Solving binary $\mathcal{MQ}$ with Grover's algorithm

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You can be a Quantum cryptanalysist too

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Convential wisdom

With stable $\sim$4000 qubit quantum computer:

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2. Bits-of-security of all symmetric crypto is halved by Grover's algorithm.
   Example: pre-image of SHA1 takes 2^{80} time instead of 2^{160}.

This is almost true.

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This is completely wrong:

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This is completely wrong:
137 qubits are enough!

This talk

1. Binary $\mathcal{MQ}$
2. Grover's algorithm
3. Writing an oracle

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Binary $\mathcal{MQ}$

Underling hard problem of Gui

Find $x_i = 0, 1$ such that

$$x_1x_3 + x_1x_2 \text{ is even}$$
$$x_1x_2 + x_2x_3 + x_3x_4 \text{ is odd}$$
$$x_1 + x_1x_3 + x_2x_3 \text{ is even}$$

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Here $n = 4$ variables and $m = 3$ equations.

Binary $\mathcal{MQ}$

Underling hard problem of Gui

Find $x_i \in \mathbb{F}_2$ such that

$$x_1x_3 + x_1x_2 = 0$$
$$x_1x_2 + x_2x_3 + x_3x_4 = 1$$
$$x_1 + x_1x_3 + x_2x_3 = 0$$

Here $n = 4$ variables and $m = 3$ equations.

Binary $\mathcal{MQ}$

Underling hard problem of Gui

Find $x_i \in \mathbb{F}_2$ such that

$$x_1x_3 + x_1x_2 + 1 = 1$$
$$x_1x_2 + x_2x_3 + x_3x_4 = 1$$
$$x_1 + x_1x_3 + x_2x_3 + 1 = 1$$

Here $n = 4$ variables and $m = 3$ equations.

Binary $\mathcal{MQ}$

Underling hard problem of Gui

Find $x_i \in \mathbb{F}_2$ such that

$$\begin{aligned} x_1x_3 + x_1x_2 + x_5 &= 1 \\ x_1x_2 + x_2x_3 + x_3x_4 &= 1 \\ x_1 + x_1x_3 + x_2x_3 + x_5 &= 1 \\ x_5 &= 1 \end{aligned}$$

Here $n = 5$ variables and $m = 4$ equations.

Binary $\mathcal{MQ}$

Underling hard problem of Gui

Find $x_i \in \mathbb{F}_2$ such that

$$\begin{aligned} x_1(x_3 + x_2) + x_5 &= 1 \\ x_1x_2 + x_2x_3 + x_3x_4 &= 1 \\ x_1(1 + x_3) + x_2x_3 + x_5 &= 1 \\ x_5 &= 1 \end{aligned}$$

Here $n = 5$ variables and $m = 4$ equations.

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Single needle problem

Given *any* $f : \mathbb{F}_2^n \to \mathbb{F}_2$ *(by a Boolean circuit) which is valued* 0 *everywhere except for one* w_1*,*
find *this* w_1 *with* $f(w_1) = 1$*.*

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Classically: expected 2^{n-1} calls to f.

Quantum (Grover): if f can be given* as a quantum circuit then it takes $2^{\frac{n}{2}}$ calls to f.

Grover teaser

```
-- |Reflects (in place) over the standard uniform superposition of qubits.
-- This is used as the second part of the Grover iteration.
reflect_over_a :: [Qubit] -> Circ ()
reflect_over_a qs = do
  with_basis_change (forM_ qs hadamard_at) $ do
    with_basis_change (forM_ qs qnot_at) $ do
      with_basis_change (hadamard_at $ head qs) $ do
        qnot_at (head qs) `controlled` tail qs

-- |Grover's algorithm.  'oracle' is a circuit that should map exactly
-- 'm' 'n'-qubit-words to |1> and the others to |0>.  'grover' returns
-- a circuit that creates a superposition over all 'n'-qubit words that
-- are mapped to |1> by the oracle.
grover :: ([Qubit] -> Circ (Qubit)) -> Int -> Int -> Circ ()
grover oracle n m = do
  -- Create a standard uniform superposition over all qubits.
  qs <- qinit $ replicate n False
  forM_ qs hadamard_at
  iterations qs
  return ()
    where
      n_iters = floor $ pi / 4 / asin(sqrt((fromIntegral m) / (2^n)))
      iterations :: [Qubit] -> Circ ([Qubit])
      iterations = nbox "grover-iteration" n_iters $ \qs -> do
        -- First, the adapted oracle.  The following will send a computational
        -- basis vector w to -w if its tagged by the original oracle and
        -- leave it in place otherwise.
        with_computed (oracle qs) $ \r -> do
          gate_Z_at (head qs) `controlled` r
        -- Then, reflect over the standard uniform superposition.
        reflect_over_a qs
        return qs
```

Quantum circuits

No time to define, suffice to know

1. Every invertible Boolean circuit is a quantum circuit

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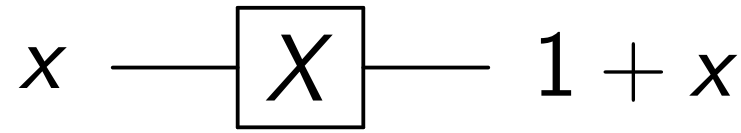
Because of the invertibility requirement,

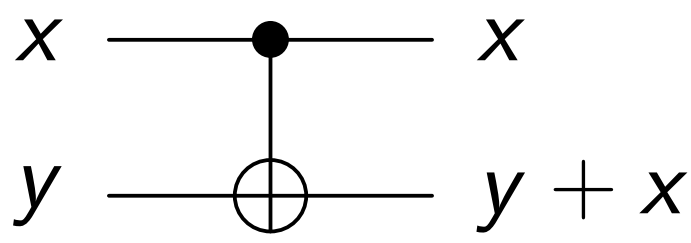
some things are way slower on a quantum computer than on a classical computer!

Gates (1): NOT gate



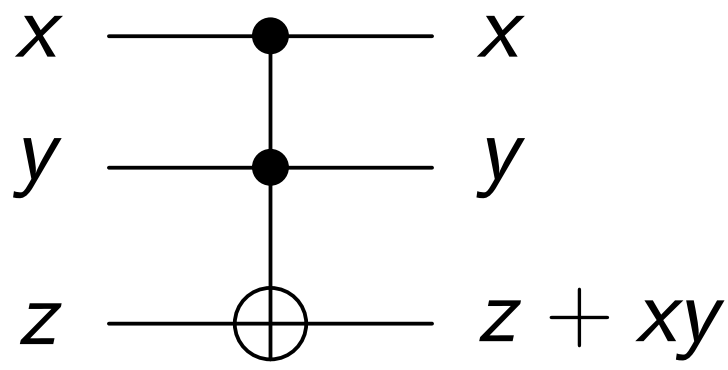
Quantum people like to call this the X-gate to confuse you.

Gates (2): CNOT gate



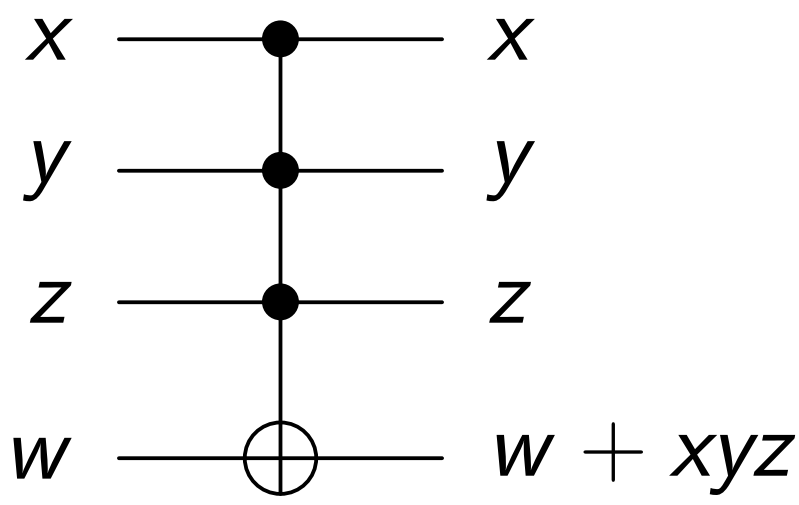
This is the invertible version of the XOR-gate.

Gates (3): Tofolli gate



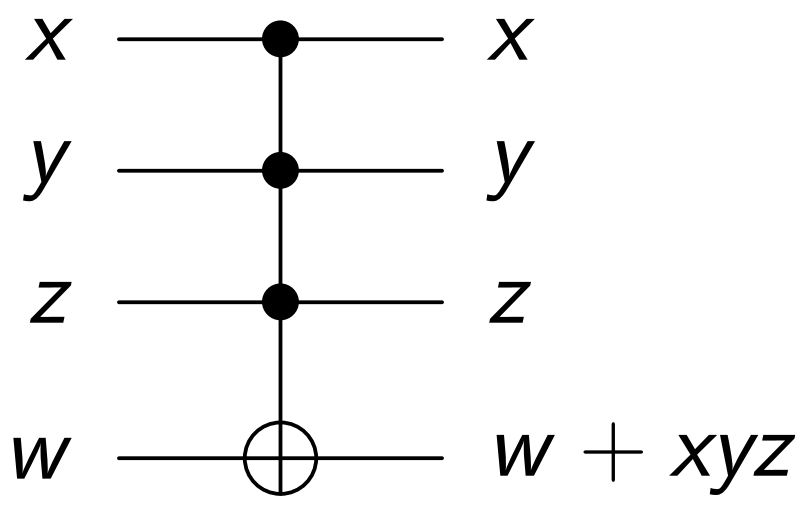
This is the invertible version of the AND-gate.

Gates (4): bigger Tofolli gates



This is for bigger AND-gates.

Gates (4): bigger Tofolli gates



This is for bigger AND-gates.

Clasically big AND-gates are easy. For quantum computers it's more costly.

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Running example

$$x_1(1 + x_2 + x_3) + x_2x_3 = 1$$
$$x_2(1 + x_3) = 1$$

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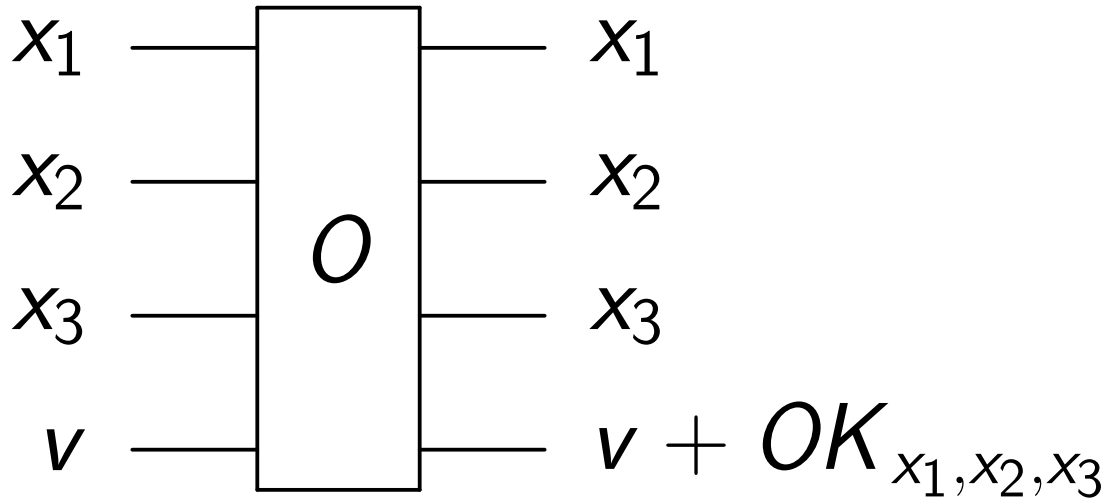
we must construct a circuit O with



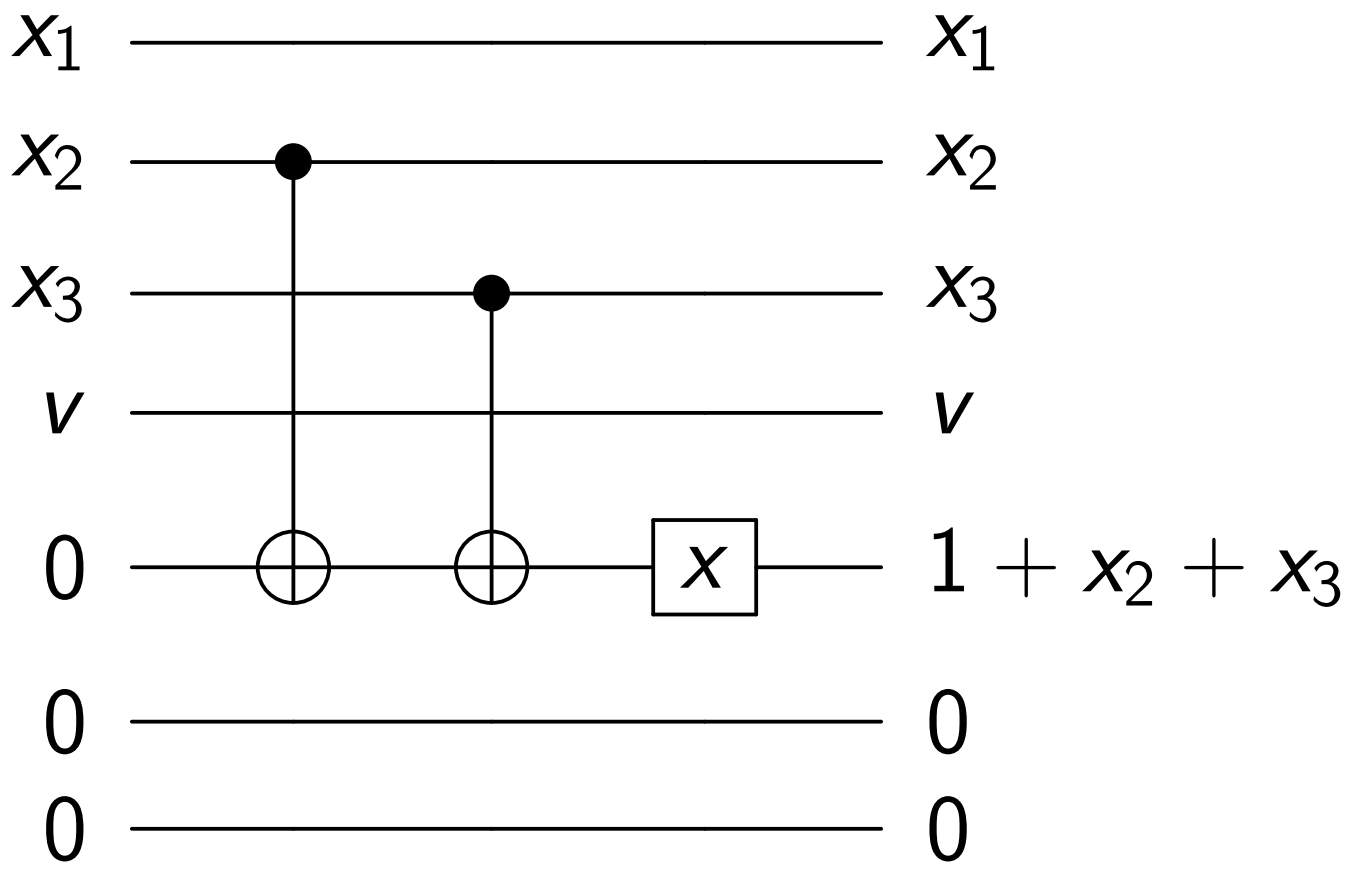
Building the circuit (1)

We will use 3 extra qubits:

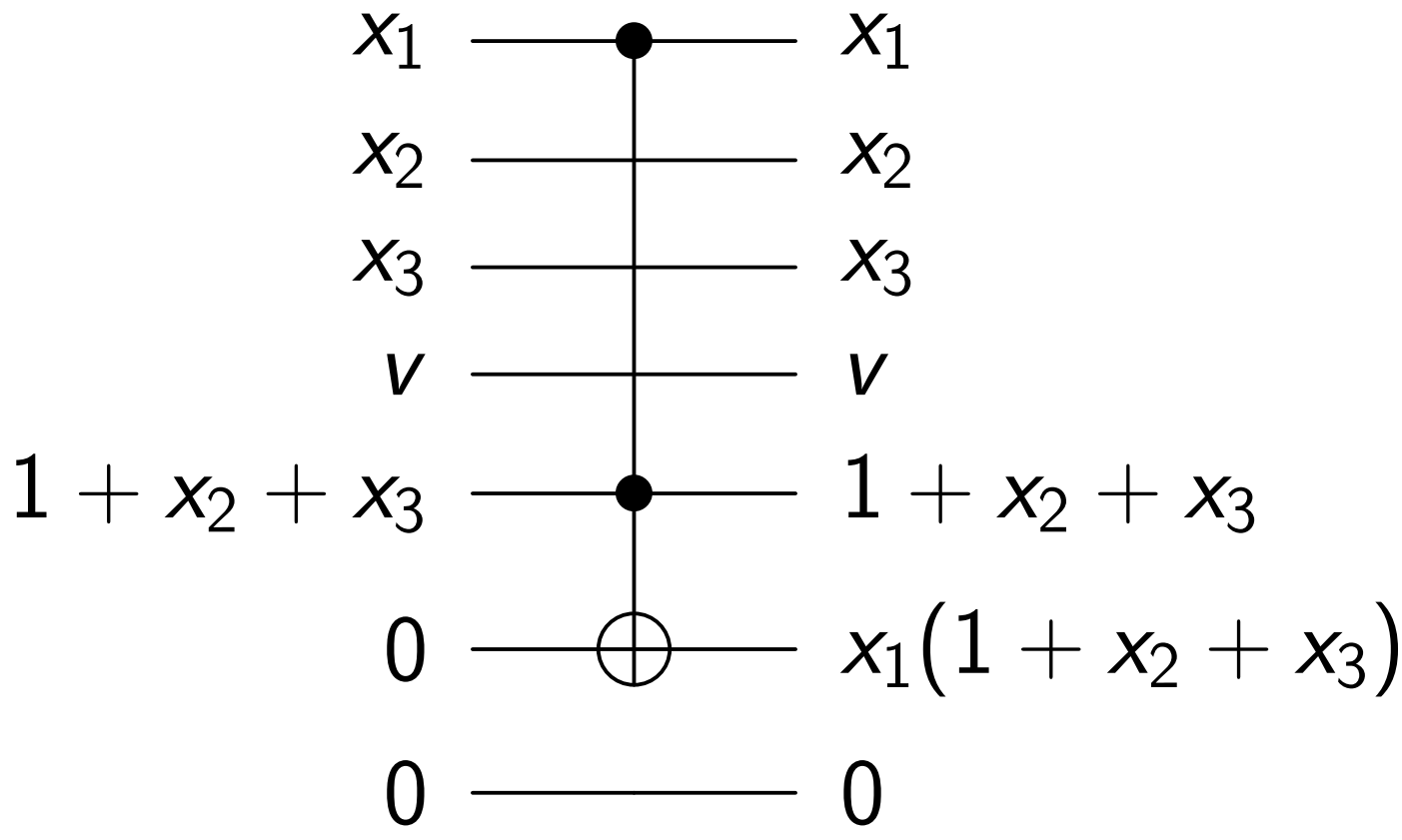
1. One for each equation to store its value.
2. One temporary.

So $n + m + 1$ total in general.

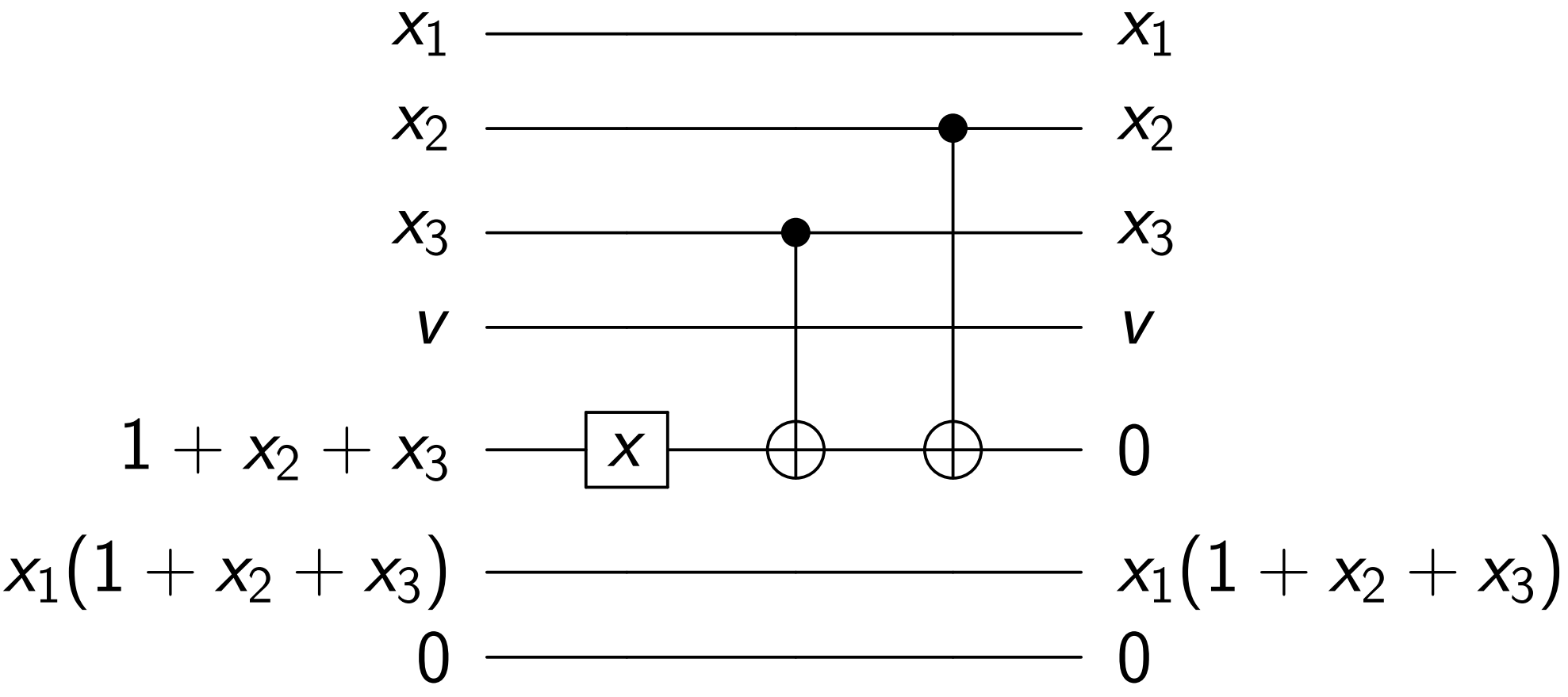
Building the circuit (2)



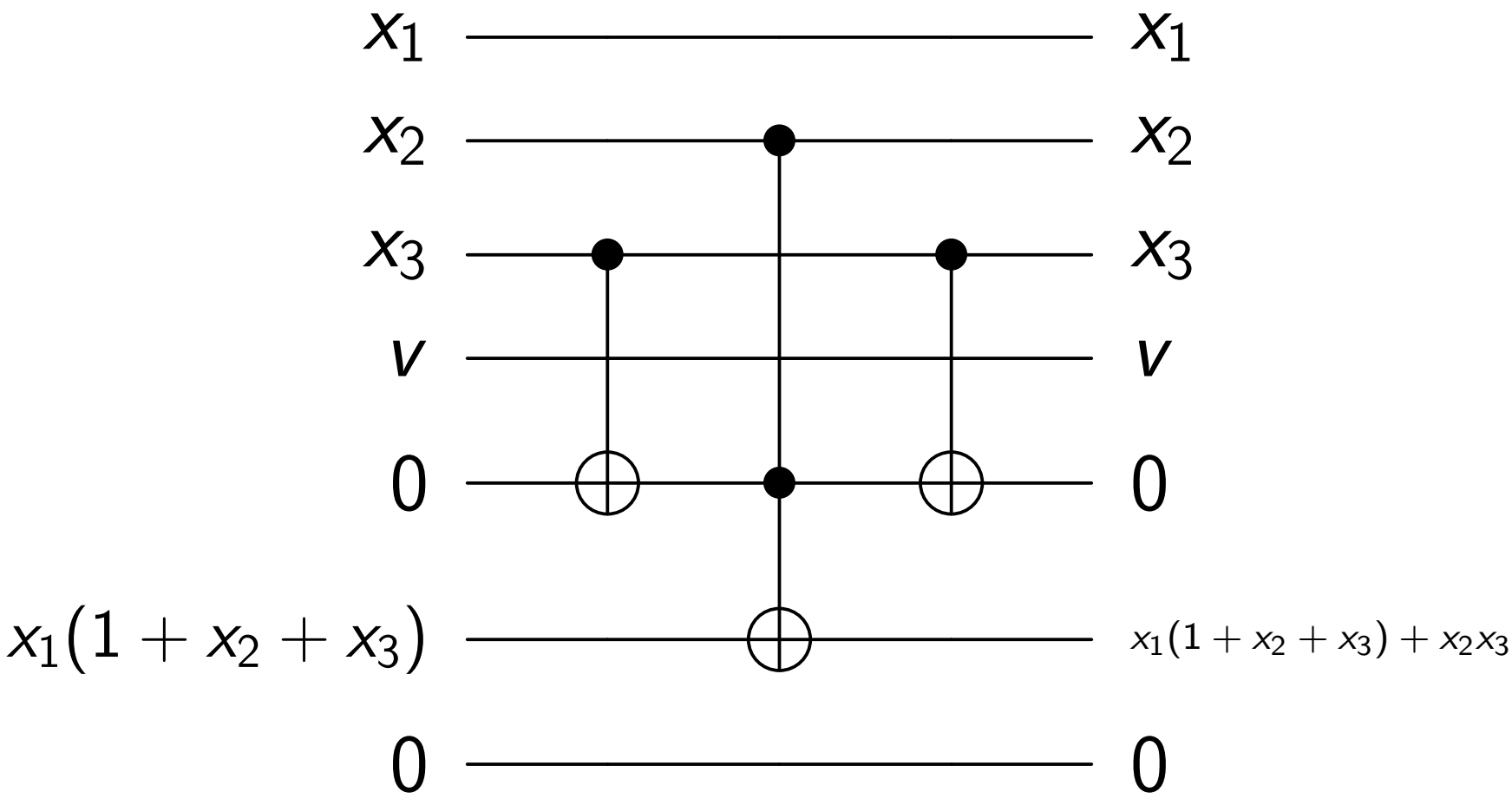
Building the circuit (3)



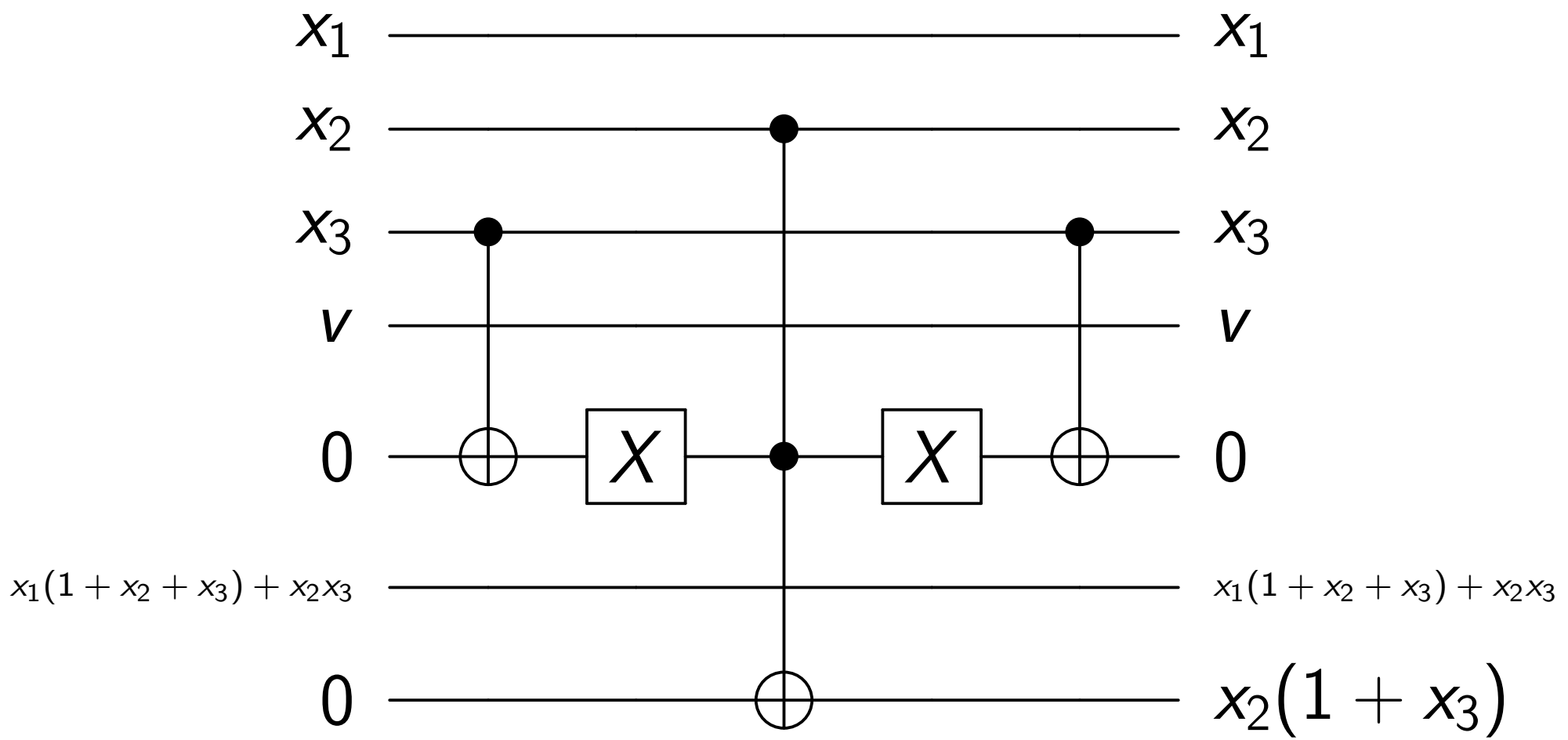
Building the circuit (4)



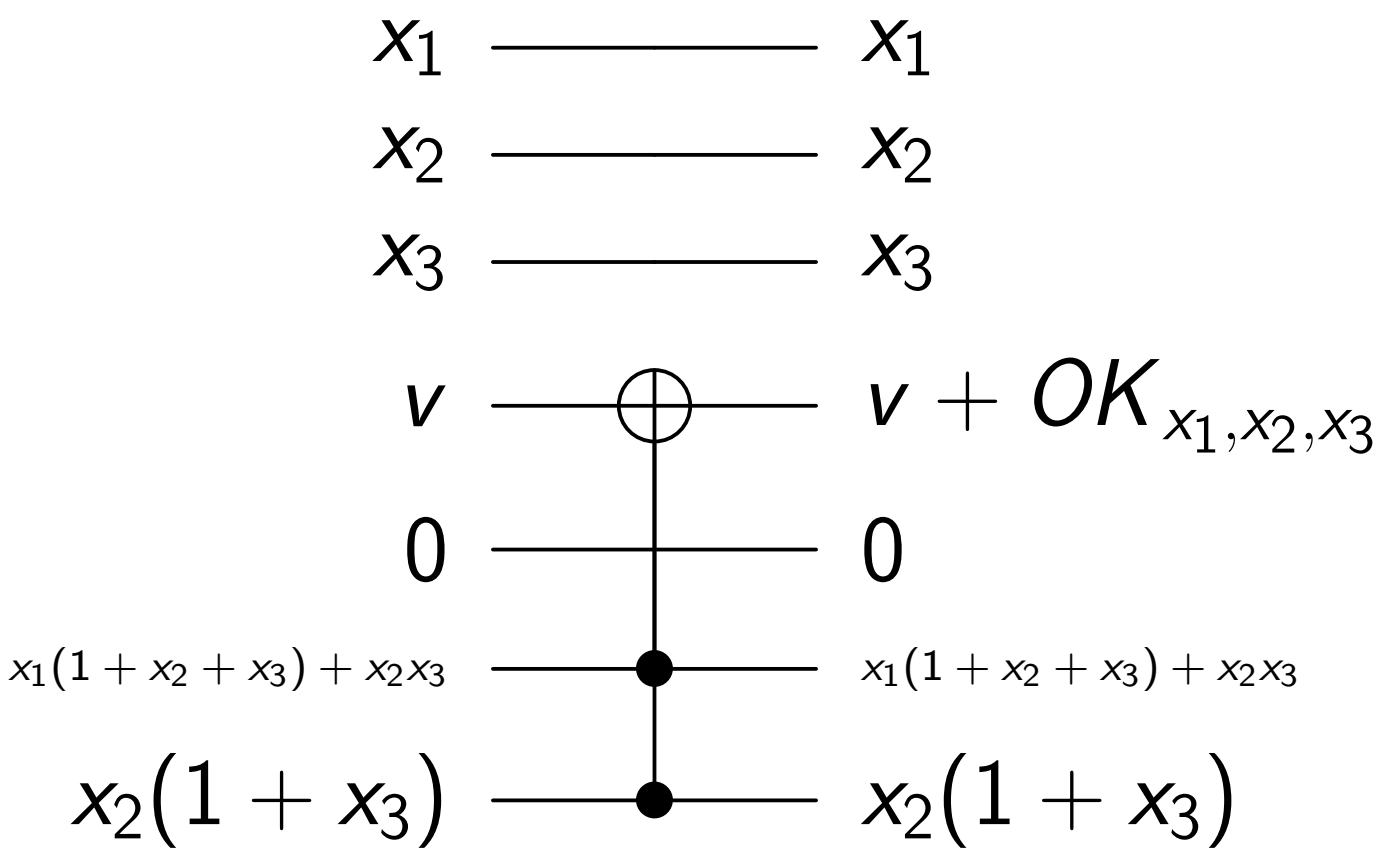
Building the circuit (5)



Building the circuit (6)



Building the circuit (7)



```haskell
-- Compute y_i^(k) from the coefficients lambda_ii^(k), ..., lambda_in^(k)
-- and the assignment x_1, ..., x_n.
compute_y :: [Bool] -> [Qubit] -> Circ (Qubit)
compute_y cs xs = withM (qinit False) $ \t ->
  unless (null cs) $ do
    when (head cs) $ qnot_at t
    zipWithM_ (\c x ->  when c (qnot_at t `controlled` x)) (tail cs) (tail xs)

-- Computes E^(k) from the "triangle" lambda^(k)_ij (1 <= j <= n)
-- and the assignment x_1, ..., x_n.
compute_e :: [Qubit] -> [[Bool]] -> Circ (Qubit)
compute_e xs css =
  withM (qinit False) $ \e ->
    forM_ (zip css (init $ tails xs))  $ \(cs, xs') ->
      with_computed (compute_y cs xs') $ \t ->
        qnot_at e `controlled` (t, head xs')

-- The first (straight-forward) oracle.  Computes whether the
-- assignment x_1, ..., x_n satisfies the given system of equations.
oracle1 :: [[[Bool]]] -> [Qubit] -> Circ (Qubit)
oracle1 csss xs =
  withM (qinit False) $ \r -> do
    label (r:xs) ("r":["x" ++ (show i) | i <- [1..length xs]])
    es <- mapM (compute_e xs) csss
    label es ["E" ++ (show i) | i <- [1..length es]]
    qnot_at r `controlled` es
```

Breaking $b\mathcal{MQ}$ $n = 80$ $m = 84$

	first
qubits	168
time	

Breaking $b\mathcal{MQ}$ $n = 80$ $m = 84$

	first	second
qubits	168	90
time		

Breaking $b\mathcal{MQ}$ $n = 80$ $m = 84$

	first	second
qubits	168	90
time	1,430,025,554,865,881,938	2,861,116,040,048,158,450

Breaking $b\mathcal{MQ}$ $n = 80$ $m = 84$

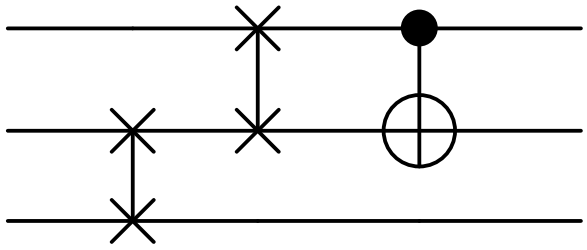
	first	second
qubits	168	90
time	2^{60}	2^{61}

Breaking $b\mathcal{MQ}$ $n = 80$ $m = 84$

	first	second	(third)
qubits	168	90	6484
serial time	2^{60}	2^{61}	2^{60}
parallel time	2^{60}	2^{61}	$\sim 2^{50}$

Effecient quantum counter

Create efficient counter from primitive polynomial over $\mathbb{F}_2$. E.g. $x^3 + x + 1$:



$$|111\rangle \mapsto |101\rangle \mapsto |100\rangle \mapsto |010\rangle$$
$$\mapsto |001\rangle \mapsto |110\rangle \mapsto |011\rangle \mapsto |111\rangle$$

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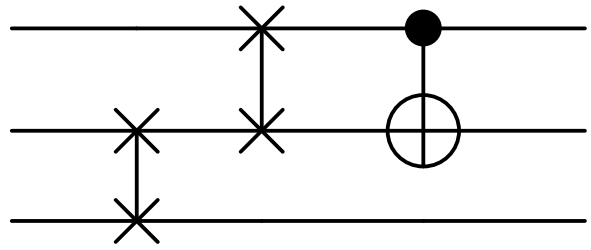

$$|111\rangle \mapsto |101\rangle \mapsto |100\rangle \mapsto |010\rangle$$
$$\mapsto |001\rangle \mapsto |110\rangle \mapsto |011\rangle \mapsto |111\rangle$$

```
oracle2 :: [[[Bool]]] -> [Qubit] -> Circ (Qubit)
oracle2 csss xs = do
  ctr <- init_counter $ length csss
  label xs ["x" ++ (show i) | i <- [1..length xs]]
  forM_ csss $ \css ->
    with_computed (compute_e xs css) $ controlled $ inc_counter ctr
  check_counter ctr
```

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4. Writing efficient oracles underrated?