Existence and Uniqueness for a Steady State Algal Bloom Model

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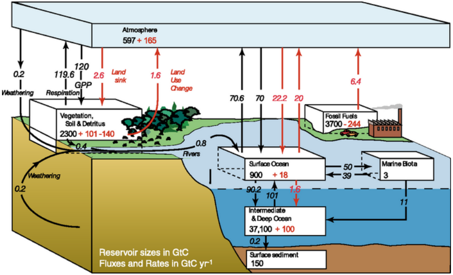
November 4, 2014

Motivation

- Carbon cycle includes exchange of CO_2 at the ocean-atmosphere interface
- Ocean absorbs carbon from the air (as CO_2), where it is a nutrient for floating plankton doing photosynthesis
- Carbon enters plankton. Perhaps a bigger fish eats the plankton. Either way, it ends up in the deep ocean as dead organisms.
- Surface layer of the ocean takes up almost half of the CO_2 produced by humans (maybe... and will it continue...)



Some chemistry:

- CO_2 is soluble in water
- Dissolved CO_2 reacts with water to form carbonic acid. This reaction is reversible.
- Whether the ocean surface takes up carbon or releases it depends on the CO_2 flux

$$F = k(p\mathrm{CO}_2^{oc} - pCO_2^{at}),$$

  (k is transfer coefficient). Negative means CO_2 is being taken up by the ocean.
- Once CO_2 is in the upper layer of the ocean, there are two mechanisms to transport it to the ocean's interior:
  1. Solubility pump via mixing ocean currents
  2. Biological pump:
     - ★ Begins with uptake of CO_2 by phytoplankton
     - ★ Organic carbon sinks as dead organisms or feces.
     - ★ There are processes to return the organic carbon to dissolved CO_2, but it happens more slowly, hence the biological pump is a carbon sink.

Algal Blooms

Phytoplankton need two things: light and high levels of nutrients

- Phytoplankon do well in coastal upwelling zones
- Particularly in freshwater, algal blooms occur from pollution runoff and are harmful to local ecosystem
- Typically only involve one (or a few) types of a phytoplankton species, and may discolor water
- Algal blooms are an indicator of climate change

Algal Bloom Model

Proposed by Klausmeier and Litchman (2001)

- Reduced **Nutrient-Phytoplankton-Zooplankton (NPZ) model**
- Assume: phytoplankton move passively, uniformly distributed horizontally, constant nutrients, death rate
- Lambert-Beer law for light
- Once plankton is dead, it sinks out of the system

$$\frac{\partial P}{\partial t}+u\frac{\partial P}{\partial x}+v\frac{\partial P}{\partial y}+(w+w_s)\frac{\partial P}{\partial z}=D_h\left(\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial y^2}\right)+D_v\frac{\partial^2 P}{\partial z^2}+\left(\frac{\partial P}{\partial t}\right)_{\text{bio}}$$

with

$$\begin{aligned}
\left(\frac{\partial P}{\partial t}\right)_{\text{bio}} &= f(I)g(N)P-h(P)Z-i(P)P \\
\left(\frac{\partial Z}{\partial t}\right)_{\text{bio}} &= \gamma h(P)Z-j(Z)Z \\
\left(\frac{\partial N}{\partial t}\right)_{\text{bio}} &= -f(I)g(N)P+(1-\gamma)h(P)Z+i(P)P+j(Z)Z
\end{aligned}$$

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- Once plankton is dead, it sinks out of the system

$$\frac{\partial P}{\partial t} + \cancel{u\frac{\partial P}{\partial x}} + \cancel{v\frac{\partial P}{\partial y}} + (w + w_s)\frac{\partial P}{\partial z} = D_h\cancel{\left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2}\right)} + D_v\frac{\partial^2 P}{\partial z^2} + \left(\frac{\partial P}{\partial t}\right)_{\text{bio}}$$

with

$$\left(\frac{\partial P}{\partial t}\right)_{\text{bio}} = f(I)\mathbf{g}P - \cancel{h(P)Z} - i(P)P$$

Algal Bloom Model

Advection-diffusion equation for $P(z,t)$:

$$\frac{\partial P}{\partial t} = D_v \frac{\partial^2 P}{\partial z^2} - (w + w_s)\frac{\partial P}{\partial z} + (gf(I) - i(P))\, P$$

BC's: Phytoplankton does not move across the top or bottom surface of the ocean:

$$D_v \frac{\partial P}{\partial z} - (w + w_s)P = 0, \qquad z = 0, L.$$

Light availability: Lambert-Beer Law

$$f(I)(z,t) = I_0 \exp\left(-K_{bg} z - k \int_0^z P(y,t)\, dy\right).$$

Algal Bloom Model

Ebert, Arrayás, Temme and Sommeijer (2001): Rescale and nondimensionalize

$$t' = K_{bg}^2 D_v t, \qquad z' = K_{bg} z, \qquad , P'(z', t') = rP(z, t).$$

New advection-diffusion equation for $P(z, t)$:

$$\frac{\partial P}{\partial t} = \frac{\partial^2 P}{\partial z^2} - C \frac{\partial P}{\partial z} + A\left(j(P) - B\right) P$$

with BCs

$$\frac{\partial P}{\partial z} - CP = 0, \qquad z = 0, L.$$

with

$$J(P)(z, t) = \exp\left(-z - \int_0^z P(y, t)\, dy\right).$$

Parameters

- $0 < A < \infty$
- $0 < B < 1$
- $C \in \mathbb{R}$
- $0 < L < \infty$.

Existence and uniqueness

Consider the steady-state equation

$$\rho'' - C\rho' + A\left(e^{-z-\int_0^z \rho(y)\,dy} - B\right)\rho = 0 \tag{1}$$

with conditions

$$\left[\rho' - C\rho\right]_{z=0,L} = 0, \qquad \rho(z) \geq 0 \text{ for all } 0 \leq z \leq L. \tag{2}$$

Results:

- (Ebert, Arrayás, Temme and Sommeijer, 2001) There exists an $L^* > 0$ such that the BVP has a nontrivial solution for all $L < L^*$.
- (Huisman, Arrayás, Ebert, Sommeijer, 2002) solved equation numerically to show that under certain light conditions, the phytoplankton develops a stationary density profile.
- (Jones, M.) If L has a nontrivial solution, then it must be unique.

Proof of uniqueness

Steady state equation:

$$\rho'' - C\rho' + A\left(e^{-z-\int_0^z \rho(y)\,dy} - B\right)\rho = 0$$

Let

$$r(z) = e^{-z-\int_0^z \rho(y)\,dy} \tag{3}$$

so that

$$r'(z) = (-1 - \rho(z))\,r(z). \tag{4}$$

Facts:

1. $r(0) = 1$
2. $r(z)$ is monotone decreasing
3. For an individual solution, we may now view the boundaries $0 \leq z \leq L$ as moving from $r = 1$ to $r = r(L) \in (0, 1)$.

Proof of uniqueness

Equation:

$$\rho'' - C\rho' + A\left(e^{-z-\int_0^z \rho(y)\,dy} - B\right)\rho = 0$$

New expression:

$$r(z) = e^{-z-\int_0^z \rho(y)\,dy}$$

Let $q = \rho'$ to attain the following first-order system:

$$\rho' = q \tag{5}$$

$$q' = Cq - A(r - B)\rho \tag{6}$$

$$r' = -(1+\rho)r, \qquad ' = \frac{d}{dz} \tag{7}$$

BCs:

$$\left[\rho' - C\rho\right]_{z=0,L} = 0 \qquad \Longrightarrow \qquad [q - C\rho]_{r=1,r(L)} = 0$$

Proof of uniqueness

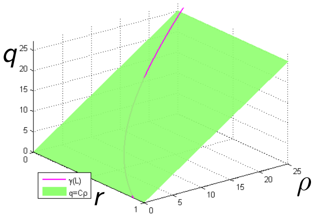
More about the BCs

BCs:

$$\left[\rho' - C\rho\right]_{z=0,L} = 0 \qquad \Longrightarrow \qquad [q - C\rho]_{r=1,r(L)} = 0$$

How to visualize this:

- Picture (ρ, q, r)-system, $0 \le r \le 1$
- $\rho' - C\rho = 0$ at $z = 0$ and $z = L$ is equivalent to a solution lying on the line $q = C\rho$ on the planes $\{r = 1\}$ and when $r = r(L)$
- For any solution, use a to refer to the initial condition of that solution so that $(\rho(0, a), q(0, a), r(0, a)) = (a, Ca, 1)$.

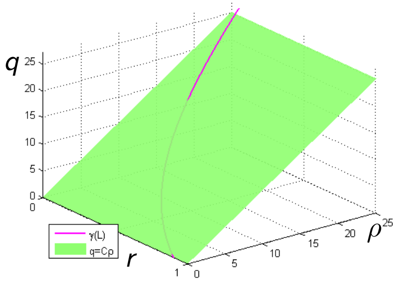


Proof of uniqueness

We can then define a curve formed by solutions to (5)-(7) at a chosen value of $x_0 \in [0, L]$ by

$$\gamma(z_0) = \{(\rho(z_0, a), q(z_0, a), r(z_0, a)) \ : \ a > 0\}. \tag{8}$$

Showing uniqueness is equivalent to showing that the curve $\gamma(L)$ intersects the plane $\{q = C\rho\}$ only once.



Proof of uniqueness

Tangent vector field along $\gamma(z)$:

$$(\delta\rho(z,\alpha), \delta q(z,\alpha), \delta r(z,\alpha)) := \left(\frac{\partial\rho(z,a)}{\partial a}\bigg|_{a=\alpha}, \frac{\partial q(z,a)}{\partial a}\bigg|_{a=\alpha}, \frac{\partial r(z,a)}{\partial a}\bigg|_{a=\alpha} \right).$$

The vector field $(\delta\rho, \delta q, \delta r)$ satisfies the equations of variations:

$$\begin{aligned} \delta\rho' &= \delta q \\ \delta q' &= C\,\delta q - A(r-B)\,\delta\rho - A\rho\,\delta r \\ \delta r' &= -r\,\delta\rho - (1+\rho)\,\delta r. \end{aligned}$$

Parametrize $\gamma(0)$ as $\gamma(0) = \{(a, Ca, 1) \;:\; a > 0\}$, so initial condition for $(\delta\rho, \delta q, \delta r)$ is $(1, C, 0)$.

Proof of uniqueness

Let $\zeta(z,a)$ be the third component of the cross-product

$$(\rho(z,a), q(z,a), r(z,a)) \times (\delta\rho(z,a), \delta q(z,a), \delta r(z,a)),$$

i.e.

$$\zeta(z,a) = \rho\,\delta q - q\,\delta\rho|_{(z,a)}.$$

Its derivative along a solution with respect to z is

$$\zeta'(z,a) = C\zeta(z,a) - A\rho^2(z,a)\,\delta r(z,a). \tag{9}$$

Linear differential equation with $\zeta(0) = 0$:

$$\zeta(z,a) = -Ae^{Cz}\int_0^z e^{-Cs}\rho^2(s,a)\,\delta r(s,a)\,ds. \tag{10}$$

Does $\zeta(z,a)$ have a fixed sign? The sign of ζ is determined by δr:

- if for all $s \in (0,z)$ we know $\delta r(s,a) > 0$, then $\zeta(z,a) < 0$
- **if for all $s \in (0,z)$ we know $\delta r(s,a) < 0$, then $\zeta(z,a) > 0$.**

Proof of uniqueness

Equation

$$\delta r' = -r\,\delta\rho - (1+\rho)\,\delta r \tag{11}$$

is linear and thus

$$\delta r(z,a) = -e^{-\int_0^z (1+\rho(y,a))\,dy} \int_0^z e^{\int_0^s (1+\rho(y,a))\,dy}\, r(s,a)\,\delta\rho(s,a)\,ds. \tag{12}$$

Since $r(z) = e^{-z-\int_0^z \rho(y)\,dy}$, this is

$$\delta r(z,a) = -r(z,a) \int_0^z \delta\rho(s,a)\,ds. \tag{13}$$

The sign of δr depends on the behavior of $\delta\rho$ in a manner similar to that of ζ and δr.

Proof of uniqueness

Facts:

$$\begin{aligned} \delta\rho' &= \delta q \\ \delta\rho(0,a) &= 1. \end{aligned}$$

If $\delta\rho(z_0,a)=0$ (z_0 is the first zero for ρ), then

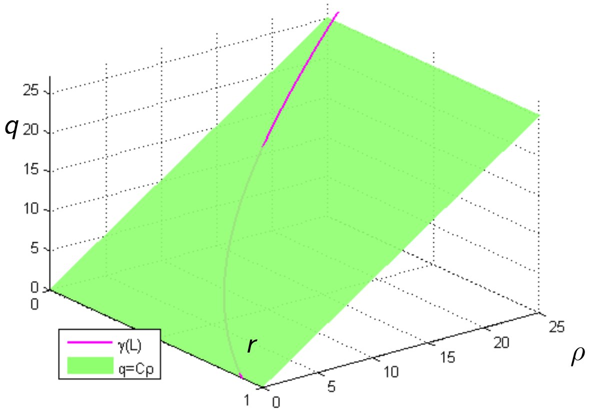
$$\delta r(z_0,a) = -r(z_0,a)\int_0^{z_0}\delta\rho(s,a)\,ds < 0. \tag{14}$$

Since $\delta q(z_0,a)\le 0$, for $\zeta(z_0,a)$:

$$\begin{aligned} \rho\,\delta q - q\,\delta\rho\big|_{z=z_0} &= -Ae^{Cz_0}\int_0^{z_0} e^{-Cs}\rho^2(s,a)\,\delta r(s,a)\,ds \\ \rho(z_0,a)\,\delta q(z_0,a) &= -Ae^{Cz_0}\int_0^{z_0} e^{-Cs}\rho^2(s,a)\,\delta r(s,a)\,ds \\ (-) &= (+) \end{aligned}$$

So for all $z\in(0,L)$, $\delta\rho(z,a)>0$. So $\delta r(z,a)<0$ for all $z\in(0,L)$, and consequently $\zeta(z,a)>0$.

Proof of uniqueness



For all $a \in (0, \alpha)$, we see $\rho' < C\rho$ when $z = L$, and for all $a > \alpha$, $\rho' > C\rho$ when $z = L$. (Parameter values are $A = 10$, $B = 0.5$, $C = 1$ and $L = 0.1$.)

Proof of uniqueness

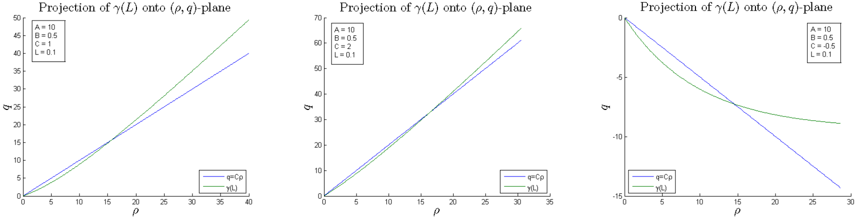


Figure: Three plots showing $\gamma(L)$ and $q = Cp$ for three different values of C.

This result on $\zeta(z, a)$ allows us to control how solutions intersect the plane $\{q = Cp\}$. In particular, projecting in the (p, q)-plane, the tangent vector $(\delta\rho(z, a), \delta q(z, a))$ must always point "up" to the region $q > Cp$. Hence for any choice of L, the curve $\gamma(L)$ can intersect $\{q = Cp\}$ transversely with the curve moving from $\{q < Cp\}$ to $\{q > Cp\}$ as the choice of initial condition a increases. Such an intersection is necessarily unique.

Conclusion: if we know the depth L, then we know there is only one possible density profile.

Desired conclusion: if we know the surface density, we know the depth.

Thank you

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