Hopf bifurcation for Welander’s piecewise smooth model of ocean circulation

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Welander Model

\[ T = k_1 (T_d - T) - k_T \rho^2 \]
\[ S = k_1 (S_d - S) - k_S \rho S \]
\[ \rho = -\alpha T + \beta S \]

The function \( k(\rho) \):

\[ k(\rho) = \begin{cases} 
  k_1 & \rho < \epsilon \\
  k_2 & \rho \geq \epsilon 
\end{cases} \]

Rest point for \( k = 0 \):
\( (T, S) = (1, 1) \)

Rest point for \( k = 1 \):
\( (T, S) = \left( \frac{1}{2}, \frac{(1 + \beta)}{\alpha} \right) \)

Welander chose scientifically reasonable values and dimensionless variables and constants:

\[ T = 1 - k_1 \rho^2 \]
\[ S = \beta (1 - S) - k_S \rho S \]
\[ \rho = -\alpha T + \beta S \]

\[ k(\rho) = \begin{cases} 
  0 & \rho < \epsilon \\
  1 & \rho \geq \epsilon 
\end{cases} \]

\( \alpha = 0.8 \)
\( \beta = 0.5 \)

Pierre Welander, Dynamics of Atmospheres and Oceans 6 (1982).
\[ T = 1 - T' - k(T - T') \]
\[ S = \beta S - kS(1 - S) \]
\[ \rho = -\alpha T + S \]

Welander Model

\( k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ -1, & \rho > \varepsilon \end{cases} \)

\( \Delta S = -\alpha T - x = \rho - \varepsilon \)

\( x = 0 \)

\( x = 0.002 \)

red = temperature
blue = salinity

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Richard McGehee, University of Minnesota
Welander Model

\[ T = 1 - \lambda \rho S \]
\[ S = \theta(1 - S - \lambda \rho T) \]
\[ \rho = \omega T + S \]

\[ \lambda(\rho) = \begin{cases} 0, & \rho < \epsilon \\ \rho - \epsilon, & \rho > \epsilon \end{cases} \]

Fillipov Approach

\[ f(x) = \begin{cases} a(x), & x < a, \\ b(x), & x > a, \end{cases} \]

\[ g(x) = \begin{cases} c(x), & x < c, \\ d(x), & x > c, \end{cases} \]
As ε goes from positive to negative, two invisible tangencies pass through each other. The sliding region goes from attracting to repelling through an attracting fused focus at ε = 0.

Looks like a supercritical Hopf bifurcation.

Follow the solution from when it crosses the discontinuity line until the next time it crosses the line. Use T as a coordinate on the line.

As ε goes from positive to negative, two invisible tangencies pass through each other. The sliding region goes from attracting to repelling through an attracting fused focus at ε = 0.

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Follow the solution from when it crosses the discontinuity line (ε < 0) until the next time it crosses the line.

Use T as a coordinate on the line.
Welander Model

Section Map

\[ T \mapsto \phi(T) \]

\[ \varepsilon = 0 \]

\[ \phi_1 \]

attracting fixed point

\[ \phi_2 \]

attracting sliding interval

\[ \phi_3 \]

repelling sliding interval

\[ \phi_4 \]

repelling fixed point

red = temperature
blue = salinity

time
Hopf Bifurcation Analog

Welander Model

Pair of complex eigenvalues cross the imaginary axis.

Rest point passes from attracting to repelling.

Attracting periodic orbit spins off.

Invisible tangencies cross.

Sliding interval passes from attracting to repelling.

Attracting periodic orbit spins off.

Normal form

Super- or sub-critical

Welander's Smooth Model


References