**Budyko’s Model**

**Budyko’s Equation**

\[ \frac{dT}{dt} = f + LT \]

If \( X \) is a decent space (e.g. \( L^1([0,1]) \)), then \( L \) is a continuous linear operator on \( X \).

Equilibrium solution:
\[ T = -L^{-1}f \]

if \( L \) is invertible.

Stability depends on the spectrum of \( L \).

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**Linear Operator**

\[ LT(y) = \int_0^1 T(y(bz)) - (B + CT) \, dz \]

Let
\[ X_1 = \{ f : [0,1] \to \mathbb{R} \mid f \text{ is constant} \} \]

Then
\[ X = X_1 \oplus X_2 \]

\[ LT = \begin{pmatrix} 0 & (B + CT) \\ -B & -B \end{pmatrix}, \text{ for } T = X_1 \]

\[ LT = \begin{pmatrix} 0 & -B \\ 0 & -B \end{pmatrix}, \text{ for } T = X_2 \]

\( L \) is invertible.

**Spectrum**

- \( -B \) is an eigenvalue, with eigenspace \( X_1 \) (dimension 1)
- \( -(B + C) \) is an eigenvalue, with eigenspace \( X_2 \) (codimension 1)

The equilibrium solution is stable.
Budyko’s Model

Equilibrium

\[ T'(y) = T + \alpha', \]

where \( T = \mathcal{T}(B, \ T') \cdot \alpha' = \varphi'(B + C) \)

Solution:

Recall: \( f(\alpha) = \int_a^b \alpha(x) \, dx = \mathcal{T} \), so

\[ T = \mathcal{T} \left( \frac{\sigma}{B} \right) - \Delta, \]

where \( \Delta = \int_a^b \alpha(x) \, dx \), so

\[ T = \mathcal{T} \left( \frac{\sigma}{B} \right) - \Delta. \]

Also, \( \alpha = f - \mathcal{T}^{\mathcal{T}(B)} \cdot \alpha \cdot \left[ \mathcal{T}^{\mathcal{T}(B)} \right] \cdot \beta \mathcal{T}^{\mathcal{T}(B)} \)

\[ T'(y) = \frac{\mathcal{T}'(y)}{B} - \left( \frac{\int_0^y \alpha^k(x) \, dx}{B} \right) \]

approached with exponential rate \(-BR\)

approached with exponential rate \(-(B+C)R\)

Budyko’s Model

Ice Albedo Feedback

So far, we have assumed that the albedo function \( \alpha(y) \) is a fixed function.

What if the albedo changes as the temperature changes?

Standard assumption: Permanent ice forms if the annual average temperature is below \( T_y = -10 \degree C \) and melts if the annual average temperature is above \( T_y \).

Possible albedo function:

\[ \alpha : [0,1] \times [0,1] : \alpha(y, T) = \begin{cases} a_1 & \text{if } T(x) > T_y, \\ a_2 & \text{if } T(x) < T_y, \end{cases} \]

\[ \frac{dT}{dy} = \mathcal{Q} (\alpha(x) \mathcal{T}(y, T)) / (A + BT) + CT \]

Big Problem: This equation is no longer linear.

Bigger Problem: Without additional assumptions, the solutions are ridiculous.

Budyko’s Model

Ice Albedo Feedback

What if the albedo changes as the temperature changes?

Better assumption: There is a single ice line at \( y = y \) between the equator and the poles. The albedo is \( a_1 \) below the ice line and \( a_2 \) above it:

\[ \alpha(y, T) = \begin{cases} a_1 & y < y, \\ a_2 & y > y. \end{cases} \]

\[ \frac{dT}{dy} = \mathcal{Q} (\alpha(x) \mathcal{T}(y, T)) / (A + BT) + CT \]

For fixed \( y \), this is just the linear equation we already analyzed.

How do we determine \( y \)?
Budyko's Model

Ice Albedo Feedback

\[ \frac{dT}{dT} = Q(y)(1-\alpha(y)) - (4 + BT) + C(T - T) \]

For each fixed \( \gamma \), there is a stable equilibrium for Budyko's equation.

Additional condition: The average temperature across the ice boundary is the critical temperature \( \bar{T} \),

\[ \frac{1}{2}(\bar{T}(y+) + \bar{T}(y-)) = \bar{T} = -10 \]

Budyko's Model

Ice Albedo Feedback

\[ \frac{dT}{dT} = Q(y)(1-\alpha(y)) - (4 + BT) + C(T - T) \]

The additional condition \( \frac{1}{2}(\bar{T}(y+) + \bar{T}(y-)) = \bar{T} = -10 \)

can be written:

\[ \bar{\alpha}(\bar{\gamma}) = \frac{\bar{Q}(\bar{\gamma}(1-\alpha(y)) + \bar{d}(\bar{\gamma},\bar{\alpha}))}{\bar{T}} \]

Two equilibria (zeros of \( \bar{\alpha} \)) satisfy the additional condition.

Budyko’s Model

Dynamics of the Ice Line

\[ \frac{d\tilde{\gamma}}{dt} = h(T(\tilde{\gamma}) - T) \]

If the average temperature across the ice line is above the critical temperature, some ice will melt, moving the ice line toward the pole. If it is below the critical temperature, the ice will advance toward the equator.

Budyko’s Model

Budyko-Widiasih Model

\[ \frac{d\tilde{\gamma}}{dt} = \alpha h(\tilde{\gamma}) \]

Widiasih's equation:

\[ \frac{d\tilde{\gamma}}{dt} = \alpha h(\tilde{\gamma}) \]

State space: \([0,1] \times X\]

Widiasih's Theorem. For sufficiently small \( \gamma \), the system has an attracting invariant curve given by the graph of a function \( \Phi : [0,1] \rightarrow X \). On this curve, the dynamics are approximated by the equation

\[ \frac{d\tilde{\gamma}}{dt} = \alpha h(\tilde{\gamma}) \]
Budyko’s Model
Summary

Next Week:
An approximation yielding a simplified proof of Widiasih’s Theorem.

But First:
Fun with Budyko!

What about the greenhouse effect?

\( A + BT \) is the outgoing long wave radiation. This term decreases if the greenhouse gases increase. We view \( A \) as a parameter.

What if \( \eta \) were here?

MCRN: MCKN Paleocean equation

New system:

\[ \frac{dA}{dt} = e^{-(q - \eta)} \]

What if \( \eta \) were here?

Budyko’s Model
Budyko-Widiasih Model

\[ \frac{dH}{dt} = eh(\eta, A) \]

isoeline \( h(\eta, A) = 0 \)

Budyko-Widiasih-Paleocarbon Model

\[ \frac{dH}{dt} = eh(\eta, A), \frac{dA}{dt} = e^{-(q - \eta)} \]

stable rest point

MCRN: MCKN Paleocean equation

what if \( \eta \) were here?
The snowball Earth hypothesis is not universally accepted. Complex ocean-dwelling photosynthetic organisms appear to have survived through the time when the snowball would have occurred. Perhaps the equator never fully froze over, leaving a belt of open water covering the equator. Hence "water belt" planet, aka "Jormungand".

Budyko's Model
Water Belt Model

\[ \frac{d\eta}{dA} = c \eta (\eta - \xi) \]

Banded Iron Formations?

What if \( \eta \) were here?

Budyko's Equation

Next Time
An approximation yielding a simplified proof of Widiasih's Theorem.