Huybers’ Stochastic Glacial Process and Random Circle Maps

Jonathan Hahn

April 2015

You must use Adobe Reader to see the animations in this presentation
Random Circle Homeomorphisms\(^1\)

\((\Omega, \mathcal{F}, \mathbb{P})\) is a probability space

For every \(\omega \in \Omega\), \(f(\omega, \cdot)\) is a circle homeomorphism.

\(^1\text{as designed by C. Rodrigues & P. Ruffino (they cite Ludwig Arnold)}\)
Random Circle Homeomorphisms

$(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space

For every $\omega \in \Omega$, $f(\omega, \cdot)$ is a circle homeomorphism.

Example:

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>$\mathbb{P}(\omega)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.25</td>
</tr>
<tr>
<td>2</td>
<td>.25</td>
</tr>
<tr>
<td>3</td>
<td>.5</td>
</tr>
</tbody>
</table>

$f(1, \cdot)$

$f(2, \cdot)$

$f(3, \cdot)$

$^{1}$as designed by C. Rodrigues & P. Ruffino (they cite Ludwig Arnold)
Random Circle Homeomorphisms

\( \theta : \Omega \rightarrow \Omega \) is a measure-preserving ergodic function w.r.t \( P \).

1. \( P(\theta^{-1}(A)) = P(A), \ A \in \mathcal{F} \)
2. If \( \theta^{-1}(E) = E \), \( P(E) = 0 \) or 1
\( \theta : \Omega \rightarrow \Omega \) is a measure-preserving ergodic function w.r.t \( P \).

1. \( P(\theta^{-1}(A)) = P(A), \ A \in \mathcal{F} \)
2. If \( \theta^{-1}(E) = E \), \( P(E) = 0 \) or \( 1 \)

---

Example:
\( \Omega = \{1, 2, 3, 4\} \)

<table>
<thead>
<tr>
<th>( \omega )</th>
<th>( P(\omega) )</th>
<th>( \theta(\omega) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.25</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>.25</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>.25</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>.25</td>
<td>1</td>
</tr>
</tbody>
</table>

\( f(4, \cdot) = f(3, \cdot) \)
Random Circle Homeomorphisms

“Random dynamics” of $f$:

$$f^n(ω, x) = f(θ^{n−1}ω, ·) ∘ ··· ∘ f(θω, ·) ∘ f(ω, x)$$

“cocycle”
Example (cycle)
Example (no cycle?)
Random Circle Homeomorphisms

Make $F(\omega, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ the lift of $f(\omega, \cdot)$ such that $F(\omega, 0) \in [0, 1)$. 
Example (Lift Iteration)
Random Circle Homeomorphisms

\[ F^n(\omega, x) = F(\theta^{n-1}\omega, \cdot) \circ \cdots \circ F(\theta\omega, \cdot) \circ F(\omega, x) \]

The rotation number is:

\[ \rho(F, \theta, \omega, x) = \lim_{n \to \infty} \frac{F^n(\omega, x) - x}{n} \]
Theorem: If $\rho$ exists, it does not depend on the starting point, $x$. 
Random Circle Homeomorphisms

Let $\Theta : \Omega \times S^1 \rightarrow \Omega \times S^1$, $\Theta^n(\omega, x) = (\theta^n\omega, f^n(\omega, x))$.

Let $\mu = \mathbb{P}(d\omega)\nu_\omega(ds)$ be an invariant probability measure w.r.t. $\Theta$.

Theorem:

$$\rho = \lim_{n \to \infty} \frac{F^n(\omega, x) - x}{n} = \mathbb{E} \int_{S^1} (F^n(\omega, x) - x) \nu_\omega(s) ds \quad \mathbb{P}\text{-a.s.}$$

where $x = \pi^{-1}(s)$

This is due to Birkhoff’s ergodic theorem. It implies that $\rho$ is independent of $\omega$.

Is $\rho$ independent of $\theta$? Is it easy (or even possible) to find $\mu$?
Let $\Theta : \Omega \times S^1 \rightarrow \Omega \times S^1$, $\Theta^n(\omega, x) = (\theta^n\omega, f^n(\omega, x))$.

Let $\mu = \mathbb{P}(d\omega)\nu_\omega(ds)$ be an invariant probability measure w.r.t. $\Theta$.

Theorem:

$$\rho = \lim_{n \to \infty} \frac{F^n(\omega, x) - x}{n} = \mathbb{E} \int_{S^1} (F^n(\omega, x) - x) \nu_\omega(s) ds \quad \mathbb{P}\text{-a.s.}$$

where $x = \pi^{-1}(s)$

This is due to Birkhoff’s ergodic theorem.

It implies that $\rho$ is independent of $\omega$.

Is $\rho$ independent of $\theta$?

Is it easy (or even possible) to find $\mu$?
Huybers’ Model of Ice Volume

V: ice volume  
\( \eta \): growth of ice (a random variable)  
T: deglaciation threshold  
t: time in years (0, 1, 2, ...)

\[
V_t = V_{t-1} + \eta_t \quad \text{if } V_t \geq T_t, \text{ terminate}
\]

\[
T_t = at + b + \theta(t)
\]
Huybers’ Model of Ice Volume

$V$: ice volume

$T$: deglaciation threshold

$\eta$: growth of ice (a random variable)

$t$: time in years ($0, 1, 2, ...$)

$V_t = V_{t-1} + \eta_t$ if $V_t \geq T_t$, terminate

$T_t = at + b + \theta(t)$

- “terminate”: reset $V_t$ linearly to zero (over 10,000 years)
Huybers’ Model of Ice Volume

\( V: \) ice volume \hspace{1cm} \( T: \) deglaciation threshold

\( \eta: \) growth of ice (a random variable) \hspace{1cm} \( t: \) time in years(0, 1, 2, ...)

\[
V_t = V_{t-1} + \eta_t \quad \text{if} \quad V_t \geq T_t, \quad \text{terminate}
\]

\[
T_t = at + b + \theta(t)
\]

- “terminate”: reset \( V_t \) linearly to zero (over 10,000 years)
- \( V_t \) is a discrete stochastic process
Huybers’ Model of Ice Volume

$V$: ice volume  \hspace{1cm}  T$: deglaciation threshold
$
\eta$: growth of ice (a random variable) \hspace{1cm}  t$: time in years(0, 1, 2, ...)

\[
V_t = V_{t-1} + \eta_t \quad \text{if} \quad V_t \geq T_t, \quad \text{terminate}
\]

\[
T_t = at + b + \theta(t)
\]

• “terminate”: reset $V_t$ linearly to zero (over 10,000 years)
• $V_t$ is a discrete stochastic process
• $V_t$ is not a Markov process
Example Run
Huybers’ Simplified

\[ V_t = V_{t-1} + \eta_t \quad \text{if} \quad V_t \geq T_t, \quad \text{terminate} \]

\[ T_t = at + b + \theta(t) \]

• Make \( T_t \) periodic with a period of \( N \) years.

\[ V_t = V_{t-1} + \eta_t \quad \text{if} \quad V_t \geq T_t, \quad \text{terminate} \]

\[ T_t = b + \sin\left(\frac{t}{2\pi N}\right) \]
Define $g$ to be the map sending a termination time to the next one.

Suppose $U_{t_0}(t)$ is the volume $V_t$ with initial condition $V_{t_0} = 0$.

$$g(t_0) = \min\{ t > t_0 : U_{t_0}(t) \geq T_t \} + 10000$$
Define $g$ to be the map sending a termination time to the next one.

Suppose $U_{t_0}(t)$ is the volume $V_t$ with initial condition $V_{t_0} = 0$.

$$g(t_0) = \min\{t > t_0 : U_{t_0}(t) \geq T_t\} + 10000$$

Alternatively, let $V_t$ have the initial condition $V_0 = 0$:

$$g(t_0) = t_0 + \min\{t > 0 : V_t \geq T_{t+t_0}\} + 10000$$
Define $g$ to be the map sending a termination time to the next one.

Suppose $U_{t_0}(t)$ is the volume $V_t$ with initial condition $V_{t_0} = 0$.

$$g(t_0) = \min\{ t > t_0 : U_{t_0}(t) \geq T_t \} + 10000$$

Alternatively, let $V_t$ have the initial condition $V_0 = 0$:

$$g(t_0) = t_0 + \min\{ t > 0 : V_t \geq T_{t+t_0} \} + 10000$$

Since $T$ is periodic,

$$g(t + N) = g(t) + N$$
Example of Return Map
Huybers’ Made Continuous

Theorem:
Let $X \sim \mathcal{N}(\mu_1, \sigma_1^2)$, $Y \sim \mathcal{N}(\mu_2, \sigma_2^2)$.
Then $X + Y \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$
Suppose $\eta_t \sim \mathcal{N}(\mu, \sigma^2)$. Then $\eta_1 + \eta_2 + \ldots + \eta_t \sim \mathcal{N}(t\mu, t\sigma^2)$

If $V_0 = 0$, $V_t \sim \mathcal{N}(t\mu, t\sigma^2)$
Suppose $\eta_t \sim \mathcal{N}(\mu, \sigma^2)$. Then $\eta_1 + \eta_2 + \ldots + \eta_t \sim \mathcal{N}(t\mu, t\sigma^2)$

If $V_0 = 0$, $V_t \sim \mathcal{N}(t \mu, t \sigma^2)$

What continuous stochastic process has this property?
Suppose $\eta_t \sim \mathcal{N}(\mu, \sigma^2)$. Then $\eta_1 + \eta_2 + \ldots + \eta_t \sim \mathcal{N}(t\mu, t\sigma^2)$

If $V_0 = 0$, $V_t \sim \mathcal{N}(t\mu, t\sigma^2)$

What continuous stochastic process has this property?

- Brownian motion with a drift (A Gaussian Lévy Process)
Suppose $\eta_t \sim \mathcal{N}(\mu, \sigma^2)$. Then $\eta_1 + \eta_2 + \ldots + \eta_t \sim \mathcal{N}(t\mu, t\sigma^2)$.

If $V_0 = 0$, $V_t \sim \mathcal{N}(t\mu, t\sigma^2)$

What continuous stochastic process has this property?

- Brownian motion with a drift (A Gaussian Lévy Process)
- $V_t = \mu t + \sigma W_t$ where $W_t$ is Brownian motion.
Example Run
If $V_t = \mu t + \sigma W_t$, we define the return map $g$ by

$$g(t_0) = t_0 + \min\{t > 0 : V_t \geq T_{t+t_0}\}$$
Circle Map

If $V_t = \mu t + \sigma W_t$, we define the return map $g$ by

$$g(t_0) = t_0 + \min\{t > 0 : V_t \geq T_{t+t_0}\}$$

$g$ induces a discrete Markov process $(X_0, X_1, X_2, \ldots)$ of termination times:

$$P[X_{n+1} \in A|X_n] = P[g(X_n) \in A]$$
If $V_t = \mu t + \sigma W_t$, we define the return map $g$ by

$$g(t_0) = t_0 + \min\{t > 0 : V_t \geq T_{t+t_0}\}$$

$g$ induces a discrete Markov process $(X_0, X_1, X_2, \ldots)$ of termination times:

$$\mathbb{P}[X_{n+1} \in A|X_n] = \mathbb{P}[g(X_n) \in A]$$

“Rotation number” for $g$:

$$\rho(g) = \lim_{n \to \infty} \frac{X_n - X_0}{n}$$
“pdf” of $g$
Does this stochastic process circle map make sense?
How does $\rho(g)$ depend on $T$, $\mu$, and $\sigma$?
Can we reconcile these definitions of random circle maps?
What about random circle maps which are not homeomorphisms?