A LETTER: THE LOG-BRUNN-MINKOWSKI INEQUALITY FOR COMPLEX BODIES

LIRAN ROTEM

We will use the following terminology: A real body $K \subseteq \mathbb{R}^n$ is the unit ball of a norm $\|\cdot\|$ on $\mathbb{R}^n$, i.e. a convex, origin symmetric, compact set with non-empty interior. Similarly, a complex body $K \subseteq \mathbb{C}^n$ is the unit ball of a norm $\|\cdot\|$ on $\mathbb{C}^n$. By identifying $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ we see that every complex body is also a real body, but not vice versa. In fact, a complex body $K \subseteq \mathbb{C}^n$ is a real body which is also symmetric with respect to complex rotations, i.e. if $z \in K$ implies that $e^{i\theta}z \in K$ for all $\theta \in \mathbb{R}$.

For a real body $K$, the support function of $K$ is defined as $h_K(\theta) = \|\theta\|_K = \sup_{x \in K} \langle x, \theta \rangle$. Given two such bodies $K$ and $T$ and a number $0 \leq \lambda \leq 1$, we define the logarithmic mean of $K$ and $T$ by

$$L_\lambda(K, T) = \left\{ x \in \mathbb{R}^n : \langle x, \theta \rangle \leq h_K(\theta)^{1-\lambda}h_T(\theta)^\lambda \text{ for all } \theta \in \mathbb{R}^n \right\},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean inner product.

The log-Brunn-Minkowski inequality states that $|L_\lambda(K, T)| \geq |K|^{1-\lambda}|T|^\lambda$, where $|\cdot|$ denotes the (Lebesgue) volume. It was conjectured by Böröczky, Lutwak, Yang and Zhang ([2]), who proved it for complex bodies $K \subseteq \mathbb{C}^n$ is a real body which is also symmetric with respect to complex rotations, i.e. if $z \in K$ implies that $e^{i\theta}z \in K$ for all $\theta \in \mathbb{R}$.

For a real body $K$, the support function of $K$ is defined as $h_K(\theta) = \|\theta\|_K = \sup_{x \in K} \langle x, \theta \rangle$. Given two such bodies $K$ and $T$ and a number $0 \leq \lambda \leq 1$, we define the logarithmic mean of $K$ and $T$ by

$$L_\lambda(K, T) = \left\{ x \in \mathbb{R}^n : \langle x, \theta \rangle \leq h_K(\theta)^{1-\lambda}h_T(\theta)^\lambda \text{ for all } \theta \in \mathbb{R}^n \right\},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean inner product.

The log-Brunn-Minkowski inequality states that $|L_\lambda(K, T)| \geq |K|^{1-\lambda}|T|^\lambda$, where $|\cdot|$ denotes the (Lebesgue) volume. It was conjectured by Böröczky, Lutwak, Yang and Zhang ([2]), who proved it for $K, T \subseteq \mathbb{R}^2$.

Saroglou proved ([6]) that the inequality holds when $K$ and $T$ are $n$-dimensional real bodies which are unconditional with respect to the same basis.

The goal of this note is to explain why the log-Brunn-Minkowski inequality holds for complex bodies:

**Theorem 1.** For complex bodies $K, T \subseteq \mathbb{C}^n$ and $0 \leq \lambda \leq 1$ we have $|L_\lambda(K, T)| \geq |K|^{1-\lambda}|T|^\lambda$.

Theorem 1 will follow from a result of Cordero-Erausquin ([3]). In his work, Cordero-Erausquin proved a generalization of the Blaschke-Santaló inequality in the complex case. Specifically, he proved that for complex bodies $K, T \subseteq \mathbb{C}^n$ we have

$$|K \cap T| |K^\circ \cap T| \leq |B^2_{2n} \cap T|,$$

where $K^\circ$ is the polar body to $K$ and $B^2_{2n} \subseteq \mathbb{C}^n$ is the unit Euclidean ball. As a side note we remark that proving the same inequality for general real bodies is an open problem – see [4] for a partial result and a short discussion.

Cordero-Erausquin’s proved the inequality $(\ast)$ as a corollary of a general theorem about complex interpolation - see Theorem 3 below. The main point of this letter is the observation that the same general theorem also implies Theorem 1. This was apparently known to Cordero-Erausquin himself, but not to other researchers in the community who haven’t studied the complex case carefully. Theorem 1 may be a strong indication that the log-Brunn-Minkowski conjecture is true in general. Alternatively, it may indicate the existence of a rich theory of geometric inequalities in the complex case.

Let us briefly recall the definition of complex interpolation. We will give the construction for the finite-dimensional case, following the presentation of [3], and refer the reader to [1] for a more detailed account. Set $S = \{ z \in \mathbb{C} : 0 < \text{Re} \, z < 1 \}$, and define

$$\mathcal{F} = \left\{ f : S \to \mathbb{C}^n : f \text{ is bounded and continuous on } \overline{S} \text{ and analytic on } S \text{ such that } \lim_{t \to \pm \infty} f(it) = \lim_{t \to \pm \infty} f(1+it) = 0 \right\}.$$

Given two norms $\|\cdot\|_0$ and $\|\cdot\|_1$ on $\mathbb{C}^n$, we define a norm on $\mathcal{F}$ by

$$\|f\|_{\mathcal{F}} = \max \left\{ \sup_{t \in \mathbb{R}} \|f(it)\|_0, \sup_{t \in \mathbb{R}} \|f(1+it)\|_1 \right\}.$$
Finally, for \( \lambda \in [0, 1] \), we define the interpolated norm \( \|\cdot\|_\lambda \) by
\[
\|x\|_\lambda = \inf \{ \|f\|_F : f \in F, f(\lambda) = x \}.
\]
It is not hard to see that for \( \lambda = 0, 1 \) we recover the original norms \( \|\cdot\|_0, \|\cdot\|_1 \). The only other result we will need from the standard theory of complex interpolation is the following:

**Proposition 2.** Let \( \|\cdot\|_0, \|\cdot\|_1 \) be norms on \( \mathbb{C}^n \) and let \( \|\cdot\|_\lambda \) be the interpolated norms. Then
\[
\|z\|_\lambda \leq (\|z\|_0^{1-\lambda} (\|z\|_1^\lambda)
\]
for every \( z \in \mathbb{C}^n \).

This inequality, with its simple proof, may be found for example in [5] as equation (7.26)*.

If \( K \) is the unit ball of \( \|\cdot\|_0 \) and \( T \) is the unit ball of \( \|\cdot\|_1 \) we will write \( C_\lambda(K, T) \) for the unit ball of \( \|\cdot\|_\lambda \). Proposition 2 implies that \( h_{C_\lambda(K, T)}(z) \leq h_K(z)^{1-\lambda} h_T(z)^\lambda \) for all \( z \in \mathbb{C}^n \), and hence \( C_\lambda(K, T) \subseteq L_\lambda(K, T) \).

The main theorem of [3] is the following:

**Theorem 3.** The function \( \lambda \mapsto |C_\lambda(K, T)| \) is log-concave on \([0, 1]\).

It is now easy to deduce Theorem 1, as
\[
|L_\lambda(K, T)| \geq |C_\lambda(K, T)| \geq |C_0(K, T)|^{1-\lambda} \cdot |C_1(K, T)|^\lambda = |K|^{1-\lambda} |T|^\lambda.
\]

**References**


