Dynamic Oscillations in Paleoclimate Theory.

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Earth’s History

Over the last 4.5 millions years there have been oscillations of Earth’s temperature over time. Approximately 1.2 million years ago, Earth moved through a transition into longer and deeper glacial-interglacial cycles.
Glacial-Interglacial Cycles

We can use ice core data (here from Vostok) to learn more about the last 800 kyrs.

Despite the eccentricity signal being weak during this time period, the $\delta^{18}O$ signal shows a strong 100,000 year frequency signal.
Using Oscillators to explain Pleistocene

Definitions

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- slow-fast dynamics
- homoclinic orbit
- a unstable focus.
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**Excitability** An excitable system has a stable fixed point- but external forcing can *excite* the system can cause a excursion away from and back to the stable fixed point.

**Note** There is sometimes a nice connection between excitable systems and slow/fast relaxation oscillators through a particular parameter. (Example forthcoming)
(A) Oscillator Structured around a Slow Manifold

The first example we’ll consider is a coupled system of two variables.

\[ \begin{align*}
\dot{a} &= b - \frac{1}{40}(a^3 - 25a^2 + 80) = b - \frac{1}{40}(a(a - 5)(a + 5)) + 2 \\
\dot{b} &= \epsilon(a_c - a)
\end{align*} \] 

for \( 0 < \epsilon \ll 1 \)

This is a slow/fast system. \( a \) is the fast variable while \( b \) is the slow variable. We will analyse this system by first finding the null-clines of each variable and plotting the phase portrait of the system.
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\] (1)

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This is a slow/fast system. \(a\) is the fast variable while \(b\) is the slow variable. We will analyse this system by first finding the null-clines of each variable and plotting the phase portrait of the system.

MCRN Webinar: Fast/Slow Dynamics on Thursdays at 1pm eastern time.
Fast/Slow Stable System
Fast/Slow Stable System

\[
a' = b - \frac{1}{40} (a^3 - 25a + 80) \\
b' = \text{Epsilon} (Ac - a)
\]

Epsilon = 0.01
Ac = 4
Fast/Slow Relaxation Oscillator
Fast/Slow Relaxation Oscillator

\[ a' = b - \frac{1}{40}(a^3 - 25a + 80) \]
\[ b' = \text{Epsilon} \cdot (A_c - a) \]

Epsilon = 0.01
A_c = 1

Cursor position: (-7.47, 2.06)
(D) Fast/Slow Excitable System
(C) Relaxation Oscillator emanating from a focus

Van der Pol Equations

\[ \ddot{x} + \mu(x^2 - a)\dot{x} + x = 0 \]  \hspace{1cm} (2)

OR

\[ \begin{align*}
\dot{x} &= y \\
\dot{y} &= -x + \mu y(a - x^2)
\end{align*} \]  \hspace{1cm} (3)
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2D limit cycle can’t exist in

- simple harmonic oscillator
- gradient system (\( \dot{x} = -\nabla V(x) \))
- Lyapunov systems
When do limit cycles exist in 2D?

**Theorem**

**Poincare-Bendixson Theorem** If the trajectory is confined to a compact region, \( R \), and there are no fixed points in \( R \) then there exists a limit cycle, \( C \), somewhere in \( R \).
(B) Oscillations with unstable homoclinic Orbit

The best common example of something like this that I could find was Smale’s Horseshoe map. But that’s too complicated to discuss here. There are also examples in three species biological models.
Back to the Pleistocene: 4 Models for glacial-interglacial

1. Saltzman
2. Palliard
3. Palliard-Parrenin
4. Crucifix
Saltzman and Maasch Model

\( M = \) Milankovitch forcing at 65°N at summer solstice.
\( X = \) ice volume
\( Y = \) atmospheric CO\(_2\).
\( Z = \) North Atlantic Deepwater Formation.
\( \dot{} = \) time derivative.
All variables are deviations from the mean.

\[
\begin{align*}
\dot{X} & = -X - Y - uM(t) \\
\dot{Y} & = -pZ + rY + sZ^2 - Z^2Y \\
\dot{Z} & = -q(X + Z)
\end{align*}
\] (4)
Saltzman and Maasch Model

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\]

Notes: Maasch and Saltzman show there exists a parameter shift which induces a change from a stable equilibrium solution to 100kyr cycles. The parameter shift is

\[ p = 0.8 \rightarrow 1, \quad q = 1.2, \quad r = 0.7 \rightarrow 0.8, \quad s = 0.8, \quad \text{and} \quad u = 0.7 \]

where \( p \) and \( r \) vary linearly in time.
Saltzman and Maasch

\[
\begin{align*}
\frac{dI}{dt} &= \alpha_1 - (c\alpha_2)\mu - \alpha_3 I - k_0\alpha_2 \theta - k_R\alpha_2 F_I(t) \\
\frac{d\mu}{dt} &= \beta_1 - (\beta_2 - \beta_3 \mu + \beta_4 \theta^2)\mu - (\beta_5 - \beta_6 \theta)\theta + F_\mu(t) \\
\frac{d\theta}{dt} &= \gamma_1 - \gamma_2 I - \gamma_3 \theta
\end{align*}
\] (SM90)

and

\[
\begin{align*}
\frac{dI}{dt} &= \alpha_1 - (c\alpha_2)\mu - \alpha_3 I - k_0\alpha_2 \theta - k_R\alpha_2 F_I(t) \\
\frac{d\mu}{dt} &= \beta_1 - (\beta_2 - \beta_3 \mu + \beta_4 \mu^2)\mu - \beta_5 \theta + F_\mu(t) \\
\frac{d\theta}{dt} &= \gamma_1 - \gamma_2 I - \gamma_3 \theta
\end{align*}
\] (SM91)

Notes Crucifix argues that this model uses variations in $F_\mu$ to capture the Mid-Pliestocene Transition. Perhaps this is what Saltzman eventually presents in his books. But it is not what is presented in the 1990 paper.
Paillard’s ice age model

\[
\frac{dx}{dt} = \frac{x_R(y) - x}{\tau_R(y)} - \frac{F(t)}{\tau_f}
\]

\(y\) = discrete variable. \(i \rightarrow g \rightarrow G\). \(i\)-Interglacial. \(g\)-mild glacial. \(G\)-deep glacial. Transitions are determined by the values of the astronomical forcing, \(F(t)\), or the value of \(x\).

\(x_R(y)\) and \(\tau_R(y)\) = characteristic relaxation values and time constants respectively. They depend on the discrete value of \(y\).
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Notes

- Physical meaning of \(y\) has to do with the Atlantic ocean circulation state. Deep sinking water, intermediate over-turning and shut-down of circulation.
- Contains concepts of fast/slow, but isn’t an oscillator b/c shift from \(g\) to \(G\) is due to external forcing.
- Mid-Pleistocene transition is show to exist by adding to the tectonic forcing terms.

Samantha Oestreicher (UMN)
\[ \frac{dI}{dt} = \frac{1}{\tau_I} (-a\mu - bF(t) + c - I) \]

\[ \frac{dA}{dt} = \frac{1}{\tau_A} (I - A) \]

\[ \frac{d\mu}{dt} = \frac{1}{\tau_\mu} (dF(t) - eI + fH(-D) + g - \mu) \]

\[ D = hI - iA + j \]

**I** = Ice Volume  
**A** = Antarctic continental ice sheet  
**\mu** = atmospheric CO\textsubscript{2}.  
**\tau_\text{A}** and **\tau_I** are slow time constants.  
**\tau_\mu** is fast time constant. **H(−D)** is a Heaviside function.
Palliard-Parrenin

\[
\begin{align*}
\frac{dI}{dt} &= \frac{1}{\tau_I}(-a\mu - bF(t) + c - I) \\
\frac{dA}{dt} &= \frac{1}{\tau_A}(I - A) \\
\frac{d\mu}{dt} &= \frac{1}{\tau_\mu}(dF(t) - eI + fH(-D) + g - \mu) \\
D &= hI - iA + j
\end{align*}
\]

**I** = Ice Volume

**\(A\) = Antarctic continental ice sheet

**\(\mu\) = atmospheric CO\(_2\).**

\(\tau_A\) and \(\tau_I\) are slow time constants.

\(\tau_\mu\) is fast time constant. \(H(-D)\) is a Heaviside function.

**Note:** \(H\) physically representing ventilation in Southern ocean. CO\(_2\) is released into atmosphere when \(D < 0\), which drives deglaciation.
Oscillations are structurally created by sub-critical Hopf bifurcations.

As a result we get a relaxation oscillator that is structured around an unstable equilibrium point without being a fast/slow system.
Crucifix’s VDP model

The Van der Pol Equation we considered earlier:

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -x + \mu y(a - x^2)
\end{align*}
\]  

Crucifix’s Alteration of Van der Pol’s Equations:

\[
\begin{align*}
\frac{dx}{dt} &= (-y + \beta + \gamma F(t))/\tau \\
\frac{dy}{dt} &= -\alpha(y^3/3 - y - x)/\tau
\end{align*}
\]
Crucifix’s VDP model

The Van der Pol Equation we considered earlier:

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\dot{x} &= y \\
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\end{align*}\] (5)

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\[
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\end{align*}
\]

Notes

- slow manifold \( x = \frac{y^3}{3} - y \).
- \( \beta \) controls position of fixed-point on the slow manifold (and so the ratio of times spent in glacial or interglacial).
- designed to challenge the arguments about predictability of ice ages.
Crucifix VDP vs. VDP

The details of the equation and the size of the parameters makes a big difference in what kind of oscillator we see. There is, perhaps, a close relationship between fast/slow oscillators and sub-critical Hopfs bifurcation oscillators?
**Stochastic effects**

- “weak stochastic forcing on an oscillator causes a fading out of the memory of exact initial conditions” - Crucifix pg 16
- This happens a lot with neutral stability in a free oscillator.
- “Stochastic forcing disperses the system states around the different attractors that are compatible with the forcing.” - Crucifix pg 17
Stochastic effects

A. noise on excitable slow-fast system

Stochastic forcing may excite an excitable system.

B. noise on oscillating slow-fast system

Or delay oscillating slow-fast systems.
My Favorite Conclusions

Question
“Can dynamical systems be used for inference on paleoclimates?”

Challenge
“The modeller’s challenge is therefore to operate a model selection on more stringent criteria than just fitting some standard time series.”

Conclusion…?
“Whether the process of inference with simple dynamical systems on paleoclimate data will lead new insight in [paleoclimate understanding and modelling] still needs to be demonstrated”