Generalized Hopf Bifurcation in a Nonsmooth Ocean Circulation Model

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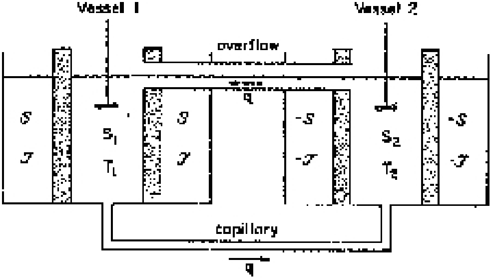
Outline

- Box Models
- The Model
- Some Nonsmooth Considerations
- The Nonsmooth Model
- Persistence of a Hopf Bifurcation

Box Models



Stommel, Henry. "Thermohaline convection with two stable regimes of flow." Tellus 13.2 (1961): 224-230.

Temperature and salinity control the flow of water between boxes. Essentially all practical ocean circulation models are box models of varying complexity.

Box Models: so what?

- Bistability
- Bistability with noise induced transitions
- Oscillations
- Oscillations moderated by noise
- Relaxation oscillation and canards
- Box models are in the purview of dynamical systems

Welander's Model

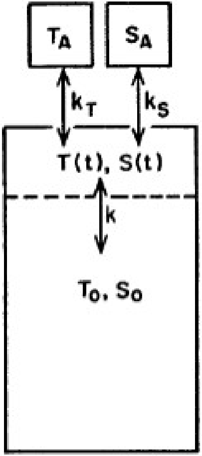
The Model:

$$\begin{aligned}\dot{T} &= k_T(T_A - T) - k(\rho)T \\ \dot{S} &= k_S(S_A - S) - k(\rho)S \\ \rho &= -\alpha T + \gamma S\end{aligned}$$

The nondimensionalized model:

$$\begin{aligned}\dot{T} &= 1 - T - k(\rho)T \\ \dot{S} &= \beta(1 - S) - k(\rho)S \\ \rho &= -\alpha T + S\end{aligned}$$

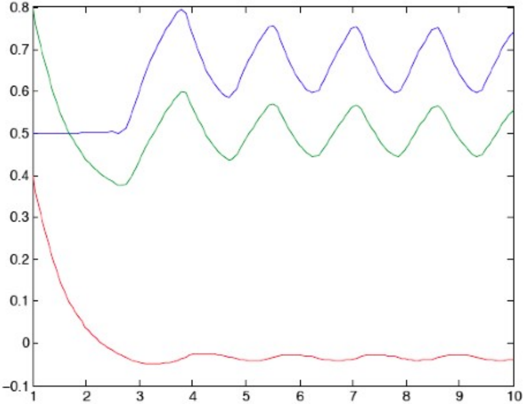
$$k(\rho) = \frac{1}{2}\left[\tanh\left(\frac{1}{a}(\rho - \varepsilon)\right) + 1\right]$$



Welander, Pierre. "A simple heat-salt oscillator." Dynamics of Atmospheres and Oceans 6.4 (1982): 233-242.

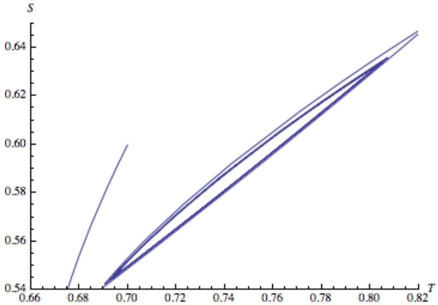
Welander's Model: so what?

Welander's model demonstrated oscillations in a 1 box convection model, which was new and exciting.



The Smooth Model

For certain values of ε, a periodic orbit exists. A Hopf Bifurcation occurs near $\varepsilon = 0$.



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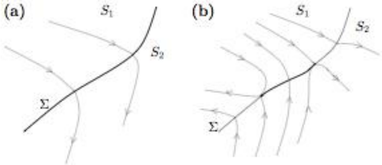
Some Nonsmooth Considerations

There is an intuitive way to piece together dynamics in a nonsmooth model with a line of discontinuity, which was originally formulated by Filippov.



Di Bernardo, Mario, et al. "Bifurcations in nonsmooth dynamical systems." SIAM review (2008): 629-701.

Taking a convex combination of the systems on either side of the discontinuity gives a way to find a flow in the sliding region.

Some Nonsmooth Considerations

The set up:
Consider the system

$$\dot{\mathbf{x}} = f(\mathbf{x}, \lambda)$$

with discontinuity boundary given defined by the zero set of a scalar function $h(\mathbf{x})$.

$$\lambda = \begin{cases} 1 & h(\mathbf{x}) > 0 \\ -1 & h(\mathbf{x}) < 0 \end{cases} \quad \text{On } h(\mathbf{x}) = 0,\ \lambda \in [-1, 1]$$

The standard Filippov formulation would be

$$\dot{\mathbf{x}} = \frac{1}{2}(1+\lambda)f^{+}(\mathbf{x}) + \frac{1}{2}(1-\lambda)f^{-}(\mathbf{x})$$

Some Nonsmooth Considerations

A sliding solution is defined as follows:

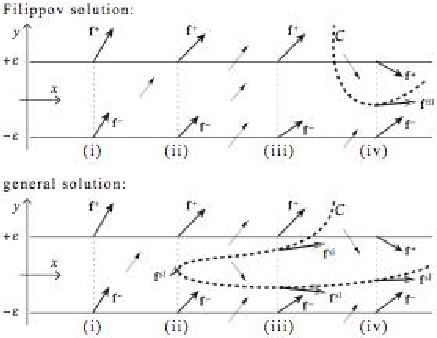
If

$$\begin{aligned} 0 &= f(\mathbf{x}, \lambda) \cdot \nabla h(\mathbf{x}) \\ 0 &= h(\mathbf{x}) \end{aligned}$$

can be solved for some $\lambda^* \in [-1, 1]$, then $\dot{\mathbf{x}} = f(\mathbf{x}, \lambda^*)$ defines a sliding solution of the system.

Note that no sliding solutions of the Filippov formulation exist in crossing regions.

Nonlinear Sliding (and why we don't like it)



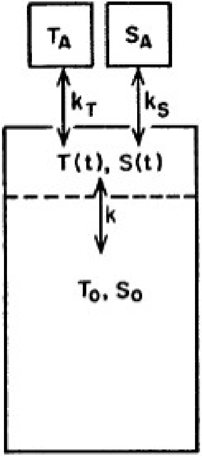
Filippov (maybe) proved in his book that a nonsmooth limit of a monotonic function does not have nonlinear sliding.

The Nonsmooth Model

The nonsmooth Welander model is

$$\begin{aligned}\dot{T} &= 1 - T - k(\rho)T \\ \dot{S} &= \beta(1-S) - k(\rho)S \\ \rho &= -\alpha T + S\end{aligned}$$

where $k(\rho) = \begin{cases} 1 & \rho > \varepsilon \\ 0 & \rho < \varepsilon \end{cases}$



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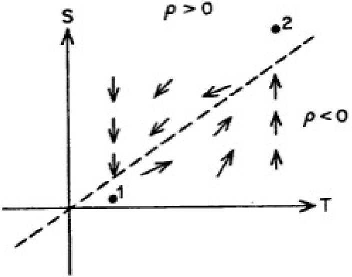
The Nonsmooth Model

The model is linear on each side of the discontinuity. It has two virtual equilibria.



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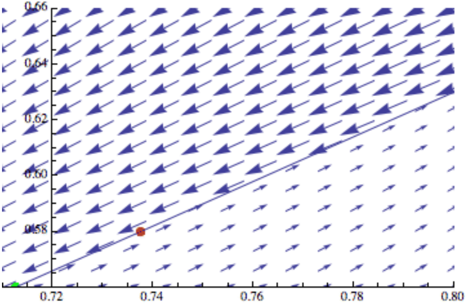
An unstable sliding region takes the role of the unstable equilibrium in the smooth system.

The Nonsmooth Model

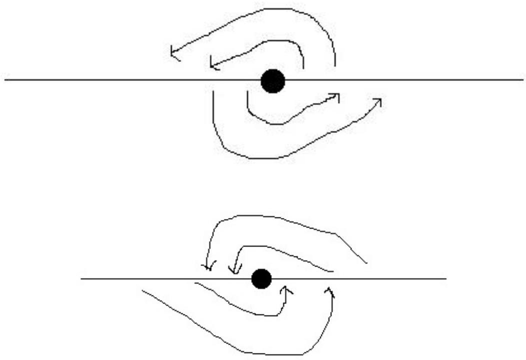
In the smooth model, a Hopf bifurcation was seen when ε varied. In the nonsmooth model, the sliding region transitions from unstable to stable as $\varepsilon = 0$.



The point at which the tangencies collide was studied by Filippov, and is called a fused focus.

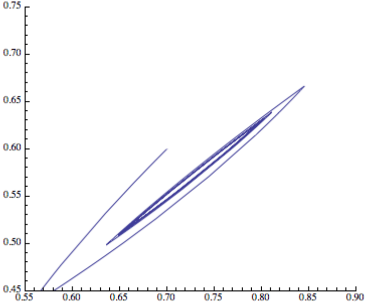
The Fused Focus

Filippov gave conditions on the stability of the fused focus.



Persistence of a Hopf Bifurcation

We see a Hopf bifurcation which corresponds to this transition of the sliding region.



Considerations for a normal form

In the smooth Hopf bifurcation, one looks for eigenvalues passing through the imaginary axis.
In a nonsmooth system, eigenvalues don't even make sense.
Instead we look for changing stability in the sliding region.
To find a normal form, one must worry about topological equivalence in nonsmooth systems, and the possible importance of sliding solutions in that equivalence.
Most of the high powered dynamical systems theorems don't apply.

Summary

- A Hopf bifurcation persists as Welander's smooth model limits to a nonsmooth system
- The Hopf bifurcation in the nonsmooth system is reliant on interactions with the discontinuity boundary, and is not due to the changing stability of an equilibrium
- The generalized bifurcation is however analogous to a bifurcation in a smooth system

References

Di Bernardo, Mario, et al. "Bifurcations in nonsmooth dynamical systems." SIAM review (2008): 629-701.

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Kuznetsov, Yu A., S. Rinaldi, and Alessandra Gragnani. "One-parameter bifurcations in planar Filippov systems." International Journal of Bifurcation and chaos 13.08 (2003): 2157-2188.

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