An Estimate of the Widiasih Parameter

Richard McGehee

Seminar on the Mathematics of Climate Change
School of Mathematics
February 1, 2012

Widiasih’s Parameter

Budyko-Sellers Model

The Albedo Function

\[ a = a_1 = 0.32 \quad \text{(water and land)} \]
\[ a = a_2 = 0.62 \quad \text{(ice)} \]

Suppose that there is a single ice boundary at latitude \( \arcsin(\eta) \).

\[ a(y) = a(y, \eta) = \begin{cases} a_1, & y < \eta, \\ a_2, & y > \eta, \end{cases} \]

\[ R \frac{dT}{dt} = Q(y) \left( 1 - a(y, \eta) \right) - (A + BT) + C (T - T) \]

Where:

\[ k(y) = \frac{Q}{B} \left( \int a(y) \, dy \right) \left( 1 - a(y, \eta) \right) + C - A \]
\[ \alpha = \frac{a_1 + a_2}{2} \]

Widiasih’s Ice Line Dynamics

State space:

\[ \{ a(\eta) \} \times E \]

\[ \frac{d\eta}{dt} = k(T(q) - T) \]

Where:

\[ k(T(q)) = \frac{Q}{B} \left( \int a(y) \, dy \right) \left( 1 - a(y, \eta) \right) + C - A \]

How small is \( \varepsilon \)?

Widiasih’s Theorem:

For an appropriate function space \( E \) and for sufficiently small \( \varepsilon \), the system has an attracting invariant curve. On the curve, the system is approximated by

\[ \frac{d\eta}{dt} = k(T(q) - T) \]

\[ R \frac{dT}{dt} = Q(y) \left( 1 - a(y, \eta) \right) - (A + BT) + C (T - T) \]

\[ T(q, t) = \frac{1}{2} \left( T(q) + T(q +) \right) \]
What about units?

\[
\frac{\partial q}{\partial t} = (1-\alpha)(1-\beta) - (d + R'T) + C(T - T')
\]

\( W/m^2 = J/m^2/sK/s \)

\( R \):  \( J/m^2/K \)

\( \alpha \) has units of \( m^2/s/\)K

It is not clear how to choose \( \varepsilon \) from first principles.

What can we learn from paleoclimate data?
Dependence on Milankovitch Cycles

\[ B_{\text{quad}}^{12} = \mathcal{Q}(\gamma)(1 - \alpha(y, \gamma)) - (\delta + BT) + C(T - T) \]

depends only on obliquity \( \beta \)
depends only on eccentricity \( \epsilon \)

\[ \mathcal{Q}(\gamma) = \frac{Q}{\sqrt{1 - \gamma^2}} \]

\[ s(\gamma, \beta) = \frac{2}{\pi} \int_{0}^{\pi} \left[ \sqrt{1 - \gamma^2 \sin \beta \cos y - \gamma \cos \beta} \right] dy \]

L2 Approximation using Legendre Polynomials

\[ s(x, \beta) = \frac{1}{\sqrt{2}} \int_{0}^{\pi} \left( \sqrt{x^2 - \gamma^2 \sin \beta \cos y - \gamma \cos \beta} \right) dy \]

\[ x \text{ is an even function of } y \text{, so a quadratic L2 approximation is} \]

\[ s(x, \beta) \approx s_2(x, \beta) \]

where

\[ s_2(x) = \frac{1}{3} \left( x^2 - 1 \right) \]

One can compute

\[ s_2(x) = \frac{1}{2} \left( 2 + \sin^2 \beta \right) \]

The current value of obliquity is about 23.5°.

\[ s(x) \approx 1 - 0.23(3x^2 - 1) \]

Compare with Tung and North’s quadratic approximation:

\[ s(x) \approx 1 - 0.24(3x^2 - 1) \]

Dependence on Milankovitch Cycles

Widiasih’s Parameter

Dependence on Milankovitch Cycles

Widiasih Ice Line Model

\[ \frac{dh}{dt} = \epsilon h + \alpha(x) \]

\[ \mathcal{Q}(\gamma) \left( \gamma \left( \gamma \left( 1 - u \right) + C(1 - u) + \int_{0}^{\pi} s(x, \beta) d\beta \right) \right) \]

quadratic in \( \eta \)
cubic in \( \gamma \)

We can solve \( h(x, \epsilon, \beta) = 0 \) for \( \eta = f(x, \beta) \) at the stable rest point.

We can then use Laskar’s paleoclimate computations of \( x(t) \) and \( \beta(t) \) to get the ice line as a function of time \( \eta(t) = f(x(t), \beta(t)) \).

One can also compute the global mean temperature (GMT) as a function of time.
Results of Model Simulation

There is a delay of about 2500 years between the model and the data.

Widiasih's Parameter

Widiasih Ice Line Model

\[
\frac{dh}{dt} = \varepsilon h(t), \quad \theta(t) = \theta(t-\alpha)
\]

So far we have computed only the moving equilibrium. The actual dynamics will have a delay between the “equilibrium” and the “response.” What value of \( \varepsilon \) will create a delay of 2.5 kiloyears?

Change time units to kiloyears. Let \( \kappa = 3.16 \times 10^{10} \) be the number of seconds in a kiloyear.

\[
\frac{dh}{dt} = -\varepsilon h(t), \quad \theta(t) = \theta(t-\alpha)
\]

Since the deviation from the rest point is relatively small and since the major contribution from the Milankovitch cycles is the obliquity at a period of 41 Kyr, we consider the linear equation

\[
\frac{dh'}{dt} = -\varepsilon h'(t), \quad \theta(t) = \theta(t-\alpha)
\]

where \( \lambda = \arctan(\alpha / \omega) \) and \( \omega = 2\pi / 41 \)

and where \( h_2 \) is the current value of the ice line (about 0.95).

Widiasih's Parameter

Linear Approximation

\[
\frac{dz}{dt} = -\lambda (z - \cos(\alpha))
\]

Steady-state solution

\[
z = \cos(\alpha) + \cos(\omega t - \alpha)
\]

where \( \omega = \arctan(\alpha / \lambda) \)

Note that \( \alpha \) is the phase shift corresponding to the delay by which the response lags the forcing. A delay of 2.5 Kyr corresponds to \( \alpha = 2.5 \pi / 41 \). Hence

\[
\lambda = \arccot \alpha = -\arccot(h_2), \quad \omega = \arccot(\alpha / \lambda)
\]

One can show that \( b(\theta) \approx -31 \) and hence that

\[
\alpha \approx 4 \times 10^{10}
\]

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Conclusion

\[
\frac{dh}{dt} = \varepsilon h(t), \quad \theta(t) = \theta(t-\alpha)
\]

\[\varepsilon \approx 4 \times 10^{-13}\]

Acknowledgements

This presentation was all joint work with Esther Widiasih.
(proprint coming soon)

A similar value for \( \varepsilon \) was computed by Esther and Jayna Resman using direct dynamical simulations of the equation.