The effect of CO$_2$ on Earth’s Radiation Budget

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M.I. Budyko (1969)*

\[ R \frac{dT}{dt} = Q_s(y)(1 - \alpha(y)) - (A + BT) - C \left( T - \int_0^1 T \, dy \right) \]

Outgoing Longwave Radiation OLR

- \( A, B > 0 \)
- \( T \nearrow \Rightarrow \text{OLR} \nearrow \)
- \( \text{CO}_2 \leftrightarrow A \quad ? \)

“Big ideas come from small models.”

May 23, 2011, Snowbird, UT

Ray Pierrehumbert

M.I. Budyko (1969)

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Outgoing Longwave Radiation OLR

- \( A, B > 0 \)
- \( T \uparrow \Rightarrow \text{OLR} \uparrow \)
- \( \text{CO}_2 \leftrightarrow A \quad ? \)

*Principles of Planetary Climate*
Cambridge, 2010
(PPC)

Ray Pierrehumbert
• Top-of-the-atmosphere considerations
• Role of greenhouse gasses in reducing OLR
Radiation: characterized by direction of propagation and frequency

frequency $\nu$ (Hz)

wavelength $\lambda$ (m)

wavenumber $n = \nu/c \ m^{-1}$

$\nu\lambda = c = 3 \times 10^8 \ m/s$

(PPC, p. 137)
Blackbody radiation

- radiation reacts so strongly with matter that it achieves thermodynamic equilibrium at same temperature as the matter ("perfect absorbers and emitters").

Planck’s Function:

\[ B(\nu, T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1} \]

- \( k \) = Boltzman thermodynamic constant
- \( h \) = Planck’s constant
- \( c \) = speed of light

- F.D. \( \uparrow \) with \( T \) at all \( n \)
- F.D. small at large \( n \) for high \( T \)

- Treat Earth as blackbody, even though core \( T \sim 6000 \) K: sufficiently dense outer shell acts as blackbody
Blackbody radiation

Stefan-Boltzmann Law

Total power exiting from each unit area of the surface of a blackbody:

\[
F = \int_0^\infty \pi B(\nu, T) \, d\nu = \sigma T^4,
\]

\[
\sigma = \frac{2\pi^5 k^4}{(15c^2h^3)} \approx 5.67 \cdot 10^{-8} \text{ W/m}^2\text{K}^4
\]

units of \( F = \) units of OLR = W/m\(^2\)
Radiation balance of planets: An idealized example

- only source of energy heating the planet is absorption of light from planet’s host star.
- the planetary albedo is spatially uniform.
- the planet is spherical and has a distinct liquid or solid surface which radiates like a perfect blackbody.
- the temperature is uniform over the surface of the planet.
- the planet’s atmosphere is perfectly transparent to the electromagnetic energy emitted by the surface

\[ a = \text{planet's radius} \quad T_* = \text{star's temperature} \]
\[ r_* = \text{star's radius} \quad r = \text{distance to star} \]

- Total flux impinging on planet \( \sigma T_*^4 r_*^2 / r^2 = \text{solar constant} \ L_* \)
- Energy absorption: \( \pi a^2 L_* (1 - \alpha) \)
- Energy loss: \( 4\pi a^2 \sigma T^4 \)

Equilibrate: \( \sigma T^4 = \frac{L_*}{4} (1 - \alpha) \)
\[ T = \frac{1}{\sqrt{2}} (1 - \alpha)^{1/4} \sqrt{\frac{r_*}{r}} T_* \]

Planet loses energy through emission at a lower wavenumber than that at which it receives energy from the star.
$T = \frac{1}{\sqrt{2}}(1 - \alpha)^{1/4} \sqrt{\frac{r_*}{r}} T_*$

Planck densities:

![Graph showing normalized spectrum with wavenumber (cm$^{-1}$) on the x-axis and normalized spectrum on the y-axis for various celestial bodies, including Pluto, Jupiter, Earth, and Mercury, with an arrow pointing to the Sun labeled 'infrared (longwave)'.
Greenhouse gas (G.G.): Basic ideas

• Earth radiates in the infrared
• If the atmosphere was transparent to infrared: \( \text{OLR} = \sigma T_s^4 \)
• Mix in G.G. with unit mass concentration \( q \)
  Assume G.G. (i) transparent to solar radiation  
  (ii) opaque to infrared at sufficiently high concentrations

\[ p_s = \text{surface pressure} \]
\[ T_s = \text{surface temp} \]
\[ \text{Assume } T_s = T(p_s) \]

absorption coefficient \( \kappa = \kappa(\nu, p, T) \text{ m}^2/\text{kg} \)
\[ \kappa \Delta p q / g > 1 \implies \text{column acts like blackbody} \]

• \( \kappa p_s q / g < 1 \implies \text{atmosphere optically thin} \)
• \( \kappa p_s q / g \gg 1 \implies \text{atmosphere optically thick} \)
Greenhouse gas (G.G.): Optically thick case \( \kappa p_s q / g \gg 1 \)

\[ \Delta p_1 \]

OLR escapes only from top slab

OLR determined by \( T_3 \)

\( \Delta p_1 = p_{rad} \) characterizes the pressure at which OLR escapes

\[
\text{OLR} \approx \sigma T (p_{rad})^4 < \sigma T_{eq}^4
\]

Figure from R. Pierrehumbert, Planetary radiation and planetary temperature, *Physics Today*, January 2011, 33-38.
Greenhouse gas (G.G.): Basic ideas

“In some sense, the whole subject of climate comes down to an ever-more sophisticated hierarchy of calculations of the curve $\text{OLR}(T_s)$.”

--R. Pierrehumbert, PPC, p. 146

(solar constant $1370 \text{ W/m}^2$, albedo 0.3)

(PPC, p. 147)
Greenhouse gas (G.G.): Basic ideas

\[ I_s = \sigma T_0^4 \left[ 1 - m \tanh(19 T_0^6 \times 10^{-16}) \right] \]

atmospheric transmission factor

Fig. 1. A plot of the computed and observed values of the annual infrared emission to space in 10° latitude belts. The computed values were obtained from Eq. (6). The observed values are those given by Sellers (1965), modified slightly by recent satellite measurements.

Greenhouse gas (G.G.): Basic ideas

- $\sigma T_{rad}^4 = \frac{1}{4}(1 - \alpha)L_*$
- $p_{rad} \downarrow$ as G.G. \uparrow

(Figure from PPC, p. 147)
Greenhouse gas (G.G.): Basic ideas

The Earth’s observed zonal-mean OLR for January, 1986 (solid curve).

\[ \sigma T_s^4 \] (dashed curve).
Greenhouse gas (G.G.): Basic ideas

![Graph showing the effect of atmospheric composition on OLR](image_url)

**Figure 6.1.** The effect of atmospheric composition on OLR, with atmospheric and surface temperature held fixed at January climatological values, 1960–1980.

OLR spectrum: Toy example with one (fictitious) G.G. oobleck

\( p \)

\( T_{\text{strat}} \quad T_s \)

\( p = 0 \)

\( \kappa(n) \)

\( \kappa_0 \)

\( n \)

\( (T_s = 280 \text{ K, } T_{\text{strat}} = 200 \text{ K}) \)

(Figure from PPC, p. 218)
OLR spectrum: Toy example with one (fictitious) G.G. *em

(T_s = 280 K, T_{strat} = 200 K)

(Figure from PPC, p. 218)
Absorption coefficient for pure CO$_2$ at $T = 293$ K, $p = 1000$ mb

Infrared emission peak for Earth centered on $\sim n = 670$ cm$^{-1}$
Absorption coefficient for CO$_2$ at 1 bar and 300K (bottom). Corresponding OLR for the three concentrations (top).

“The rate at which absorption decays with distance determines the rate at which the OLR decreases as the greenhouse gas concentration is made larger.”

--R. Pierrehumbert, PPC, p. 219
The $OLR$ vs $CO_2$ concentration for $CO_2$ in a dry air atmosphere. $T_s = 273$K held fixed. This curve gives the amount of absorbed solar radiation needed to maintain the surface temperature at freezing.

$OLR$ goes down approximately in proportion to the logarithm of the $CO_2$ concentration.

(PPC, p. 265)
The left panel compares a computed global-mean, annual-mean emission spectrum for Earth (blue) with observations from the satellite-borne AIRS instrument (red); both are superimposed over a series of Planck distributions.

\[ R \frac{\partial T}{\partial t} = Qs(y)(1 - \alpha(y)) - (A + BT) - C \left( T - \int_0^1 T \, dy \right) \]

\( T \ 5^\circ \times 5^\circ \)

\( OLR \ 15^\circ \times 15^\circ \)

Fig. 2. Scatter plot of OLR versus surface temperature from 30\(^\circ\)N to 90\(^\circ\)N from the 10-year data set. The scale on the right indicates the percent of total. Note that there is a cosine of latitude weighting to account for the differing grid point areas.

Insolation = $OLR$

$Q = \sigma T^4$

$Q = 240 \Rightarrow T \approx 255K$

\[ Q = \sigma T^4 - \overline{G} = \sigma T^4 - 150 \Rightarrow T \approx 288K \]

Let time vary:

\[ R \frac{dT}{dt} = Q - OLR = Q - (\sigma T(t)^4 - G(t)) \]

\[ G(t) = \overline{G} + \beta \ln \left( \frac{C}{C_0} \right) \]

\[ C = C(t) = \text{atmospheric concentration of CO}_2 \text{ (ppm)} \]

\[ R \frac{dT}{dt} = Q - OLR = Q - (\sigma T(t)^4 - G(t)) \]

\[ G(t) = \bar{G} + \beta \ln \left( \frac{C}{C_0} \right) \]

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Linear approximation: \( T_0 = 273 \), \( T \) \( ^\circ \)C, \( T/T_0 \ll 1 \)

\[ T^4 = (T_0 + T)^4 = T_0^4 (1 + T/T_0)^4 \approx T_0^4 (1 + 4T/T_0) \]

\[ OLR = \sigma T^4 - G \approx \sigma T_0^4 + 4\sigma T_0^3 T - \left( \bar{G} + \beta \ln \left( \frac{C}{C_0} \right) \right) = BT + A, \]

\[ A = A(t) = \sigma T_0^4 - \left( \bar{G} + \beta \ln \left( \frac{C(t)}{C_0} \right) \right) \]

Basic equations for radiative flux*

\[ dI^\uparrow = -\kappa \rho_r I^\uparrow \sec \theta \, dz + \kappa \rho_r I_b \sec \theta \, dz \]


\[ z \text{ height above surface} \]
\[ \kappa \text{ absorption coefficient} \]
\[ \rho_r \text{ density of radiating gas} \]
\[ I_b \text{ blackbody radiation flux from Planck} \]

Energy absorbed from the incoming beam by the absorbing gas.
Energy emitted by the layer of gas.

- assume: given \( z, T \) and \( p \) are known
- consider \( \kappa = \kappa(\nu, z) \)
- consider \( I_b = I_b(\nu, z) \)
Basic equations for radiative flux

\[ dI^\uparrow = -\kappa \rho_r I^\uparrow \sec \theta \, dz + \kappa \rho_r I_b \sec \theta \, dz \]

new independent variable: mass of radiating gas per unit area

\[ u = \int_z^\infty \rho_r \, dz = \int_z^\infty c \rho \, dz, \quad du = -\rho_r \, dz \]

\( \rho \) total density of atmosphere at height \( z \)
\( c = \rho_r / \rho \) fractional concentration of radiating gas
\( u = 0 \) at t.o.a

\[ dI^\uparrow = \kappa I^\uparrow \sec \theta \, du - \kappa I_b \sec \theta \, du \]

- \( \kappa = \kappa(u) \)
- \( I_b = I_b(u) \)
Basic equations for radiative flux

\[ dI^\uparrow = \kappa I^\uparrow \sec \theta du - \kappa I_b \sec \theta du \]

\[ u = \int_z^\infty \rho_r dz = \int_z^\infty c \rho dz \]

Transmission function between “heights” \( u_0 < u_1 \):

\[ \tau(u_0, u_1) = \exp \left( -\sec \theta \int_{u_0}^{u_1} \kappa du \right) \]

- \( \kappa \approx 0 \Rightarrow \tau \approx 1 \)
- \( \kappa \gg 1 \Rightarrow \tau \approx 0 \)

Solution:

\[ I^\uparrow(u) = I^\uparrow(u_1) \tau(u, u_1) + \sec \theta \int_u^{u_1} \kappa(v) I_b(v) \tau(u, v) dv \]

upward flux at the lower boundary layer \( u_1 \) times the transmission between \( u \) and \( u_1 \)

blackbody flux emitted by each layer of gas between \( u \) and \( u_1 \) times the transmission between \( u \) and the height of the emitting layer
Basic equations for radiative flux

Solution: \[ I_\uparrow(u) = I_\uparrow(u_1)\tau(u, u_1) + \sec \theta \int_{u}^{u_1} \kappa(v)I_b(v)\tau(u, v)dv \]

Remove explicit dependence on absorption coefficient: Integrate by parts!

\[ I_\uparrow(u) = I_\uparrow(u_1)\tau(u, u_1) + I_b(u) - I_b(u_1)\tau(u, u_1) + \int_{u}^{u_1} \tau(u, v)\frac{dI_b(v)}{dv}dv \]

Downward flux: \[ dI_\downarrow = -\kappa I_\downarrow \sec \theta du + \kappa I_b \sec \theta du \]

\[ I_\downarrow(u) = I_\downarrow(u_0)\tau(u_0, u) + I_b(u) - I_b(u_0)\tau(u_0, u) - \int_{u_0}^{u} \tau(v, u)\frac{dI_b(v)}{dv}dv \]
Basic equations for radiative flux

\[ I^\uparrow(u) = I^\uparrow(u_1)\tau(u, u_1) + I_b(u) - I_b(u_1)\tau(u, u_1) + \int_u^{u_1} \tau(u, v) \frac{dI_b(v)}{dv} dv \]

\[ I^\downarrow(u) = I^\downarrow(u_0)\tau(u_0, u) + I_b(u) - I_b(u_0)\tau(u_0, u) - \int_{u_0}^{u} \tau(v, u) \frac{dI_b(v)}{dv} dv \]

\( u_1 \) value of \( u \) at lower boundary (usually the surface)

\( u_0 \) value of \( u \) at upper boundary (usually t.o.a, so \( u_0 = 0 \))

Approximation: \( I^\downarrow(0) = 0 \) \( I^\uparrow(u_1) = I_b(u_1) \)

- solar infrared flux that reaches Earth very small relative to OLR
- Earth’s surface radiates as (almost) perfect blackbody in the infrared
Basic equations for radiative flux

\[
I^{\uparrow}(u) = I_b(u) + \int_u^{u_1} \tau(u, v) \frac{dI_b(v)}{dv} dv
\]

\[
I^{\downarrow}(u) = I_b(u) - I_b(0) \tau(0, u) - \int_0^u \tau(v, u) \frac{dI_b(v)}{dv} dv
\]

Infrared radiation does not come from a single level—a bit is contributed from each level (each having its own $T$).
A bit of this is absorbed at each intervening level of the atmosphere.
Radiation is emitted in all directions.
Rate of emission and absorption strongly depends on frequency.
Several greenhouse gases.

Currently, Earth receives about 1 W/m$^2$ more from solar absorption than it emits to space as infrared (due to rapid rise of greenhouse gases).

Earth’s temperature has not yet risen enough to restore the energy balance: takes time to first warm up the oceans and melt the ice. (PPC, p. 152)