Hopf bifurcation for Welander's piecewise smooth model of ocean circulation

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Mathematics of Climate Seminar
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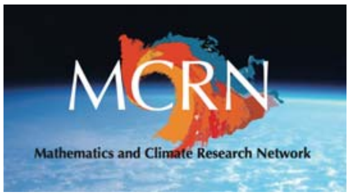


Welander Model

mixing with atmosphere

temperature → $\dot T = k_T(T_A - T) - k(\rho)T$

salinity → $\dot S = k_S(S_A - S) - k(\rho)S$

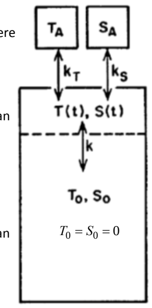
density → $\rho = -\alpha T + \gamma S$

mixing with deep ocean

atmosphere

shallow ocean

deep ocean



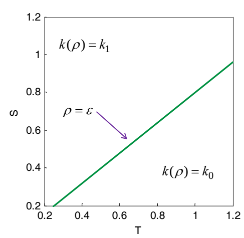
Pierre Welander, *Dynamics of Atmospheres and Oceans* **6** (1982).

Welander Model

$$\dot T = k_T(T_A - T) - k(\rho)T$$
$$\dot S = k_S(S_A - S) - k(\rho)S$$
$$\rho = -\alpha T + \gamma S$$

The function k

$$k(\rho) = \begin{cases} k_0, & \rho < \varepsilon \\ k_1, & \rho > \varepsilon \end{cases}$$



Welander Model

$$\dot T = k_T(T_A - T) - k(\rho)T$$
$$\dot S = k_S(S_A - S) - k(\rho)S$$
$$\rho = -\alpha T + \gamma S$$

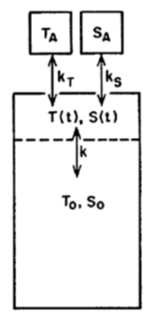
Welander chose scientifically reasonable values and dimensionless variables and constants

$$\dot T = 1 - T - k(\rho)T$$
$$\dot S = \beta(1 - S) - k(\rho)S$$
$$\rho = -\alpha T + S$$

$$k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases}$$

$\alpha = 0.8$

$\beta = 0.5$



Welander Model

$$\dot T = 1 - T - k(\rho)T$$
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$$\rho = -\alpha T + S$$

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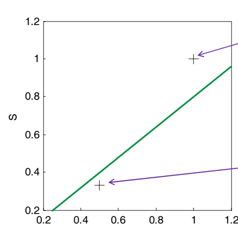


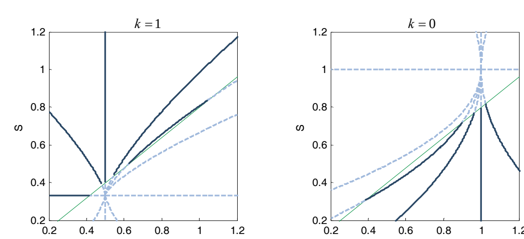
Rest point for $k = 0$: $(T, S) = (1, 1)$

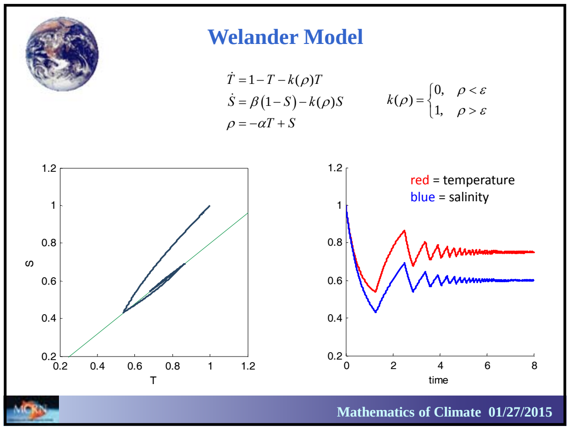
Rest point for $k = 1$: $(T, S) = (1/2, \beta/(1+\beta))$

Welander Model

$$\dot T = 1 - T - k(\rho)T$$
$$\dot S = \beta(1 - S) - k(\rho)S$$
$$\rho = -\alpha T + S$$

$$k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases}$$




Welander Model
$\dot{T} = 1 - T - k(\rho)T$
$\dot{S} = \beta(1-S) - k(\rho)S$
$\rho = -\alpha T + S$
$k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases}$
S
T
time
red = temperature
blue = salinity
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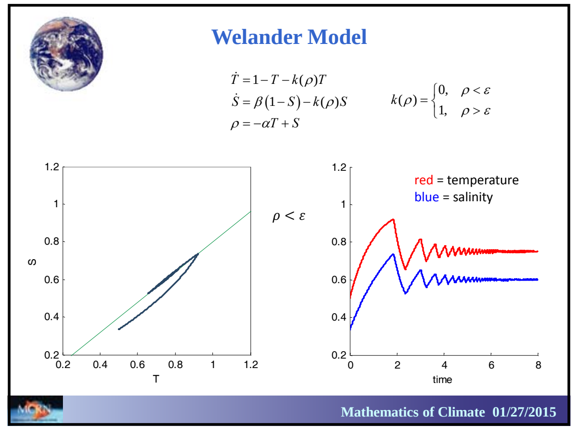
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$\rho < \varepsilon$
S
T
time
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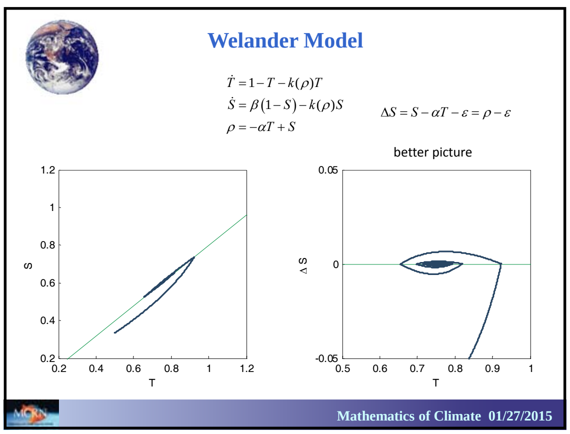
Welander Model
$\dot{T} = 1 - T - k(\rho)T$
$\dot{S} = \beta(1-S) - k(\rho)S$
$\rho = -\alpha T + S$
$\Delta S = S - \alpha T - \varepsilon = \rho - \varepsilon$
better picture
S
T
Δ S
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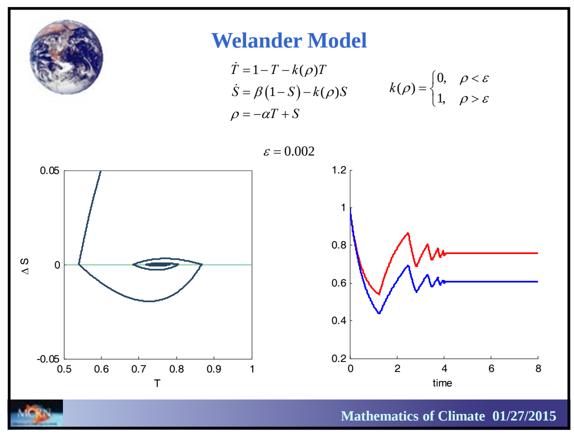
Welander Model
$\dot{T} = 1 - T - k(\rho)T$
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$\rho = -\alpha T + S$
$k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases}$
$\varepsilon = 0.002$
Δ S
T
time
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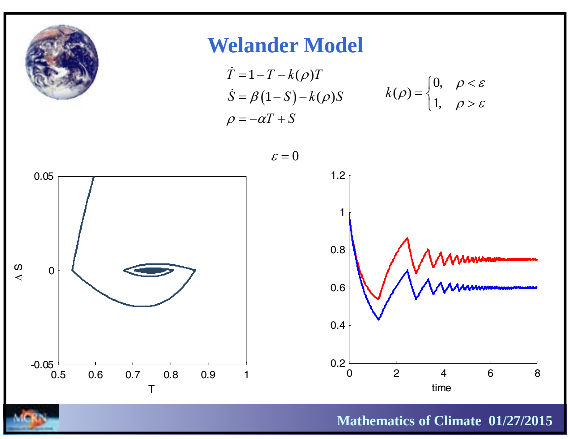
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$\dot{T} = 1 - T - k(\rho)T$
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$\rho = -\alpha T + S$
$k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases}$
$\varepsilon = 0$
Δ S
T
time
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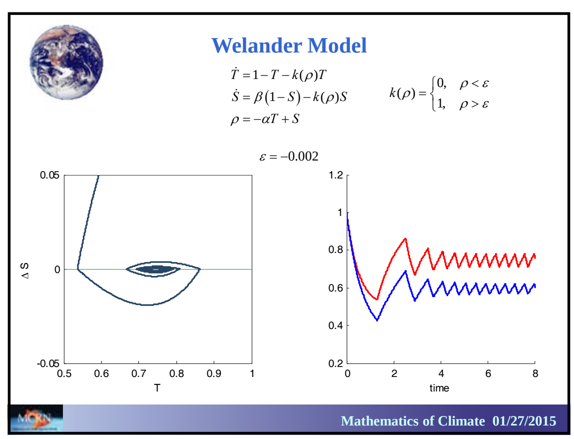
Welander Model
$\dot{T} = 1 - T - k(\rho)T$
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$\rho = -\alpha T + S$
$k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases}$
$\varepsilon = -0.002$
Δ S
T
time
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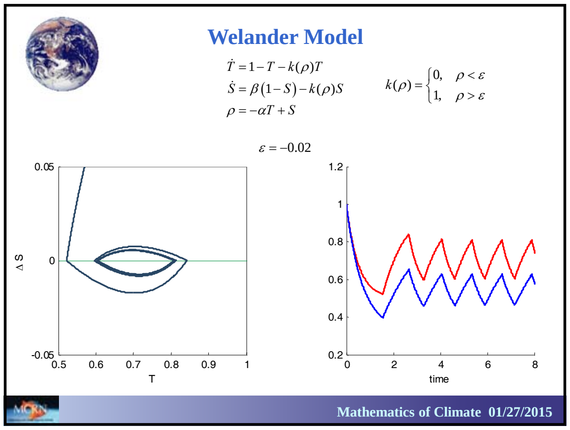
Welander Model
$\dot{T} = 1 - T - k(\rho)T$
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$\rho = -\alpha T + S$
$k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases}$
$\varepsilon = -0.02$
Δ S
T
time
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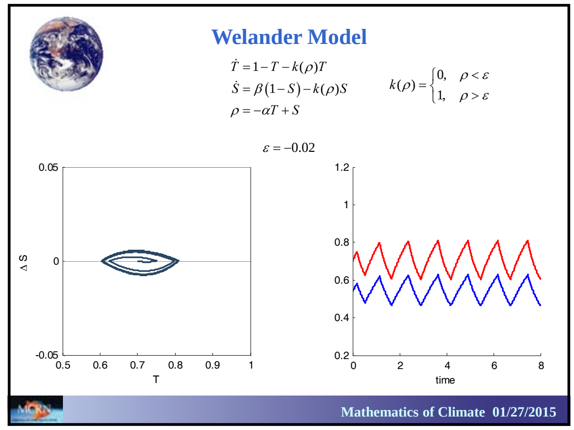
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$\dot{T} = 1 - T - k(\rho)T$
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$\varepsilon = -0.02$
Δ S
T
time
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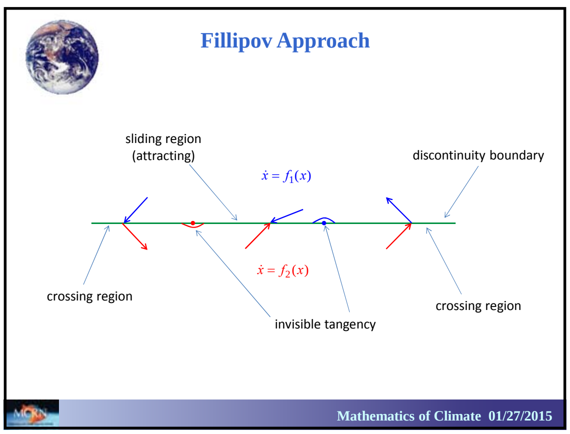
Fillipov Approach
sliding region (attracting)
discontinuity boundary
$\dot{x} = f_1(x)$
$\dot{x} = f_2(x)$
crossing region
invisible tangency
crossing region
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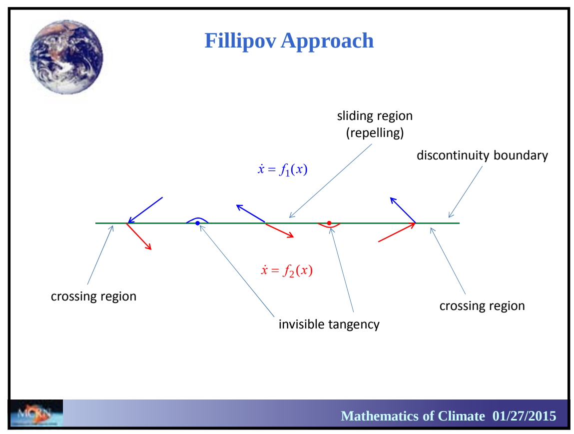
Fillipov Approach
sliding region (repelling)
discontinuity boundary
$\dot{x} = f_1(x)$
$\dot{x} = f_2(x)$
crossing region
invisible tangency
crossing region
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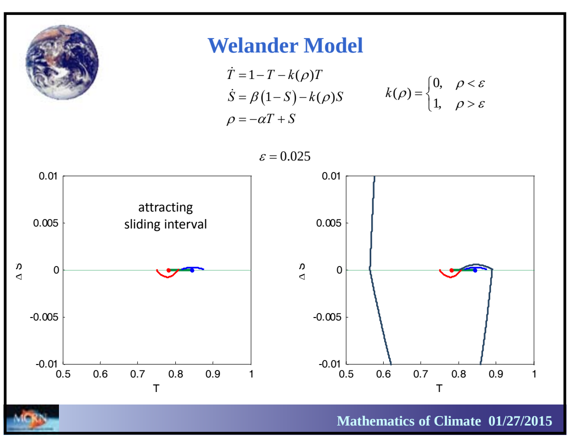
Welander Model
$\dot{T} = 1 - T - k(\rho)T$
$\dot{S} = \beta(1-S) - k(\rho)S$
$\rho = -\alpha T + S$
$k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases}$
$\varepsilon = 0.025$
attracting sliding interval
T
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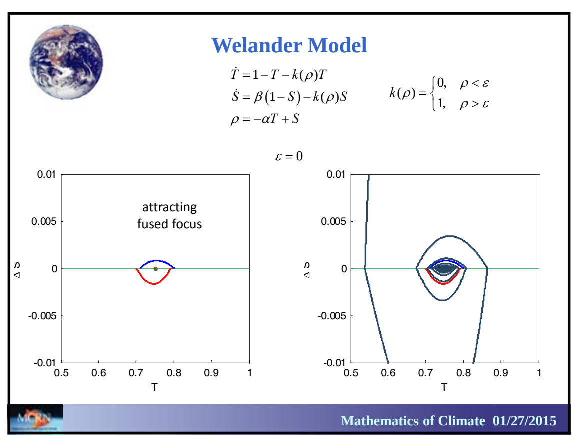
Welander Model
$\dot{T} = 1 - T - k(\rho)T$
$\dot{S} = \beta(1-S) - k(\rho)S$
$\rho = -\alpha T + S$
$k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases}$
$\varepsilon = 0$
attracting fused focus
T
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Welander Model

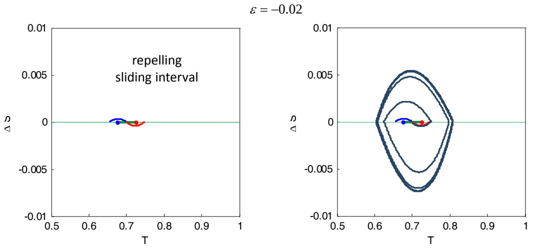
$\dot{T} = 1 - T - k(\rho)T$

$\dot{S} = \beta(1 - S) - k(\rho)S$

$\rho = -\alpha T + S$

$k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases}$

$\varepsilon = -0.02$



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Welander Model

$\dot{T} = 1 - T - k(\rho)T$

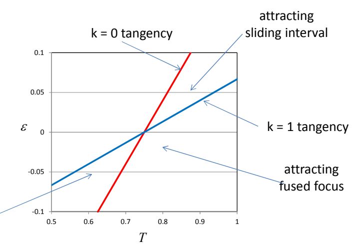
$\dot{S} = \beta(1 - S) - k(\rho)S$

$\rho = -\alpha T + S$

$k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases}$

As ε goes from positive to negative, two invisible tangencies pass through each other. The sliding region goes from attracting to repelling through an attracting fused focus at $\varepsilon = 0$.

Looks like a supercritical Hopf bifurcation.



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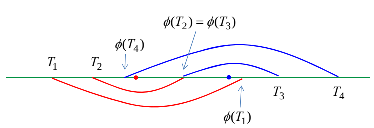
Section Map

Follow the solution from when it crosses the discontinuity line until the next time it crosses the line.

$T \mapsto \phi(T)$

Not 1-1 for an attracting sliding region



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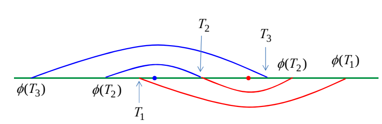
Section Map

$T \mapsto \phi(T)$

Multi-valued for an repelling sliding region



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Welander Model

$\dot{T} = 1 - T - k(\rho)T$

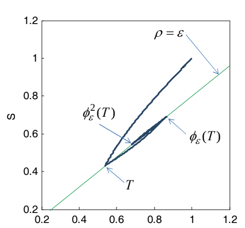
$\dot{S} = \beta(1 - S) - k(\rho)S$

$\rho = -\alpha T + S$

$k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases}$

Follow the solution from when it crosses the discontinuity line ($\rho = \varepsilon$) until the next time it crosses the line. Use T as a coordinate on the line.

$T \mapsto \phi_\varepsilon(T)$



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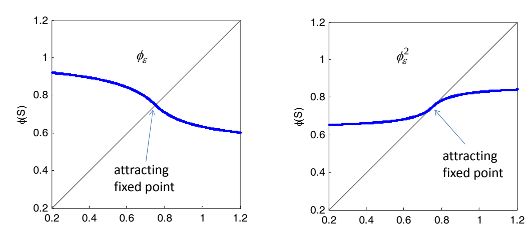
Welander Model

Section Map

$T \mapsto \phi_\varepsilon(T)$

$\varepsilon = 0$



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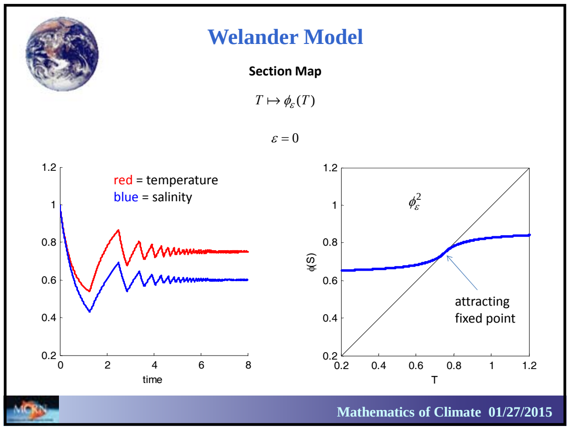
Welander Model
Section Map
$T \mapsto \phi_\varepsilon(T)$
$\varepsilon = 0$
red = temperature
blue = salinity
time
ϕ_ε^2
ϕ(S)
attracting
fixed point
T
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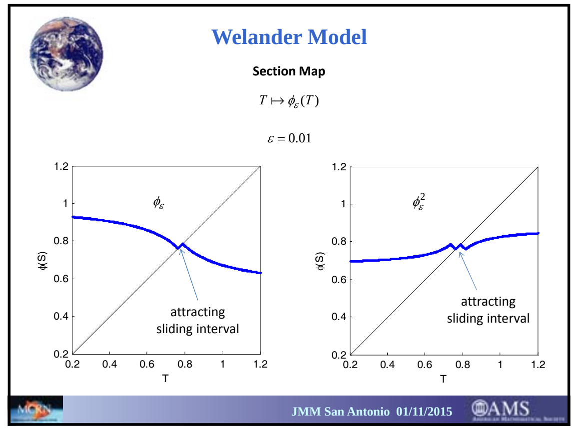
Welander Model
Section Map
$T \mapsto \phi_\varepsilon(T)$
$\varepsilon = 0.01$
ϕ_ε
ϕ(S)
attracting
sliding interval
T
ϕ_ε^2
attracting
sliding interval
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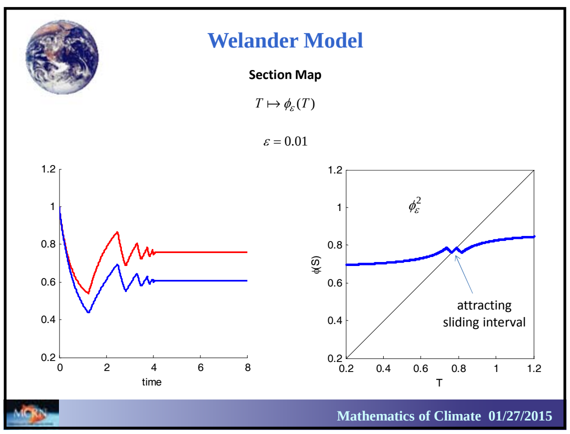
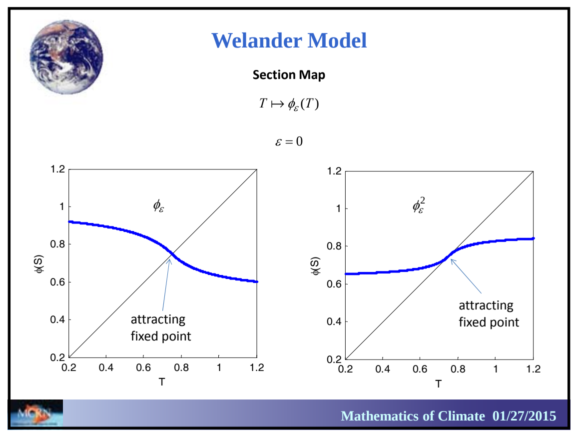

Welander Model
Section Map
$T \mapsto \phi_\varepsilon(T)$
$\varepsilon = 0.01$
time
ϕ_ε^2
ϕ(S)
attracting
sliding interval
T
Mathematics of Climate 01/27/2015


Welander Model
Section Map
$T \mapsto \phi_\varepsilon(T)$
$\varepsilon = 0$
ϕ_ε
ϕ(S)
attracting
fixed point
T
ϕ_ε^2
attracting
fixed point
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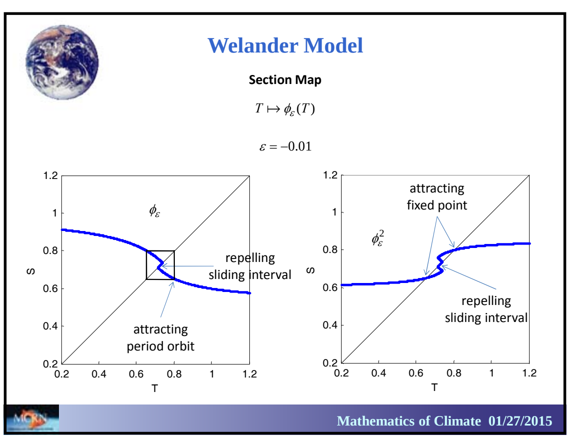
Welander Model
Section Map
$T \mapsto \phi_\varepsilon(T)$
$\varepsilon = -0.01$
ϕ_ε
S
repelling
sliding interval
attracting
period orbit
T
ϕ_ε^2
attracting
fixed point
repelling
sliding interval
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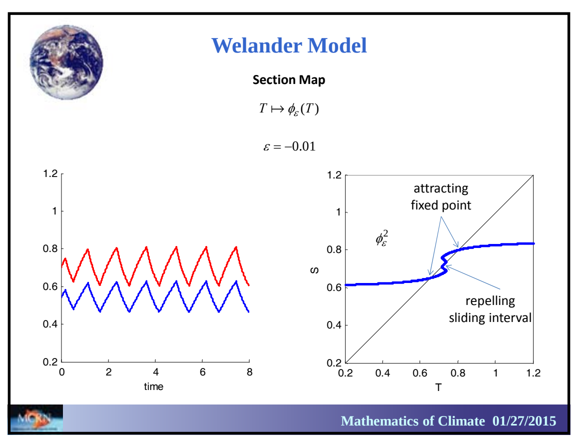
Welander Model
Section Map
$T \mapsto \phi_\varepsilon(T)$
$\varepsilon = -0.01$
time
ϕ_ε^2
S
attracting
fixed point
repelling
sliding interval
T
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Welander Model

Hopf Bifurcation Analog

Pair of complex eigenvalues cross the imaginary axis.	⟺	Invisible tangencies cross.
Rest point passes from attracting to repelling.	⟺	Sliding interval passes from attracting to repelling.
Attracting periodic orbit spins off.	⟺	Attracting periodic orbit spins off.

Welander Model

Hopf Bifurcation Analog

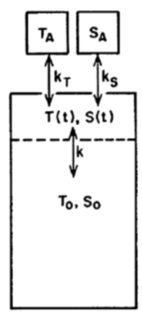
$$\dot{x} = f(x,\mu), \qquad x \in \mathbb{R}^2, \quad \mu \in \mathbb{R}, \quad f(0,\mu) = 0$$

eigenvalues of $D_1 f(0,\mu)$: $\mu \pm i\omega$

Pair of complex eigenvalues cross the imaginary axis.	⟺	Invisible tangencies cross.
Normal form $\dot{z} = (\mu + i\omega)z - c\lvert z\rvert^2 z$	⟺	?
Super- or sub-critical	⟺	?

Welander Model

$$\dot{T} = 1 - T - k(\rho)T$$
$$\dot{S} = \beta(1-S) - k(\rho)S \qquad k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases}$$
$$\rho = -\alpha T + S$$



Welander's Smooth Model

$$k_a(\rho) = \frac{1}{\pi}\tan^{-1}\left(\frac{\rho - \varepsilon}{a}\right) + \frac{1}{2}$$

Note that $k_a \to k$ as $a \to 0$.

Welander Model

Welander's Smooth Model

$$\dot{T} = 1 - T - k_a(\rho)T$$
$$\dot{S} = \beta(1-S) - k_a(\rho)S \qquad k_a(\rho) = \frac{1}{\pi}\tan^{-1}\left(\frac{\rho - \varepsilon}{a}\right) + \frac{1}{2}$$
$$\rho = -\alpha T + S$$

Questions

Is there a Hopf bifurcation for the smooth system?

In what sense does the smooth system limit to the discontinuous system?

Does the behavior of the discontinuous system determine the behavior of the smooth system?

Welander Model

References

Pierre Welander, A simple heat-salt oscillator, *Dynamics of Atmospheres and Oceans* **6** (1982) 233-242.

Yu. A. Kuznetsov, S. Rinaldi and A. Gragnani, One-parameter bifurcations in planar Filippov systems, *International Journal of Bifurcation and Chaos* **13** (2003) 2157-2188.

A. F. Filippov, *Differential Equations with Discontinuous Righthand Sides*, Kluwer, 1988.