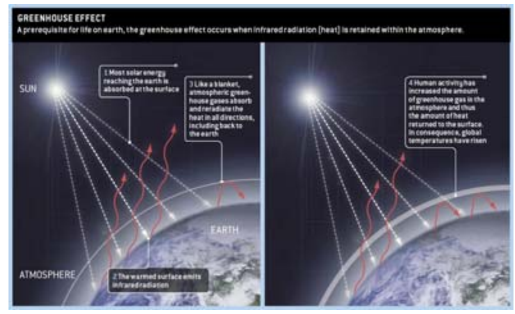
Energy Balance Models

Richard McGehee

Seminar on the Mathematics of Climate Change
School of Mathematics
November 2, 2011

Energy Balance Models

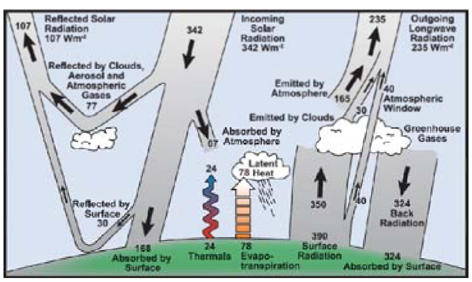
Earth's Energy Balance



Gary Stix, *Scientific American* September 2006, pp.46-49

Energy Balance Models

Earth's Energy Balance



Historical Overview of Climate Change Science, IPCC AR4, p.96
`http://ipcc-wg1.ucar.edu/wg1/Report/AR4WG1_Print_CH01.pdf`

Energy Balance Models

Insolation

Insolation = **In**coming **sol**ar radi**ation**

average solar intensity for Earth's orbit: 1368 W/m^2

radius of the Earth: ρ meters
cross sectional area: $\pi\rho^2$ m^2
intercepted power: 1368 $\pi\rho^2$ Watts
surface area: $4\pi\rho^2$ m^2

average insolation: 1368/4 W/m^2 = 342 W/m^2

Energy Balance Models

References

Classic Papers:

M. I. Budyko, The effect of solar radiation variation on the climate of the Earth, *Tellus* **21** (1969), 611-619.

W. D. Sellers, A Global Climatic Model Based on the Energy Balance of the Earth-Atmosphere System, *Journal of Applied Meteorology* **8** (1969), 392-400.

Recent Interpretation:

K.K. Tung, Topics in Mathematical Modeling, Princeton University Press, 2007. (Chapter 8)

Energy Balance Models

Homogeneous Earth

$$R\frac{dT}{dt} = Q(1-\alpha) - (A + BT)$$

T = global mean temperature (°C)
Q = mean solar input (W/m^2)
α = mean albedo
$A+BT$ = outward radiation (linear approximation)
R = heat capacity of Earth's surface

Tung's values:

T = global mean temperature (°C)
Q = 343 W/m^2
A = 202 W/m^2
B = 1.9 W/(m^2 °C)

$\alpha = \alpha_1 = 0.32$ (water and land)
$\alpha = \alpha_2 = 0.62$ (ice)

Energy Balance Models

Homogeneous Earth

$$R\frac{dT}{dt} = Q(1-\alpha) - (A + BT)$$

Equilibrium temperature

$$T_{eq} = \frac{Q(1-\alpha) - A}{B}$$

ice free Earth: $\alpha = \alpha_1$, T_{eq} = 16.4 °C
snowball Earth: $\alpha = \alpha_2$, T_{eq} = -37.7 °C

According to Tung, glaciers form if $T < T_c$ = -10 °C and melt if $T > T_c$.

Since 16.4 > -10, no glacier would form on an ice free Earth.
Since -37.7 < -10, no glacier would melt on a snowball Earth.

Energy Balance Models

Inhomogeneous Earth

$$R\frac{\partial T}{\partial t} = Qs(y)\big(1-\alpha(y)\big) - (A + BT) + C\left(\overline{T} - T\right)$$

Now the annual average surface temperature T is a function of y = sine(latitude).

The albedo α is a function of y .

The outward radiation $A+BT$ is as before.

Heat transport across latitudes is assumed to be linear and is given by

$$C\left(\overline{T} - T\right)$$

where C = 3.04 W/m²/°C.

The global annual average insolation is Q, with the same value as above, while $s(y)$ is the relative insolation, normalized to satisfy

$$\int_0^1 s(y)\,dy = 1$$

Energy Balance Models

Inhomogeneous Earth

$$R\frac{\partial T}{\partial t} = Qs(y)\big(1-\alpha(y)\big) - (A + BT) + C\left(\overline{T} - T\right)$$

The variable y is chosen instead of the latitude, because the global annual mean temperature is given by

$$\overline{T}(t) = \int_0^1 T(y,t)\,dy$$

We assume symmetry with respect to the equator, so the variable y takes on values between 0 and 1.

Rate of solar energy absorption at y = sine(latitude):

$$Qs(y)\big(1-\alpha(y)\big)$$

Look for an equilibrium solution $\quad T = T^*(y)$

This equilibrium satisfies

$$\boxed{Qs(y)\big(1-\alpha(y)\big) - \big(A + BT^*(y)\big) + C\left(\overline{T^*} - T^*(y)\right) = 0}$$

Energy Balance Models

Inhomogeneous Earth

$$\boxed{Qs(y)\big(1-\alpha(y)\big) - \big(A + BT^*(y)\big) + C\left(\overline{T^*} - T^*(y)\right) = 0}$$

Integrate:

$$\int_0^1 \left(Qs(y)\big(1-\alpha(y)\big) - \big(A + BT^*(y)\big) + C\left(\overline{T^*} - T^*(y)\right)\right) dy = 0,$$

$$Q(1-\overline{\alpha}) - A - B\overline{T^*} = 0$$

where

$$\overline{\alpha} = \int_0^1 \alpha(y)s(y)\,dy,$$

Therefore, the equilibrium global mean temperature is

$$\boxed{\overline{T^*} = \frac{1}{B}\big(Q(1-\overline{\alpha}) - A\big)}$$

and the equilibrium temperature profile is

$$\boxed{T^*(y) = \frac{1}{B+C}\left(Qs(y)\big(1-\alpha(y)\big) - A + C\overline{T^*}\right)}$$

Energy Balance Models

Inhomogeneous Earth

What about $s(y)$, the relative insolation function?

$$s(y) = \frac{2}{\pi^2}\int_0^{2\pi} \sqrt{1 - \left(\sqrt{1-y^2}\sin\beta\cos\gamma - y\cos\beta\right)^2}\,d\gamma$$

where β = obliquity. (Current value is about 23.5°.)

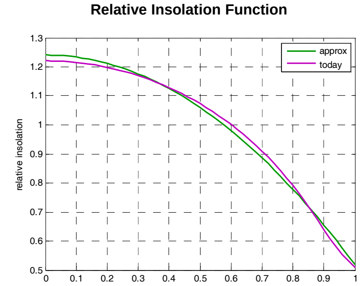
Tung and North's quadratic approximation:

$$s(y) \approx 1 - 0.241\left(3y^2 - 1\right)$$

Energy Balance Models

Inhomogeneous Earth

Relative Insolation Function



green = quadratic approximation (Tung and North)

mauve = formula using obliquity of 23.5°

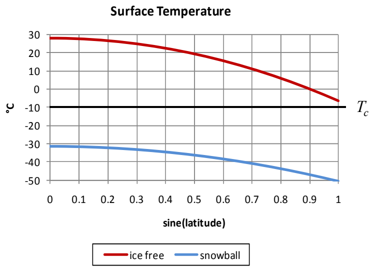
Energy Balance Models

Inhomogeneous Earth

Assume that $\alpha(y) = \alpha_i$ (constant).

$$T^*(y) = \frac{1}{B+C}\left(Qs(y)(1-\alpha_i) - A + C\overline{T^*}\right)$$

Both ice-free ($\alpha = \alpha_1 = 0.32$) and snowball ($\alpha = \alpha_2 = 0.62$) states look pretty stable.



Energy Balance Models

Inhomogeneous Earth

Now suppose that the Earth has an ice cap, so that ice covers the surface above a certain latitude, and below that latitude the surface is ice free. We can write the equation:

$$R\frac{\partial T}{\partial t} = Qs(y)\left(1-\alpha(y,\eta)\right) - (A+BT) + C\left(\overline{T} - T\right)$$

$$\alpha(y,\eta) = \begin{cases} \alpha_1, & y<\eta, \\ \alpha_2, & y>\eta, \end{cases}$$

where the ice boundary is at latitude $\arcsin(\eta)$.

The same technique finds an equilibrium solution, i.e., look for a solution

$$T = T_\eta^*(y)$$

satisfying

$$\boxed{Qs(y)\left(1-\alpha(y,\eta)\right) - \left(A + BT_\eta^*(y)\right) + C\left(\overline{T}_\eta^* - T_\eta^*(y)\right) = 0}$$

Energy Balance Models

Inhomogeneous Earth

$$\boxed{Qs(y)\left(1-\alpha(y,\eta)\right) - \left(A + BT_\eta^*(y)\right) + C\left(\overline{T}_\eta^* - T_\eta^*(y)\right) = 0}$$

Integrate:

$$\int_0^1 \left(Qs(y)\left(1-\alpha(y,\eta)\right) - \left(A + BT_\eta^*(y)\right) + C\left(\overline{T}_\eta^* - T_\eta^*(y)\right)\right)dy = 0,$$

$$Q\left(1-\overline{\alpha}(\eta)\right) - A - B\overline{T}_\eta^* = 0$$

where

$$\overline{\alpha}(\eta) = \int_0^1 \alpha(y,\eta)s(y)\,dy = \int_0^\eta \alpha_1 s(y)\,dy + \int_\eta^1 \alpha_2 s(y)\,dy$$

$$= \alpha_1 S(\eta) + \alpha_2\left(1-S(\eta)\right) = \alpha_2 - (\alpha_2-\alpha_1)S(\eta),$$

and where

$$S(\eta) = \int_0^\eta s(y)\,dy$$

The global mean temperature is

$$\boxed{\overline{T}_\eta^* = \frac{1}{B}\left(Q\left(1-\overline{\alpha}(\eta)\right) - A\right)}$$

Energy Balance Models

Inhomogeneous Earth

Equilibrium equation (given ice line):

$$Qs(y)\left(1-\alpha(y,\eta)\right) - \left(A + BT_\eta^*(y)\right) + C\left(\overline{T}_\eta^* - T_\eta^*(y)\right) = 0$$

Global mean temperature:

$$\overline{T}_\eta^* = \frac{1}{B}\left(Q\left(1-\overline{\alpha}(\eta)\right) - A\right)$$

Solve for equilibrium temperature profile:

$$\boxed{T_\eta^*(y) = \frac{1}{B+C}\left(Qs(y)\left(1-\alpha(y,\eta)\right) - A + C\overline{T}_\eta^*\right)}$$

where

$$\alpha(y,\eta) = \begin{cases} \alpha_1, & y<\eta, \\ \alpha_2, & y>\eta. \end{cases}$$

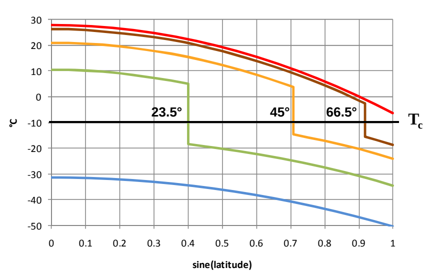
$$\overline{\alpha}(\eta) = \alpha_2 - (\alpha_2-\alpha_1)\int_0^\eta s(y)\,dy$$

Energy Balance Models

Inhomogeneous Earth

Temperature profiles for various ice lines

The standard argument seems to imply that these are all stable.



Energy Balance Models

Inhomogeneous Earth

$$\boxed{T_\eta^*(y) = \frac{1}{B+C}\left(Qs(y)\left(1-\alpha(y,\eta)\right) - A + C\overline{T}_\eta^*\right)}$$

Additional assumption: At equilibrium, the average temperature across the ice boundary is T_c = -10 °C

$$T_\eta^*(\eta-) = \frac{1}{B+C}\left(Qs(\eta)(1-\alpha_1) - A + C\overline{T}_\eta^*\right)$$

$$T_\eta^*(\eta+) = \frac{1}{B+C}\left(Qs(\eta)(1-\alpha_2) - A + C\overline{T}_\eta^*\right)$$

(s is continuous)

$$T_c = \frac{T_\eta^*(\eta-) + T_\eta^*(\eta+)}{2} = \frac{1}{B+C}\left(Qs(\eta)(1-\alpha_0) - A + C\overline{T}_\eta^*\right)$$

where

$$\alpha_0 = \frac{\alpha_1+\alpha_2}{2}$$

Energy Balance Models

Inhomogeneous Earth

Now we can solve for the ice boundary.

$$\frac{1}{B+C}\left(Qs(y)(1-\alpha_0)-A+C\overline{T}_\eta^*\right)=T_c$$

where

$$\overline{T}_\eta^*=\frac{1}{B}\left(Q\left(1-\overline{\alpha}(\eta)\right)-A\right)$$

Therefore,

$$\frac{1}{B+C}\left(Qs(\eta)(1-\alpha_0)-A+\frac{C}{B}\left(Q\left(1-\overline{\alpha}(\eta)\right)-A\right)\right)=T_c$$

which reduces to

$$h(\eta)\equiv\frac{Q}{B+C}\left(s(\eta)(1-\alpha_0)+\frac{C}{B}\left(1-\alpha_2+(\alpha_2-\alpha_1)\int_0^\eta s(y)\,dy\right)\right)-\frac{A}{B}-T_c=0$$

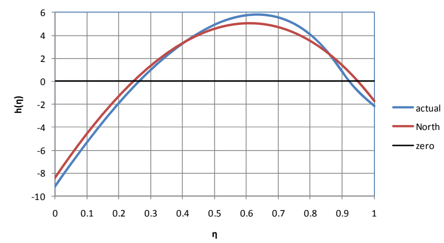
which can be solved numerically for η .

Energy Balance Models

Inhomogeneous Earth

Two equilibria satisfy $h(\eta)=0$. What about stability?

Graph of h



Energy Balance Models

Inhomogeneous Earth

$$R\frac{\partial T}{\partial t}=Qs(y)\left(1-\alpha(y)\right)-(A+BT)+C\left(\overline{T}-T\right)$$

The energy balance equation, as it stands, does not provide any dynamics for the ice boundary. There are many equilibrium states, and no way for the ice line to move. The addition condition that the average temperature across the ice boundary is T_c does not give us any way to examine stability.

Next time: Adding dynamics.