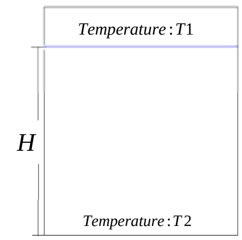
The Lorenz Equations

$$\frac{dx}{dt} = \sigma(-x+y)$$
$$\frac{dy}{dt} = rx - y - xz$$
$$\frac{dz}{dt} = -bz + xy$$

x is proportional to the intensity of convection motion.
y is proportional to the temperature difference between the ascending and descending currents.
z is proportional to the distortion of the vertical temperature profile from linearity.

The Lorenz attractor arises in a simplified system of equations describing the two-dimensional flow of fluid with uniform depth and imposed temperature difference between the upper and lower surfaces.



$$\Delta T = T1 - T2 = \text{constant}$$

The meaning of parameters in the Lorenz equations

- ***sigma*** is called the **Prandtl Number**: the ratio of momentum diffusivity (Kinematic viscosity) and thermal diffusivity.
- *r* is called the **Rayleigh Number**: determine whether the heat transfer is primarily in the form of conduction or convection.
- *b* is a geometric factor.
- A typical value of the three parameters

$$\sigma = 10, b = 8/3, r = 28$$

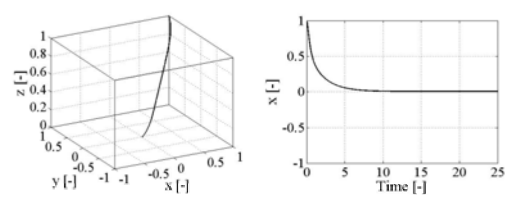
The important role of parameter r in the chaotic behavior of the Lorenz Equations

Fix sigma=10, b=8/3
When 0< r <1, the origin is globally stable.
When r >1, the origin is non-stable.
When 1< r <24.74,

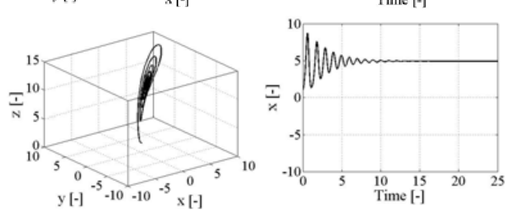
$$c_1 = (\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1)$$
$$c_2 = (-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1)$$

will be stable.
When r >24.74, the above two points c_1, c_2 will become unstable, as the well-known **Lorenz attractor**.

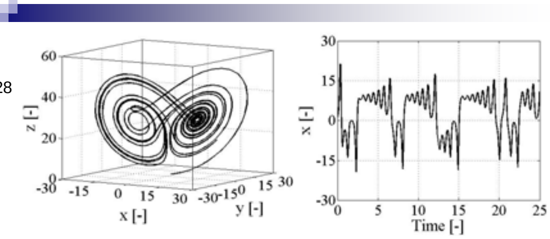
r=0.5



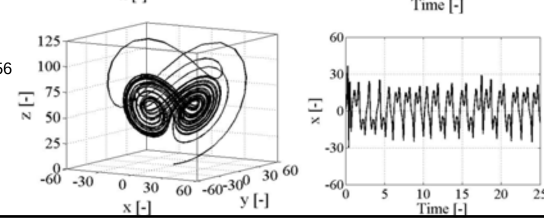
r=10



r=28



r=56



Assume the Lorenz equations is the weather we want to predict.

Question: How confident we are???
Answer: Certainly NOT 100%

Reasons: Because the initial condition and model parameters are imperfectly known by the predictors.

Solution: by construction probabilistic models of dynamical systems, one of them is the *parametric probabilistic approach*.

Parametric probabilistic *models* incorporate uncertainty by modeling certain parameters and initial condition of a prediction model by random variables. In this case, the output of the probabilistic model will also be random variables. **The mean values** of the output random variables is often interpreted as the best estimates, while **the standard deviations** can be viewed as a measure of the uncertainty in the prediction.

A parametric probabilistic model of the Lorenz equations.

$$\frac{dx}{dt}(t,a) = \sigma(a)(-x(t,a) + y(t,a))$$
$$\frac{dy}{dt}(t,a) = r(a)x(t,a) - y(t,a) - x(t,a)z(t,a)$$
$$\frac{dz}{dt}(t,a) = -b(a)z(t,a) + x(t,a)y(t,a)$$

A fundamental problem (in the practical construction):
The choice of the probability density functions of the random variables.

For convenience of computation, we simply choose uniform distribution.

Computation with the probabilistic model

Fix sigma=10, r=28, and b=8/3.
Step 1: choose a time step Δt and a number n_T of total time steps. choose a number n_s of Monte Carlo samples.

Step 2: simulate a set $\{x_s^0, y_s^0, z_s^0 \,|\, 1 \le s \le n_s\}$ of independent and identically distributed samples of random variables x^0, y^0, z^0.

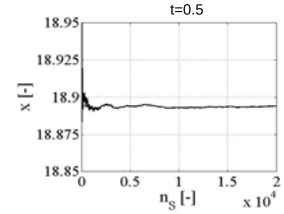
Step 3: for each $1 \le s \le n_s$, solve the deterministic Lorenz equations with the initial condition x_s^0, y_s^0, z_s^0, using first-order Euler time stepping algorithm

Step 4: statistical estimation of quantities, like mean value and standard deviation, etc.

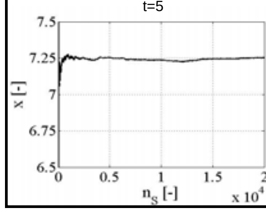
Now, choose $\Delta t = 0.01$ and $n_T = 2500$.

Statistical mean of the samples of x(t,a)
At t =0.5, t=5 and t = 25
as a function of the number of Monte Carlo samples

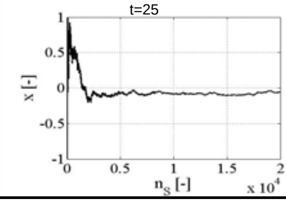
t=0.5



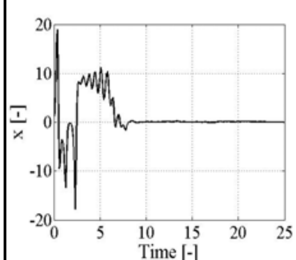
t=5



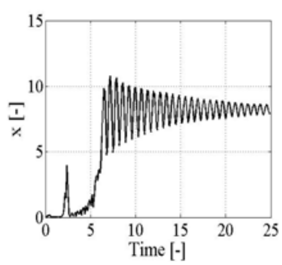
t=25



Mean and standard deviation of x(t,a) as function of time.
All results to follow have been obtained with $\mathrm{n}_s = 10{,}000$



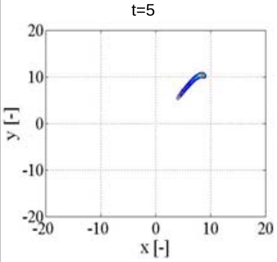
Mean



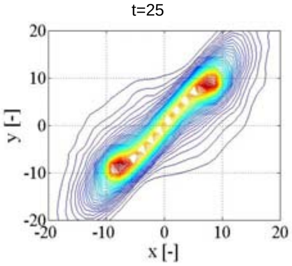
Standard deviation

Contours of the joint probability density function of x(t,a) and y(t,a)

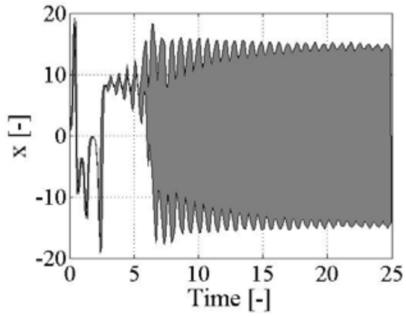
t=5



t=25



This figure shows the large sensitivity of the response of the Lorenz system to small uncertainties in the initial conditions.



95%-confidence regions for x(t,a) as a function of time

Fix sigma=10, b=8/3 as before, but let r be a random and time-depend quantity, $r(t,a) = r_0 + s(a)(t - t_0)$ with $r_0 = 28, t_0 = 0$

$s(a)$is a uniform random variable taking values in the interval [0, (56-28)/2500].

From now on, we try to predicting the absolute value of x. Let $D(t,a) = |x(t,a)|$ and decompose it as

$$D(t,a) = \overline{D}(t,a) + \widehat{D}(t,a)$$

where:

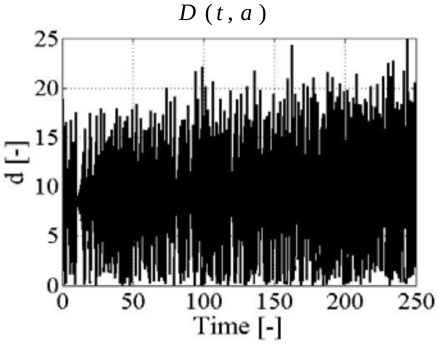
$$\overline{D}(t,a) = \frac{1}{2h}\int_{t-h}^{t+h} D(s,a)\,ds$$

$\overline{D}(t,a)$ a slowly varying moving-average component.

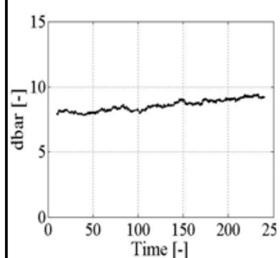
$\widehat{D}(t,a)$ is a fast varying component.

Here the total time steps $n_T = 25,000$ not 2500 as we chosen before

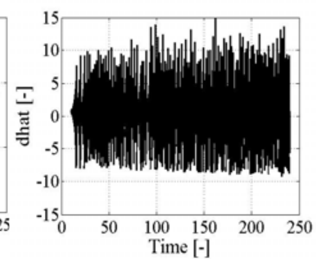
$D(t,a)$



$\overline{D}(t,a)$

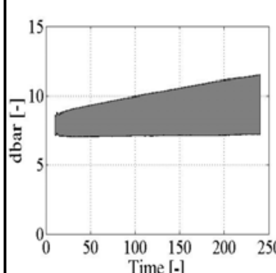


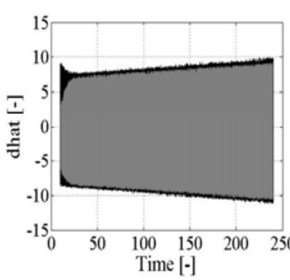
$\widehat{D}(t,a)$



We choose $h = 10$ here.

95%-confidence regions for $\overline{D}(t,a)$ and $\widehat{D}(t,a)$ as a function of time.





We can observe $\overline{D}(t,a)$ is much less sensitive to the uncertainty introduced in the model than $\widehat{D}(t,a)$, suggesting the *slow component* of the response can be predicted with good accuracy, whereas the *fast component* cannot.

From a climate change perspective, we may consider r as the CO2 concentration in the atmosphere: we know that it is increasing, but we don't know by how much. And its increasing rate is highly dependent on the decision we made in future. So when we try to predict weather or climate a random variable may be a better choice.

Furthermore we may view the *slow component* as the climate response and the *fast component* as the weather response. The results obtained above Illustrate that accurate long-term predictions may be feasible for the climate, even when they are not for the weather.

Predict regime changes in Lorenz model: result from RISE (Research Internships in Science and Engineering) program at University of Maryland, summer 2002

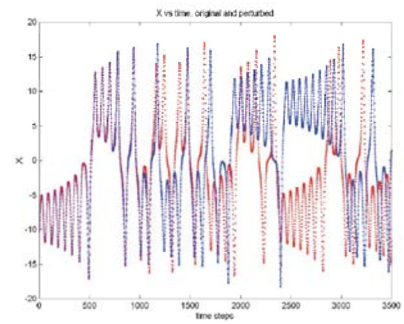
Imaging that you are a forecaster living the Lorenz attractor. We know there are two different weather regimes, which we could denote as ‘warm’ and ‘cold’.

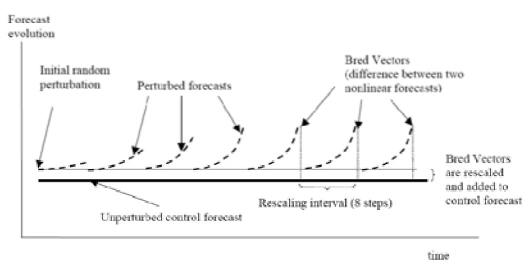
Question: can you develop simple forecasting rules to predict when changes in regime will happen and how long will they last?

A simple method: *Bred Vector*

Definition: *Bred* Vector is the periodically rescaled difference between two model runs, the second originating from slightly perturbed initial conditions.

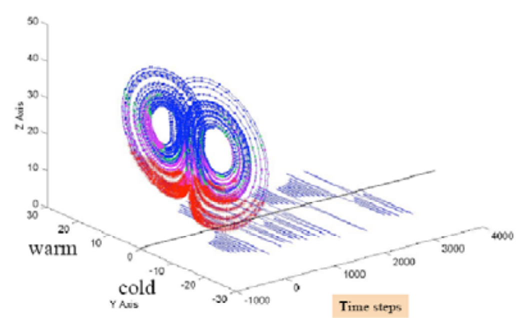
First model: the real weather
Second model: people’s predict



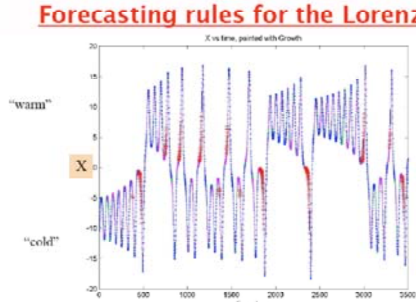


Local breeding growth rate: $g(t) = \frac{1}{n\Delta t} \ln \left(|\delta \mathbf{x}| / |\delta \mathbf{x}_0| \right)$

red points means growth rate >1.8 in 8 steps.



Forecasting rules for the Lorenz model:



Growth rate of bred vectors:

A * indicates fast growth (>1.8 in 8 steps)

Regime change: The presence of red stars (fast BV growth) indicates that the next orbit will be the last one in the present regime.

Regime duration: One or two red stars, next regime will be short. Several red stars: the next regime will be long lasting.