(G). \quad \square$$

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