Ocean Circulation Box Models

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Outline

- What are Box Models?
- Stommel's Two Box Model
- Welander's Model
- Cessi's Model

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(And why do we use them?)

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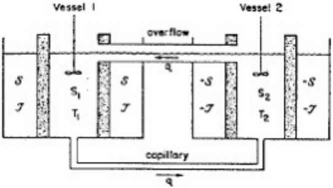
- The point is to look at large scale, long term behavior, as opposed to detailed behavior

Stommel's Two Box Model



- Two boxes are connected via a capillary tube through which water is allowed to flow. To preserve volume, there is also an overflow.

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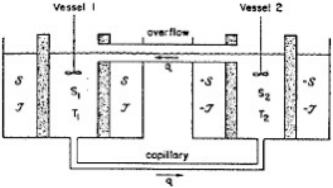

- Two boxes are connected via a capillary tube through which water is allowed to flow. To preserve volume, there is also an overflow.
- The tanks are also connected to large "oceans" via porous walls, through which water can diffuse.

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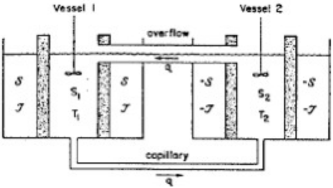

- Flow through the capillary is determined by density differences: $kq = \rho_1 - \rho_2$, i.e. $q > 0$ if the flow goes from tank 1 to tank 2, and $q < 0$ otherwise.

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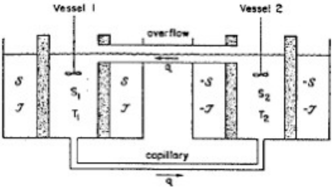

- Flow through the capillary is determined by density differences: $kq = \rho_1 - \rho_2$, i.e. $q > 0$ if the flow goes from tank 1 to tank 2, and $q < 0$ otherwise.
- Density obeys the equation of state $\rho = \rho_0(1 - \alpha T + \beta S)$, where T is temperature and S is salinity.

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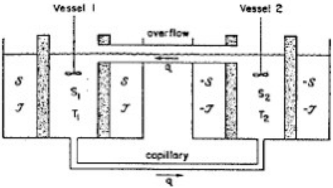

$$\begin{aligned}\frac{dT_1}{dt} &= c(\mathcal{T} - T_1) - |q|T_1 + |q|T_2 \\ \frac{dT_2}{dt} &= c(-\mathcal{T} - T_2) + |q|T_1 - |q|T_2 \\ \frac{dS_1}{dt} &= d(\mathcal{S} - S_1) - |q|S_1 + |q|S_2 \\ \frac{dS_2}{dt} &= d(-\mathcal{S} - S_2) + |q|S_1 - |q|S_2\end{aligned}$$

Stommel's Two Box Model

The symmetry of the system suggests we should look at solutions where $T_1 = -T_2$, and $S_1 = -S_2$.
Let $z = T_1 + T_2$.

$$\frac{dz}{dt} = c\mathcal{T} - c\mathcal{T} - c(T_1 + T_2) - |q|(T_1 + T_2) + |q|(T_1 + T_2) = -cz$$

So, the $T_1 = -T_2$ is invariant and attracting, so the reduction is legitimate.

Stommel's Two Box Model

Doing this dimensional reduction we get

$$\begin{aligned} \frac{dT}{dt} &= c(\mathcal{T} - T) - 2|q|T \\ \frac{dS}{dt} &= d(\mathcal{S} - S) - 2|q|S \end{aligned}$$

Let $\tau = ct$, $\delta = \frac{d}{c}$, $y = \frac{T}{\mathcal{T}}$, and $x = \frac{S}{\mathcal{S}}$.
The system becomes:

$$\begin{aligned} \frac{dy}{d\tau} &= 1 - y - |f|y \\ \frac{dx}{d\tau} &= \delta(1 - x) - |f|x \end{aligned}$$

where f is the nondimensionalized flow:

$$\lambda f = -y + Rx, \quad R = \frac{\beta \mathcal{S}}{\alpha \mathcal{T}}$$

$$f = \frac{2q}{c}, \quad \lambda = \frac{ck}{4\rho_0 \alpha \mathcal{T}}$$

Stommel's Two Box Model

Equilibrium solutions occur at

$$\begin{aligned} x &= \frac{1}{1+\frac{|f|}{\delta}} \\ y &= \frac{1}{1+|f|} \end{aligned}$$

So, we have

$$\lambda f = -y + Rx = -\frac{1}{1+|f|} + \frac{R}{1+\frac{|f|}{\delta}} = \phi(f, R, \delta)$$

The existence of multiple equilibria depend on λ, R, and δ. For certain R and δ, it is possible to have three equilibrium solutions.

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The existence of multiple equilibria depend on λ, R, and δ. For certain R and δ, it is possible to have three equilibrium solutions. Linearizing the system around the equilibria can give stability conditions. Instead of doing this in generality, we give an example.

Stommel's Two Box Model

A sample solution:

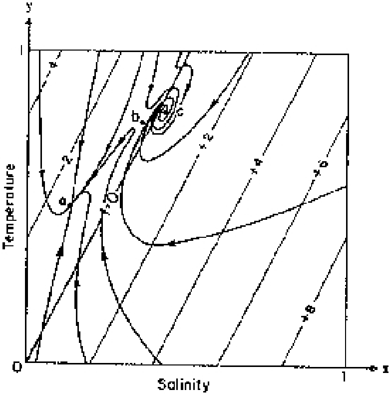

Fig. 7. The three equilibria a, b, and c for the two vessel convection experiment with $R = 2$, $\delta = 1/6$, $\lambda = 1/5$. A few sample integral curves are sketched to show the stable node *a*, the saddle *b*, and the stable spiral *c*.

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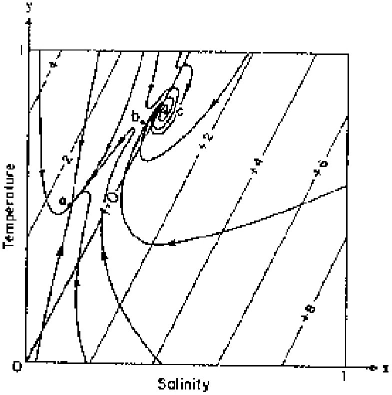

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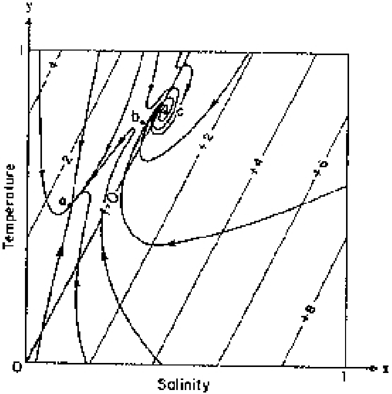

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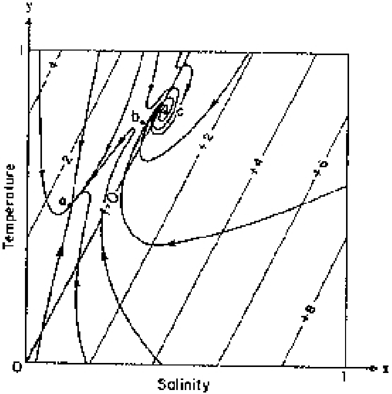
$R = 2$, $\delta = \frac{1}{6}$, $\lambda = \frac{1}{5}$.
Lines represent constant flow f.
The three equilibria can be calculated as $f = -1.1, -0.30, 0.23$. Two of the equilibria are stable, $f = -1.1$ and $f = 0.23$

Stommel's Two Box Model

Consider the two stable equilibrium points:



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a: $f < 0 \Rightarrow$ flow is from 2 to 1. This means flow from a lower temperature to a higher temperature, and from a lower salinity to a higher salinity.

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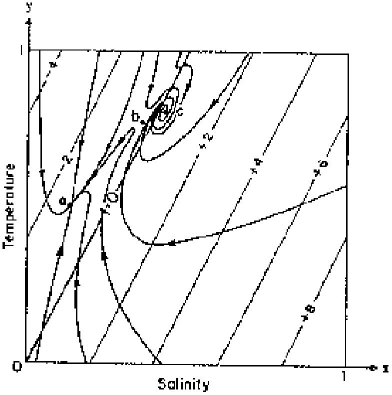

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c: $f > 0 \Rightarrow$ the flow is from 1 to 2. This implies salinity is the dominating effect.

Welander's Model

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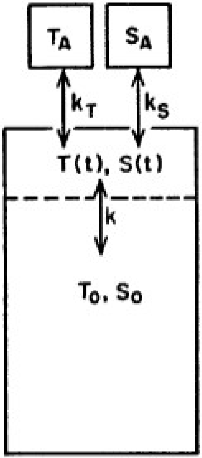
- What about oscillatory behavior? There have been several instances of periodic weakening and strengthening ocean circulation in the Earth's past.

Welander's Model



- The surface of the ocean is forced externally by the atmosphere

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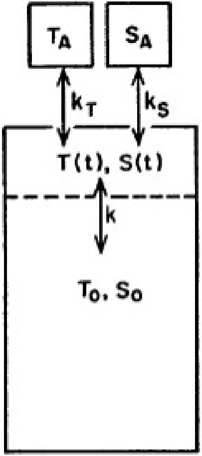

- The surface of the ocean is forced externally by the atmosphere
- The surface box is connected to the deep ocean through turbulent mixing described by the function $k \geq 0$, which depends on the density difference between the boxes.

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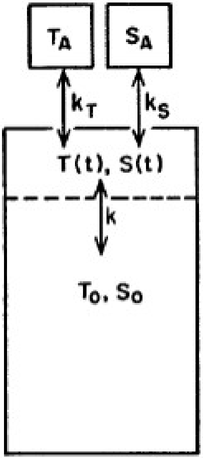

- The surface of the ocean is forced externally by the atmosphere
- The surface box is connected to the deep ocean through turbulent mixing described by the function $k \geq 0$, which depends on the density difference between the boxes.
- k is assumed to be large for positive $\Delta\rho$ and small for negative $\Delta\rho$. In reality, k is an incredibly complicated thing.

Welander's Model

We assume that $T_0 = S_0 = 0$, and that $\rho = -\alpha T + \beta S$, similar to the previous situation. The equations are then

$$\begin{aligned} \frac{dT}{dt} &= k_T(T_A - T) - k(\rho)T \\ \frac{dS}{dt} &= k_S(S_A - S) - k(\rho)S \end{aligned}$$

and equilibria occur at $\overline{T} = \frac{T_A k_T}{k_T + \overline{k}}$, $\overline{S} = \frac{k_S S_A}{k_S + \overline{k}}$, where $\overline{k} = k(\overline{\rho})$.

Welander's Model

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- This is a two dimensional system. Any rectangle with $|T| > T_A$ and $|S| > S_A$ is invariant under the flow.
- Therefore, if there exists only one equilibrium, and if it is unstable, Poincare-Bendixson allows us to conclude the existence of a periodic solution.

Welander's Model

The linearized system is

$$\begin{bmatrix} w' \\ z' \end{bmatrix} = \begin{bmatrix} -(k_T + k(\overline{\rho})) + \overline{T}\alpha k'(\overline{\rho}) & -\overline{T}\gamma k'(\overline{\rho}) \\ \overline{S}\alpha k'(\overline{\rho}) & -(k_S + k(\overline{\rho})) - \overline{S}\gamma k'(\overline{\rho}) \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix}$$

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We want an equilibrium that is completely unstable, so we'd like the trace of the matrix to be positive. The condition is

$$k_S + k_T + 2k + \overline{\rho}k' < 0$$

k_S and k_T can be chosen to satisfy this condition.

Welander's Model

We can demonstrate this with a specific example:
$k = \frac{2k_1}{\pi}[\arctan m(\rho + \rho_1) + \pi/2]$
Nondimensionalizing and letting

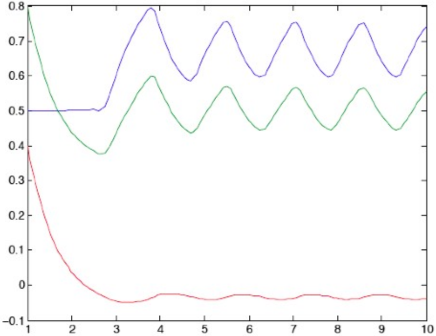
$$\frac{\alpha T_A}{\gamma S_A} = \frac{4}{5}, \quad \frac{k_T}{k_S} = 2, \quad \frac{k_1}{k_T} = \frac{1}{2}, \quad \frac{\rho_1}{\gamma S_A} = \frac{1}{30}, \quad m\gamma S_A = 500$$

We get the equations

$$\begin{aligned} \dot{T}^* &= 1 - T^* - k^*(\rho^*)T^* \\ \dot{S}^* &= 0.5(1 - S^*) - k^*(\rho^*)S^* \\ \rho^* &= -0.8T^* + S^* \end{aligned}$$

Welander's Model

The unstable equilibrium occurs at $T^* = 2/3$, $S^* = 1/2$, and $\rho^* = -1/30$. The periodic orbit can be found numerically:



The top line is temperature, the middle is salinity, and the bottom is density.

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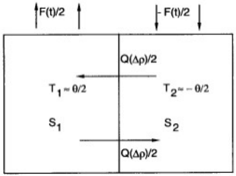
- However, it's also possible that changes in external forcing can cause a transition between the stable states found in Stommel's model.
- Cessi explores this idea by applying a time varying salinity forcing to Stommel's model.

Cessi's Model

Cessi's model is a variation on Stommel's two box model.



$$
\begin{aligned}
\rho/\rho_0 &= 1+\alpha_S(S-S_0)-\alpha_T(T-T_0)\\
\dot{T}_1 &= -t_r^{-1}\left(T_1-\tfrac{\theta}{2}\right)-\tfrac{1}{2}Q(\Delta\rho)(T_1-T_2)\\
\dot{T}_2 &= -t_r^{-1}\left(T_2+\tfrac{\theta}{2}\right)-\tfrac{1}{2}Q(\Delta\rho)(T_2-T_1)\\
\dot{S}_1 &= \tfrac{F(t)}{2H}S_0-\tfrac{1}{2}Q(\Delta\rho)(S_1-S_2)\\
\dot{S}_2 &= -\tfrac{F(t)}{2H}S_0-\tfrac{1}{2}Q(\Delta\rho)(S_2-S_1)
\end{aligned}
$$

F is a nonconstant salinity forcing.

Cessi's Model

We again reduce the dimension of the system, in the same way as in Stommel's model ($\Delta T = T_1 - T_2$, and $\Delta S = S_1 - S_2$):

$$\begin{aligned} \frac{d\Delta T}{dt} &= -t_r^{-1}(\Delta T - \theta) - Q(\Delta\rho)\Delta T \\ \frac{d\Delta S}{dt} &= \frac{F(t)}{H} S_0 - Q(\Delta\rho)\Delta S \end{aligned}$$

Nondimensionalize using the variables $x = \frac{\Delta T}{\theta}$, $y = \frac{\alpha_S \Delta S}{\alpha_T \theta}$, and $t = t_d t'$

Let $Q(\Delta\rho) = \frac{1}{t_d} + \frac{q}{v}(\Delta\rho)^2$ where t_d is the timescale for diffusion between the boxes.

The equations become

$$\begin{aligned} \dot{x} &= -\alpha(x-1) - x[1 + \mu^2(x-y)^2] \\ \dot{y} &= p(t) - y[1 + \mu^2(x-y)^2] \end{aligned}$$

Cessi's Model

$\alpha = t_d/t_r$ is a large parameter, as the diffusion time scale is much longer than the temperature relaxation time scale. So we can treat this as a fast-slow system to further reduce the dimension. Multiplying the top equation by epsilon, we get the standard slow equation:

$$\begin{aligned} \varepsilon\dot{x} &= -(x-1) - \varepsilon x[1+\mu^2(x-y)^2] \\ \dot{y} &= p(t) - y - \mu^2 y(1-y)^2 \end{aligned}$$

Changing time scales with $' = \frac{d}{d\tau}$ $(\varepsilon\tau = t)$, we get the fast equation:

$$\begin{aligned} x' &= -(x-1) - \varepsilon x[1+\mu^2(x-y)^2] \\ y' &= \varepsilon(p(t) - y - \mu^2 y(1-y)^2) \end{aligned}$$

The critical manifold is $x = 1$. When $\varepsilon = 0$, we immediately see that this manifold is normally hyperbolic. So we can reduce the dimension, and consider only the y equation:

$$\dot{y} = p(t) - y - \mu^2 y(1-y)^2$$

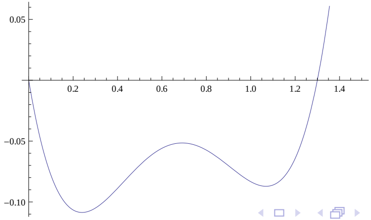
Cessi's Model

We assume that p is of the form $p(t) = \overline{p} + p'(t)$ with

$$p'(t) = \begin{cases} 0 & t \leq 0 \\ \Delta & 0 \leq t \leq \tau \\ 0 & t > \tau \end{cases}$$

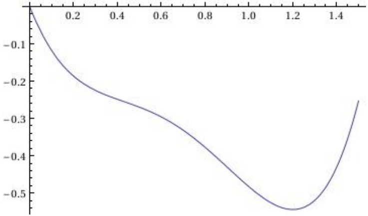
For any fixed time, the equation is of the form $\frac{dy}{dt} = -U'(y)$. In this case, $U(y)$ is a Lyapunov function in neighborhoods of the stable equilibria, as stable equilibria occur at minima of U. Integrating gives

$$U(y) = \frac{y^2}{2} + \mu^2\left(\frac{y^4}{4} - \frac{2}{3}y^3\right) + \frac{y^2}{2} - (\overline{p})y$$



Cessi's Model

After turning on the forcing, the potential function will take a new shape.

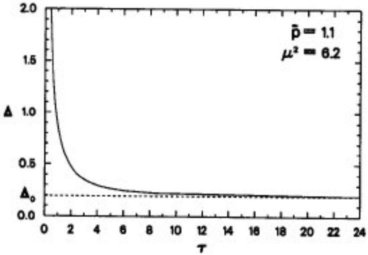


Once the potential changes, the system changes according to the integral

$$\int_{y_a}^{y} \frac{d\tilde{y}}{-[1+\mu^2(\tilde{y}-1)^2]\tilde{y}+\overline{p}+\Delta} = \int_0^{\tau} dt$$

Cessi's Model

A transition between the two stable states occurs when the potential barrier of the unstable critical point can be overcome. The ability of the system to transition will depend on the time over which the forcing is applied. For specific values of the parameters, we can numerically find this dependence.



Cessi's Model

This also shows that a critical amplitude is necessary for the transition.

$$\bar{p} + \Delta_0 = \frac{2}{3} + \frac{2}{27}\mu^2(\pm 1 + (1 - 3\mu^{-2})^{3/2})$$

This indicates that the total volume of freshwater is not the deciding factor in a transition. Instead, the rate at which the system is forced plays a critical role.

We conclude that a changing salinity forcing is capable of transitioning the system between the two stable states.

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- And more!