Energy Balance Models

Budyko-Sellers Model

\[
\frac{dT}{dt} = Q_s(y)(1-a(y)) - (\lambda + \beta T) + C(T - T_0)
\]

- \( T = T(y,t) \): annual mean surface temperature
- \( y = \sin(\text{latitude}) \)
- \( Q_s(y) = \text{relative annual mean insolation} \)
- \( A = 202 \text{ W/m}^2 \)
- \( B = 1.9 \text{ W/m}^2/\text{C} \)
- \( C = 3.04 \text{ W/m}^2/\text{C} \)

What about \( s(y) \), the relative insolation function?

\[
s(y) = \frac{2}{\beta} \int_{-\beta}^{\beta} \sqrt{1 - \frac{y^2}{\beta^2}} \, dy
\]

where \( \beta = \text{obliquity} \). (Current value is about 23.5°.)

Tung and North’s quadratic approximation:

\[
s(y) = 1 - 0.241(1 - \cos(\beta))
\]

Suppose that there is a single ice boundary at latitude \( \arcsin(\eta) \).

Energy Balance Models

Relative Insolation Function

- green = quadratic approximation (Tung and North)
- mauve = formula using obliquity of 23.5°

Energy Balance Models

What about \( \alpha(y) \), the albedo function?

\[
\alpha(y) = \begin{cases} 
\alpha_1 & y \leq \eta, \\
\alpha_2 & y > \eta, 
\end{cases}
\]

\( \alpha_1 = 0.32 \) (water and land)
\( \alpha_2 = 0.62 \) (ice)

Suppose that there is a single ice boundary at latitude \( \arcsin(\eta) \).
Energy Balance Models

Budyko-Sellers Model

\[ \frac{dT}{dt} = -Q(y)[1-a(y,y)] \cdot (A+B T) + C(T_c - T) \]

Lots of equilibrium solutions

The previous argument seems to imply that these are all stable.

Additional assumption: At equilibrium, the average temperature across the ice boundary is \( T_c = -10 \degree C \)

Energy Balance Models

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Energy Balance Models

Budyko-Widiasih Model

\[ \frac{dy}{dt} = \varepsilon(T(y) - T_c) \]

\[ \frac{dT}{dt} = -Q(y)[1-a(y,y)] \cdot (A+B T) + C(T - T_c) \]

Widiasih’s Theorem:
For an appropriate function space \( E \) and for sufficiently small \( \varepsilon \), the system has an attracting invariant curve. On the curve, the system is approximated by

Recall:

\[ \hat{h}(\eta) = \frac{Q}{\varepsilon + C} \left( \int h(y)[1-a(y,y)] \cdot (A+B y) + C(T_c - T_c) \, dy \right) \quad \frac{dx}{dy} = 0 \]
Energy Balance Models

Budyko-Widiasih Model

\[ \frac{dq}{dt} = \epsilon h(q) \]

Idea: First the temperature approaches its equilibrium, then the ice line adjusts.

Recall from last time: an equilibrium solution:

\[ T^* = \frac{1}{B+C} \left\{ \int (\sigma_0 (1 - \alpha (s, \eta)) - \Delta + C \tau^*_0) \right\} \]

where

\[ \tau^*_0 = \frac{1}{B} \int (\sigma_1 (s) - \sigma_2 (s)) \]

\[ \sigma_1 (s) = \sigma (s) - (\alpha_1 - \alpha_2) \int (\sigma (s) (1 - \alpha (s, \eta))) \]

\[ \tau^*_0 = \frac{1}{B} \int (\tau_0 (s) + \tau_1 (s)) \]

What about the greenhouse effect?

\[ A + B \] is the outgoing long wave radiation. This term decreases if the greenhouse gases increase.

We view \( A \) as a parameter.

\[ \frac{dA}{dt} = \epsilon h(q, A) \]

What if \( A \) is a dynamical variable?

Simple equation:

\[ \frac{dA}{dt} = \epsilon (q - q_0) \]

New system:

\[ \frac{dq}{dt} = \epsilon h(q, A) \]

\[ \frac{dA}{dt} = \epsilon (q - q_0) \]
Energy Balance Models

Augmented Budyko-Widiasih Model

\( \frac{d\eta}{dt} = c_\eta(\eta, A) \frac{dA}{dt} + \frac{dA}{dt} \left( \eta - \eta_c \right) \)

Energy Balance Models

Augmented Budyko-Widiasih Model

Snowball – Hothouse Oscillations

Energy Balance Models

What about the Jormungand model?

Stable Jormungand State

Energy Balance Models

What about the Jormungand model?

Jormungand Oscillations
Energy Balance Models

What about the Jormungand model?

Big Ice Ages

unstable rest point

Energy Balance Models

Lots to Ponder