Bifurcation Analysis for Minimal Complexity PaleoClimate Modeling

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A Low-Order Dynamical Model of Global Climatic Variability Over the Full Pleistocene

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The Model

\[ \dot{I}' = a_0 I' - a_1 \mu' - a_2 M(t) \]
\[ \dot{\mu}' = b_1 \mu' - (b_2 - b_3 N') N' - b_4 N'^2 \mu' \]
\[ \dot{N}' = -c_0 I' - c_2 N' \]

- \( I = \) global ice mass
- \( N = \) North Atlantic Deep Water
- \( \mu = \) Atmospheric \( \text{CO}_2 \)
- \( a_{0,1}, b_{1,2,3,4}, \) and \( c_{1,2} > 0 \)
- \( M(t) = \) Milankovitch Forcing (65° normalized to O mean and unit variance)
- Primes denote departures from an eq. state controled by possible ultraslow variation of solar constant or the tectonic state of the Earth.
The Model

Reduction:

Original Dynamical System:

\[
\begin{align*}
\dot{X} &= -X - Y - uM(t^*) \\
\dot{Y} &= -pZ + rY + sZ^2 - Z^2Y \\
\dot{Z} &= -q(X + Z)
\end{align*}
\]

Substitutions:

\[
\begin{align*}
\mu' &= \left[ \frac{c_2}{a_1 c_0} \sqrt{\frac{a_0}{b_4}} \right] Y \\
N' &= \left[ \sqrt{\frac{a_0}{b_4}} \right] Z \\
I' &= \left[ \frac{c_2}{c_0} \sqrt{\frac{a_0}{b_4}} \right] X
\end{align*}
\]

where \( p = \frac{a_1 c_0 b_2}{a_0^2 c_2} \), \( q = \frac{c_2}{a_0} \), \( r = \frac{b_1}{a_0} \), \( s = \frac{a_i b_3 c_0 \sqrt{a_0^3 b_4}}{c_2} \), and \( u = \frac{a_2 c_0 \sqrt{\frac{b_4}{a_3}}}{c_2} \).
The Model

Reference Parameters:

\[(p, q, r, s) = (1.0, 1.2, 0.8, 0.8)\]
Equilibrium Solutions

1. Let $u = 0$.

2. Set $\dot{X} = \dot{Y} = \dot{Z} = 0$ and solve

\[
0 = -X - Y - uM(t^*) \\
0 = -pZ + rY + sZ^2 - Z^2Y \\
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Equilibrium Solutions

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\begin{align*}
0 &= -X - Y - uM(t^*) \\
0 &= -pZ + rY + sZ^2 - Z^2Y \\
0 &= -q(X + Z)
\end{align*}

3. $-X = Y = Z$

4. $0 = pX - rX + sX^2 + X^3 = X(X^2 + sX + (p - r))$
Equilibrium Solutions

\[ 0 = pX - rX + sX^2 + X^3 \]
\[ = X(X^2 + sX + (p - r)) \]

\[ X_0 = 0 \]
\[ X_{1,2} = \frac{-s \pm \sqrt{s^2 - 4(p - r)}}{2} \]
Equilibrium Solutions

Thus for each point in the parameter space \((p,q,r,s)\) there are 3 eq. solutions.

For each of the 3 eq. pts are 3 eigenvalues, \(\lambda_{1,2,3}\) for which \(\text{Re}(\lambda)\) will determine the stability of that eq. pt.

\[
\begin{align*}
0 &= pX - rX + sX^2 + X^3 \\
&= X(X^2 + sX + (p - r))
\end{align*}
\]

\[
\begin{align*}
X_0 &= 0 \\
X_{1,2} &= \frac{-s \pm \sqrt{s^2 - 4(p - r)}}{2}
\end{align*}
\]
To determine eigenvalues we must consider the linearised system at a given eq. point. By definition, the linearised system is:

\[
\hat{f}(X, Y, Z) = \begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{z}
\end{bmatrix} = Df|_{x_0, y_0, z_0} \cdot \begin{bmatrix}
x - x_0 \\
y - y_0 \\
z - z_0
\end{bmatrix}
\]
Linearised System

Reduced Dynamical System:

\[
\begin{align*}
\dot{X} &= -X - Y - uM(t^*) \\
\dot{Y} &= -pZ + rY + sZ^2 - Z^2Y \\
\dot{Z} &= -q(X + Z)
\end{align*}
\]

\[
Df = \begin{bmatrix}
-1 & -1 & 0 \\
0 & r - Z^2 & (-p + 2sZ - 2YZ) \\
-q & 0 & -q
\end{bmatrix}
\]
Linearised System

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Recall

\[-X = Y = Z\]

\[
\Rightarrow \quad Df = \begin{bmatrix}
-1 & -1 & 0 \\
0 & r - X^2 & (-p - 2sX - 2X^2) \\
-q & 0 & -q
\end{bmatrix}
\]
Thus to linearise about \((-\alpha, \alpha, \alpha)\):
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\[
\hat{f}(\alpha, -\alpha, -\alpha)(X, Y, Z) = \begin{bmatrix}
\dot{X} \\
\dot{Y} \\
\dot{Z}
\end{bmatrix} = \begin{bmatrix}
-1 & -1 & 0 \\
0 & r - \alpha^2 & (-p + 2s\alpha - 2\alpha^2) \\
-q & 0 & -q
\end{bmatrix} \begin{bmatrix}
X - \alpha \\
Y - \alpha \\
Z - \alpha
\end{bmatrix}
\]

Recall if \(\text{Re}(\lambda) < 0\) for all \(\lambda\) then the eq. pt is stable.
If \(\text{Re}(\lambda) > 0\) for any \(\lambda\) then the eq. pt is unstable.
We must now solve:

\[
|Df - \lambda I| = 0
\]
Eigenvalues

- **Characteristic Polynomial:**

\[ \lambda^3 + (1 + q + X^2 - r)\lambda^2 + (q(1 + X^2 - r) - r + X^2)\lambda + q(p + 2sX + X^2 - r) \]

- For reference parameters \((p,q,r,s) = (1,1.2,0.8,0.8)\):

\[ \lambda^3 + (1.4 + X^2)\lambda^2 + (2.2X^2 - 0.56)\lambda + 3.6X^2 + 1.92X + 0.24 = 0 \]

- Solving \(\lambda\) for each of the three eq. pts:

<table>
<thead>
<tr>
<th>eq pt</th>
<th>(\lambda_1)</th>
<th>(\lambda_2)</th>
<th>(\lambda_3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 0, 0))</td>
<td>-1.7882</td>
<td>0.0194 - 0.3107i</td>
<td>0.0194 + 0.3107i</td>
</tr>
<tr>
<td>((-0.4 + 0.2i, 0.4 - 0.2i, 0.4 - 0.2i))</td>
<td>-1.6625 + 0.1665i</td>
<td>-0.2408 - 0.1305i</td>
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The system is hyperbolic for this parameter, thus the linearised system is an accurate representation for the non-linear system locally.
Thus for reference parameters \((p,q,r,s) = (1,1.2, 0.8, 0.8)\)

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The origin is spirally unstable with a 2D unstable space and a 1D stable space.

The other two points don’t have any physical meaning because the eq. pts are complex valued.
Eigenvalues

- Considering the system as a function of \( p \). Now try to understand how the stability of the system changes as \( p \) changes.
Considering the system as a function of $p$. Now try to understand how the stability of the system changes as $p$ changes.

Fix $q$, $r$ and $s$ at reference values.

Initially we can see eq pts $X_{2,3}$ are only real for $p < 0.96$

$$X_{1,2} = \frac{-s \pm \sqrt{s^2 - 4(p - r)}}{2}.$$
Bifurcations
Claim: For eq. pt. $X_2 = \frac{-s - \sqrt{s^2 - 4(p-r)}}{2}$ there exists a Hopf's Bifurcation between $p = 0.9353$ & $p = 0.9354$.

Proof: We will use the following theorem:

A Hopf's bifurcation occurs when all eigenvalues of $Df$ have $\text{Re}(\lambda_0) < 0$ except one conjugate pair $\lambda_{1,2} = i\omega$. 
For \((p,q,r,s) = (0.9353, 1.2, 0.8, 0.8)\) the eigenvalues are:

\[
\lambda = -1.71039 \quad \lambda = -0.0000220078 \pm 0.350529i
\]

For \((p,q,r,s) = (0.9354, 1.2, 0.8, 0.8)\) the eigenvalues are:

\[
\lambda = -1.7103 \quad \lambda = 0.000114597 \pm 0.350082i
\]

1. \(\text{Re}(\lambda_0) < 0\) as required.
2. The next claim is that the \(\text{Re}(\lambda_{1,2}) = 0\) at some point \(0.9353 < p < 0.9354\).
Bifurcations

- We can view the system completely as a function of $p$.

\[
D_f = \begin{bmatrix}
-1 & -1 & 0 \\
0 & r - (X(p))^2 & (-p - 2s(X(p)) - 2(X(p))^2) \\
-q & 0 & -q
\end{bmatrix}
\]

- $X(p)$ is a continuous function of $p$.
- $\text{Det}[Df]$ is thus a continuous function of $p$.
- The $\text{Re}(\lambda_i)$ are continuous with respect to $p$.
- Thus by Intermediate Value Theorem there exists a $p$, $0.9353 < p < 0.9354$ such that $\text{Re}(\lambda_{1,2}) = 0$. 
Bifurcations
Next we’ll take a quick review of the bifurcation diagrams for r and s.
Bifurcations

Real part of $\Lambda$ as a function $r$ for the equilibria $X = 0$

Real part of $\Lambda$ as a function of nonzero equilibria $X$ varying $r$
Bifurcations

Bifurcation $r$ vs. $X=0$ and $X_{1,2}$
Bifurcations

Real part of $\Lambda^2$ as a function of $X$ with varying $s$.
Varying Parameters
Below is the published solution curve for $q = 1.2$, $s = 0.8$ and $p$ and $r$ linearly varying between $0.8 \rightarrow 1$ and $0.7 \rightarrow 0.8$ respectively.
Below is **my** solution curve for \( q = 1.2, \ s = 0.8 \) and \( p \) and \( r \) linearly varying between \( 0.8 \rightarrow 1 \) and \( 0.7 \rightarrow 0.8 \) respectively.
Below is my solution curve for $q = 1.2$, $s = 0.8$ and $p$ and $r$ linearly varying between $1 \rightarrow 0.08$ and $0.8 \rightarrow 0.7$ respectively.
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Varying Parameters

- We parameterize \( p \) and \( r \):

\[
p(\alpha) = 0.8 + 0.2\alpha \quad r(\alpha) = 0.7 + 0.1\alpha
\]

- We can view the system as continuous with respect to \( \alpha \).

\[
Df = \begin{bmatrix}
-1 & -1 & 0 \\
0 & r(\alpha) - (X(\alpha))^2 & (-p(\alpha) - 2s(X(\alpha)) - 2(X(\alpha))^2) \\
-q & 0 & -q
\end{bmatrix}
\]
Varying Parameters

\[ \text{Re}(\lambda_0) < 0, \]
\[ \text{Re}(\lambda_{1,2}) > 0, \]
\[ \lambda_{1,2} \text{ in } \mathbb{C}, \]
\[ \text{for all } \alpha \text{ in } (0,1) \]

No interesting dynamics due to eigenvalues.
The only possible stable eq. pts. are when $\alpha < -2$. This is well outside the published range of $[0,1]$. 

$\Re(\lambda_0)$ for XO=0 vs. $\alpha$

$\Re(\lambda_{1,2})$ for X1 & X2 vs. $\alpha$
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$\text{Re}(\lambda_0)$ for $XO=0$ vs. $\alpha$

$\text{Re}(\lambda_{1,2})$ for $X1 & X2$ vs. $\alpha$
Conclusions

- Changing $\alpha$ seems to cause global system changes which can not be captured in the standard local bifurcation approach.
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- Changing $\alpha$ seems to cause global system changes which cannot be captured in the standard local bifurcation approach.

- Despite any errors, the main concept that Maasch and Saltzman present with respect to bifurcation values is still valid.

- It is likely that there exists a small parameter shift that would cause a large change in the oscillations of the system.