**Milankovitch Cycles**

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http://www.tqnyc.org/NYC052141/beginningpage.html

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**Solar Forcing**

Milankovitch Cycles

http://en.wikipedia.org/wiki/Milankovitch_cycles

**Climate Response (Zachos, et al)**

A. Power spectrum of climate for the last 4.5 Myr. Note the peaks at 41Kyr and 100 Kyr.

B. Power spectrum of climate for the period 25 Myr bp to 20.5 Myr bp. Note the new peak at 400 Kyr and the “split” peaks at 126Kyr and 95 Kyr.

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**Budyko’s Ice Line Model**

The annual global average insolation is $Q$. The annual average insolation as a function of latitude $\theta$, where $y = \sin \theta$, is $Q(y).$ $Q$ is largely determined by the eccentricity, but $s(y)$ is determined from a combination of the other orbital elements.

What is $s(y)$ as a function of obliquity and precession?

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**Insolation Function**

In solar coordinates:

$$S_{\rho(\beta)} = \begin{bmatrix} \cos \beta & 0 & \sin \beta \\ -\sin \beta & 0 & \cos \beta \end{bmatrix}$$

$$S(s) = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} \cos \rho \cos \gamma \\ \cos \rho \sin \gamma \\ \sin \rho \end{bmatrix}$$

Orthogonal matrix to obliquity angle

$$S_{\beta} = \begin{bmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{bmatrix}$$

Orthogonal matrix to precession angle

In solar coordinates:

$$S_{\rho(\beta)} S(s)$$
**Milankovitch Cycles**

**Instantaneous Insolation Function**

The Earth’s position with respect to the Sun, in the plane of the ecliptic $(r, \theta)$

Instantaneous insolation at the point $s$ on the Earth’s surface:

$$I = \left[ \frac{K}{4\pi r^2} \cos \theta \sin \theta \right] S_s(\rho) S_\gamma(\beta) I$$

Doing the math:

$$I(\beta, \rho, r, \theta, \phi, \gamma) = \left[ \frac{K}{4\pi r^2} \cos \phi \left( \cos \beta \cos \theta - \cos \gamma \sin \theta \right) \right]$$

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**Annual Insolation Function**

Instantaneous insolation:

$$I(\rho, r, \theta, \phi, \gamma) = \frac{K}{4\pi r^2} \cos \phi \left[ -\cos (\theta - \rho) \right]$$

Annual average:

$$T(\rho, \phi, \gamma) = \frac{K}{4\pi r^2} \int_0^{2\pi} \cos \phi \left[ -\cos (\theta - \rho) \right] d\theta$$

$$T(\phi, \gamma) = \frac{K}{4\pi r^2} \cos \phi \left[ -\cos (\theta - \rho) \right] d\theta$$

$P = \text{one year}$

Note the disappearance of $\rho$ (precession angle).

**Milankovitch Cycles**

**Annual Insolation Function**

Specific angular momentum: $\Omega = r(t) \frac{d\theta}{dt}$

Annual average:

$$T(\rho, \phi, \gamma) = \frac{K}{4\pi r^2} \int_0^{2\pi} \cos \phi \left[ -\cos (\theta - \rho) \right] d\theta$$

$$T(\phi, \gamma) = \frac{K}{4\pi r^2} \cos \phi \left[ -\cos (\theta - \rho) \right] d\theta$$

Claim: $I(-\phi) = I(\phi)$

Proof:

$$\int_0^{2\pi} \left( -\sin \beta \cos (-\phi) \cos \gamma - \cos \beta \sin (-\phi) \right) \, d\gamma = 0$$

$$\int_0^{2\pi} \left( -\sin \beta \cos \phi \cos \gamma - \cos \beta \sin \phi \right) \, d\gamma = 0$$

$$\int_0^{2\pi} \left( -\sin \beta \cos \phi \cos \gamma + \cos \beta \sin \phi \right) \, d\gamma = 0$$

$$\int_0^{2\pi} \left( -\sin \beta \cos \phi \cos \gamma - \cos \beta \sin \phi \right) \, d\gamma = 0$$

$$\int_0^{2\pi} \left( -\sin \beta \cos \phi \cos \gamma + \cos \beta \sin \phi \right) \, d\gamma = 0$$
Milankovitch Cycles
Relation to Budyko

\[ \frac{dT}{dt} = \left\{ \alpha(T) \right\} \left\{ 1 - \alpha(T) \right\} - \left\{ I(T) \right\} \left\{ H(T) \right\} \]

\[ Q(y) = T(y), \text{ where } y = \sin \varphi \]

\[ Q(y) = \frac{K}{4 \pi^2 \Omega^2} \int_{y}^{1} \sqrt{\left( \frac{1}{y} - 1 \right) \sin \beta \cos \gamma - y \cos \beta} \, dy \]

The function \( s \) is normalized so that \( \int_{0}^{1} s(y) \, dy = 1 \)

\[ Q(s) \int_{s}^{1} s(y) \, dy = \frac{K}{4 \pi^2 \Omega^2} \int_{y}^{1} \sqrt{\left( \frac{1}{y} - 1 \right) \sin \beta \cos \gamma - y \cos \beta} \, dy \]

\[ = \frac{K}{4 \pi^2 \Omega^2} \int_{0}^{1} \sqrt{\left( -\sin \beta \cos \gamma \right) \left( 1 - y \cos \beta \right)} \, dy \]

Since \( \int_{0}^{1} s(y) \, dy = 2 \) we have \( Q = \frac{K}{8 \pi \Omega} \)

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Relation to Budyko

Summary

\[ Q(y) = T(y) = \frac{K}{4 \pi^2 \Omega^2} \int_{y}^{1} \sqrt{\left( \frac{1}{y} - 1 \right) \sin \beta \cos \gamma - y \cos \beta} \, dy \]

\[ Q = \frac{K}{8 \pi \Omega} \]

\[ s(y) = \frac{2}{\pi} \int_{y}^{1} \sqrt{\left( \frac{1}{y} - 1 \right) \sin \beta \cos \gamma - y \cos \beta} \, dy \]

Note that \( Q \) depends only on the eccentricity and that \( s \) depends only on the obliquity.

Milankovitch Cycles
Relative Insolation Function

green = quadratic approximation (Tung and North)

mauve = formula using obliquity of 23.5°

red = obliquity of 24.5°

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Climate Response (Zachos, et al)

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Assuming there is a single ice line in the northern hemisphere, located at \( y = \eta \), the fast variables collapse to a one-dimensional center manifold with equation

\[
\frac{\partial \eta}{\partial t} = \epsilon h(\eta)
\]

The function \( h \) for current eccentricity and obliquity.

Once we know the ice line, we can solve for the global mean temperature

\[
\bar{T} = \frac{1}{\beta} \left( \chi(1 - \bar{\sigma}) - A \right)
\]

and the temperature at the pole

\[
T(1) = \frac{1}{\beta + C} \left( \chi(1 - \sigma) - A + C \bar{T} \right)
\]

and see how these vary with the eccentricity and the obliquity.
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Conclusions

1. Precession doesn’t matter.
2. Obliquity is more important than eccentricity.
3. Polar temperatures vary twice as much as global temperatures.