Mixed integrals and related inequalities

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Our point of departure will be Minkowski’s theorem on mixed volumes:

**Theorem 1** (Minkowski). Fix bodies $K_1, K_2, \ldots, K_m \in \mathcal{K}^n_c$. Then the function $F : (\mathbb{R}^+)^m \to [0, \infty)$, defined by

$$F(\lambda_1, \lambda_2, \ldots, \lambda_m) = \text{Vol} \left( \lambda_1 K_1 + \lambda_2 K_2 + \cdots + \lambda_m K_m \right),$$

is a homogeneous polynomial of degree $n$, with non-negative coefficients.

Here $\mathcal{K}^n_c$ is the family of compact and convex bodies in $\mathbb{R}^n$, and the addition operation $+$ is Minkowski addition, $A + B = \{a + b : a \in A, b \in B\}$.

By standard linear algebra, Minkowski’s theorem is equivalent to the existence of a map $V : (\mathcal{K}^n_c)^n \to [0, \infty)$ which is multilinear, symmetric and which satisfies $V(K, K, \ldots, K) = \text{Vol}(K)$. This map is unique, and the number $V(K_1, K_2, \ldots, K_n)$ is known as the mixed volume of $K_1, \ldots, K_n$.

Our goal is to extend Minkowski’s theorem to a functional setting. That is, we want to take $n$ functions $f_1, f_2, \ldots, f_n : \mathbb{R}^n \to [0, \infty)$ and define their “mixed volume” $V(f_1, f_2, \ldots, f_n)$. In order to do so we need to choose an appropriate family of functions, a “volume” functional on this family, and an addition operation.

For the family of functions, we choose the class of quasi-concave functions. A function $f : \mathbb{R}^n \to [0, \infty)$ is called quasi-concave if for every $x, y \in \mathbb{R}^n$ and every $0 < \lambda < 1$ we have

$$f(\lambda x + (1 - \lambda)y) \geq \min \{f(x), f(y)\}.$$ 

While not always necessary, it is very convenient to assume further that $f$ is upper semicontinuous, that $\max f = f(0) = 1$ and that $f(x) \to 0$ as $|x| \to 0$. Denote this set of functions by $\text{QC} (\mathbb{R}^n)$.

As a volume, we choose the Lebesgue integral, i.e.

$$\text{Vol}(f) = \int_{\mathbb{R}^n} f(x) dx.$$ 

Finally, for addition, we define a new addition on quasi-concave functions by

$$(f \oplus g)(x) = \sup_{y \in \mathbb{R}^n} \min \{f(y), g(x - y)\}.$$ 

We further define the product $\lambda \circ f$ for $f \in \text{QC} (\mathbb{R}^n)$ and $\lambda > 0$ by $(\lambda \circ f)(x) = f\left(\frac{x}{\lambda}\right)$. We briefly comment that these operations emerge as a limit of the natural addition operations on $\alpha$-concave functions. An explanation of this statement appears in [2] and [4].

Under the above definition, we have to following theorem:
Theorem 2. Fix \( f_1, f_2, \ldots, f_m \in \text{QC}(\mathbb{R}^n) \). Then the function \( F : (\mathbb{R}^+)^m \to [0, \infty] \), defined by

\[
F(\lambda_1, \lambda_2, \ldots, \lambda_m) = \int [(\lambda_1 \circ f_1) \oplus (\lambda_2 \circ f_2) \oplus \cdots \oplus (\lambda_m \circ f_m)]
\]

is a homogeneous polynomial of degree \( n \), with non-negative coefficients.

The proof of this result appears in [3]. As usual, this is equivalent to the existence of a multilinear, symmetric map \( V : \text{QC}(\mathbb{R}^n)^m \to [0, \infty] \) which satisfies \( V(f, f, \ldots, f) = \int f \). The number \( V(f_1, f_2, \ldots, f_m) \) will be called the mixed integral of \( f_1, f_2, \ldots, f_m \). The following theorem summarizes some of the basic properties of mixed integrals:

**Theorem 3.**

1. \( V(K_1, K_2, \ldots, K_n) = V(1_{K_1}, 1_{K_2}, \ldots, 1_{K_n}) \).
2. If \( f_i \geq g_i \), for all \( i \), then \( V(f_1, f_2, \ldots, f_n) \geq V(g_1, g_2, \ldots, g_n) \).
3. \( V \) is rotation and translation invariant.
4. Fix \( g_{m+1}, \ldots, g_n \in \text{QC}(\mathbb{R}^n) \), and define

\[
\Phi(f) = V(f[m], g_{m+1}, \ldots, g_n).
\]

\( \Phi \) satisfies a valuation type property: if \( f_1, f_2 \in \text{QC}(\mathbb{R}^n) \) and \( f_1 \lor f_2 = \max(f_1, f_2) \in \text{QC}(\mathbb{R}^n) \) as well, then

\[
\Phi(f_1 \lor f_2) + \Phi(f_1 \land f_2) = \Phi(f_1) + \Phi(f_2).
\]

Once we have a generalization of the notion of mixed volumes, it makes sense to try and generalize the important inequalities as well. For example, for \( f \in \text{QC}(\mathbb{R}^n) \) define its \( k \)-th quermassintegral to be

\[
W_k(f) = V(\underbrace{f, f, \ldots, f}_{n-k \text{ times}}, \underbrace{1_D, 1_D, \ldots, 1_D}_{k \text{ times}}),
\]

where \( D \) is the unit Euclidean ball. This notion of functional quermassintegrals was discovered independently by Bobkov, Colesanti and Fragalà ([1]). In particular, we have the notion of surface area, defined by \( S(f) = nW_1(f) \).

We now want to prove a functional isoperimetric inequality. Unfortunately, it turns out that for general quasi-concave functions it is impossible to give a lower bound for \( S(f) \) in terms of \( \int f \). Surprisingly, however, it is possible to state a functional extension of the isoperimetric inequality:

**Theorem 4.** For every \( f \in \text{QC}(\mathbb{R}^n) \) we have \( S(f) \geq S(f^*) \), where \( f^* \) is the symmetric decreasing rearrangement of \( f \).

Plugging in \( f = 1_{K_1} \), we see that this theorem really generalizes the isoperimetric inequality.

Using a slightly more complicated notion of a “generalized rearrangement”, it is possible to prove functional versions of most of the classic inequalities: Brunn-Minkowski (and its extension to mixed volumes), Alexandrov-Fenchel, and others. As a special case, we have the following extension of Theorem 4:

**Theorem 5.** For every \( f_1, f_2, \ldots, f_n \in \text{QC}(\mathbb{R}^n) \) we have \( V(f_1, f_2, \ldots, f_n) \geq V(f_1^*, f_2^*, \ldots, f_n^*) \).
For indicator functions, this reduces to the known statement that for every convex bodies $K_1, K_2, \ldots, K_n$ in $\mathbb{R}^n$ we have

$$V(K_1, K_2, \ldots, K_n) \geq \left( \prod_{i=1}^{n} \text{Vol}(K_i) \right)^{\frac{1}{n}}.$$

Finally, if one is willing to restrict the class of functions, it is possible to prove certain inequalities in a more familiar form. For example, in the class of geometric log-concave functions we have the following Alexandrov type inequalities:

**Theorem 6.** Define $g(x) = e^{-|x|}$. For every geometric log-concave function $f$ and every integers $0 \leq k < m < n$ we have

$$\left( \frac{W_k(f)}{W_k(g)} \right)^{\frac{1}{n-k}} \leq \left( \frac{W_m(f)}{W_m(g)} \right)^{\frac{1}{n-m}},$$

with equality if and only if $f(x) = e^{-c|x|}$ for some $c > 0$.

**References**


