Huybers' Stochastic Glacial Process and Random Circle Maps

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Random Circle Homeomorphisms[1]

$(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space

For every $\omega \in \Omega$, $f(\omega, \cdot)$ is a circle homeomorphism.

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[1] as designed by C. Rodrigues & P. Ruffino (they cite Ludwig Arnold)

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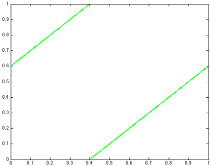
For every $\omega \in \Omega$, $f(\omega, \cdot)$ is a circle homeomorphism.

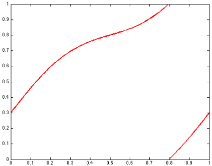
Example:
$\Omega = \{1, 2, 3\}$

ω	$\mathbb{P}(\omega)$
1	.25
2	.25
3	.5

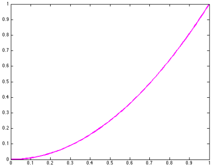
$f(1, \cdot)$



$f(2, \cdot)$



$f(3, \cdot)$



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Random Circle Homeomorphisms

$\theta : \Omega \to \Omega$ is a measure-preserving ergodic function w.r.t P.

1. $\mathbb{P}(\theta^{-1}(A)) = \mathbb{P}(A)$, $A \in \mathcal{F}$
2. If $\theta^{-1}(E) = E$, $\mathbb{P}(E) = 0$ or 1

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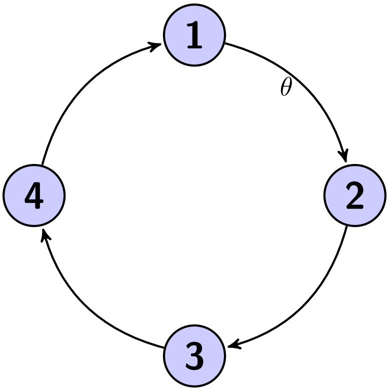
Example:
$\Omega = \{1, 2, 3, 4\}$

ω	$\mathbb{P}(\omega)$	$\theta(\omega)$
1	.25	2
2	.25	3
3	.25	4
4	.25	1

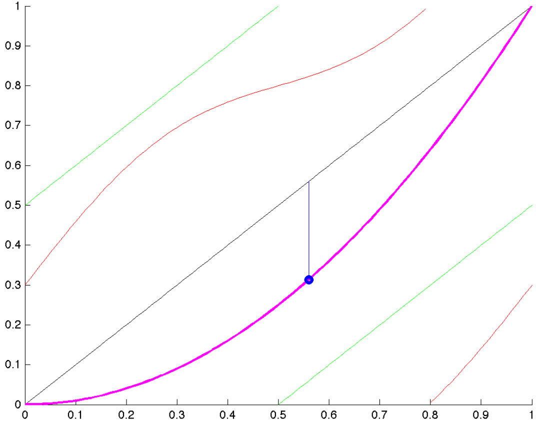
$f(4, \cdot) = f(3, \cdot)$



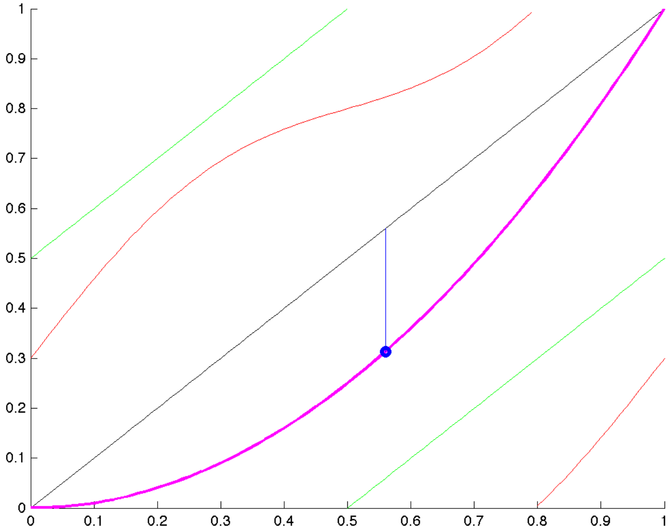
Random Circle Homeomorphisms

"Random dynamics" of f:

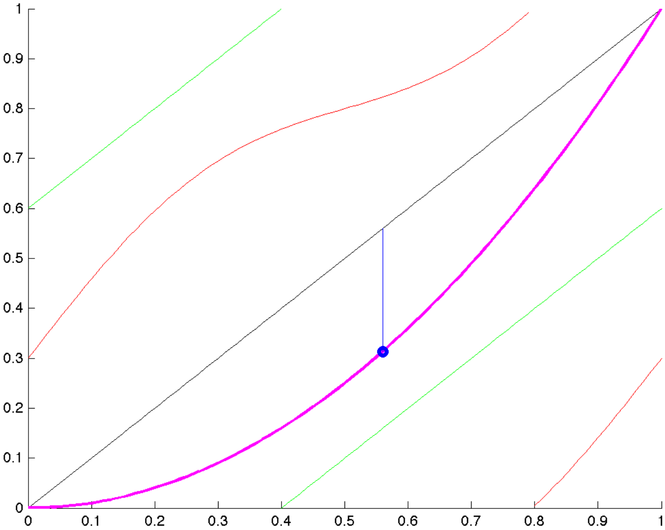
$$f^n(\omega, x) = f(\theta^{n-1}\omega, \cdot) \circ \cdots \circ f(\theta\omega, \cdot) \circ f(\omega, x) \qquad \text{"cocycle"}$$



Example (cycle)

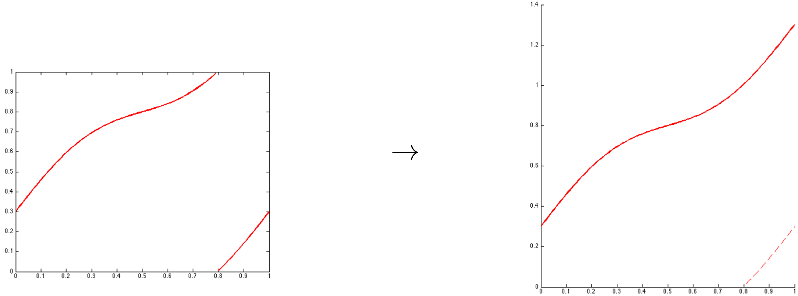


Example (no cycle?)

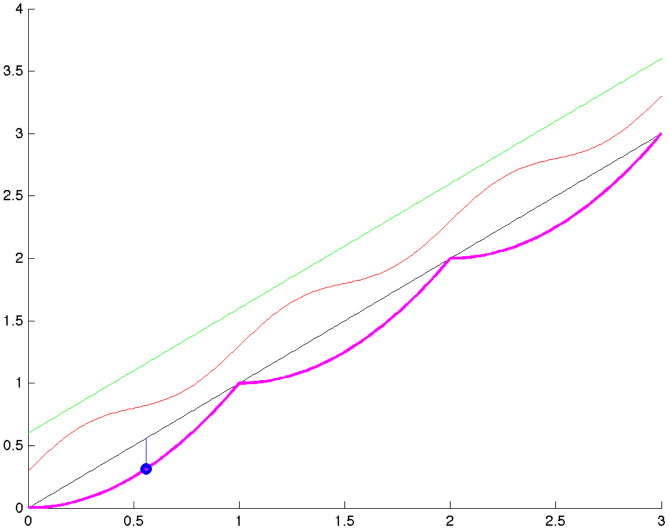


Random Circle Homeomorphisms

Make $F(\omega, \cdot) : \mathbb{R} \to \mathbb{R}$ the lift of $f(\omega, \cdot)$ such that $F(\omega, 0) \in [0, 1)$.



Example (Lift Iteration)



Random Circle Homeomorphisms

$$F^n(\omega, x) = F(\theta^{n-1}\omega, \cdot) \circ \cdots \circ F(\theta\omega, \cdot) \circ F(\omega, x)$$

The rotation number is:

$$\rho(F, \theta, \omega, x) = \lim_{n \to \infty} \frac{F^n(\omega, x) - x}{n}$$

Random Circle Homeomorphisms

Theorem: If ρ exists, it does not depend on the starting point, x.

Random Circle Homeomorphisms

Let $\Theta : \Omega \times \mathbb{S}^1 \to \Omega \times \mathbb{S}^1$, $\Theta^n(\omega, x) = (\theta^n\omega, f^n(\omega, x))$.

Let $\mu = \mathbb{P}(d\omega)\nu_\omega(ds)$ be an invariant probability measure w.r.t. Θ.

Theorem:

$$\rho = \lim_{n\to\infty} \frac{F^n(\omega, x) - x}{n} = \mathbb{E} \int_{\mathbb{S}^1} \big(F^n(\omega, x) - x\big)\nu_\omega(s)ds \qquad \mathbb{P}\text{-a.s.}$$

where $x = \pi^{-1}(s)$

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This is due to Birkhoff's ergodic theorem.
It implies that ρ is independent of ω.

Is ρ independent of θ?
Is it easy (or even possible) to find μ?

Huybers' Model of Ice Volume

V: ice volume

η: growth of ice (a random variable)

T: deglaciation threshold

t: time in years$(0, 1, 2, ...)$

$$V_t = V_{t-1} + \eta_t \qquad \text{if } V_t \geq T_t, \text{ terminate}$$
$$T_t = at + b + \theta(t)$$

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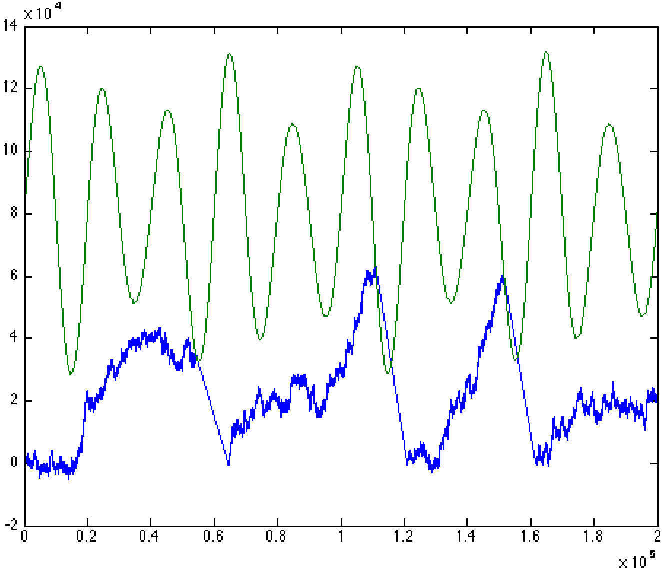
- "terminate": reset V_t linearly to zero (over 10,000 years)
- V_t is a discrete stochastic process
- V_t is not a Markov process

Example Run



Huybers' Simplified

$$V_t = V_{t-1} + \eta_t \qquad \text{if } V_t \geq T_t, \text{ terminate}$$
$$T_t = at + b + \theta(t)$$

- Make T_t periodic with a period of N years.

$$V_t = V_{t-1} + \eta_t \qquad \text{if } V_t \geq T_t, \text{ terminate}$$
$$T_t = b + \sin(\frac{t}{2\pi N})$$

Reduction to a “Circle” Map

Define g to be the map sending a termination time to the next one.

Suppose $U_{t_0}(t)$ is the volume V_t with initial condition $V_{t_0} = 0$.

$$g(t_0) = \min\{t > t_0 : U_{t_0}(t) \geq T_t\} + 10000$$

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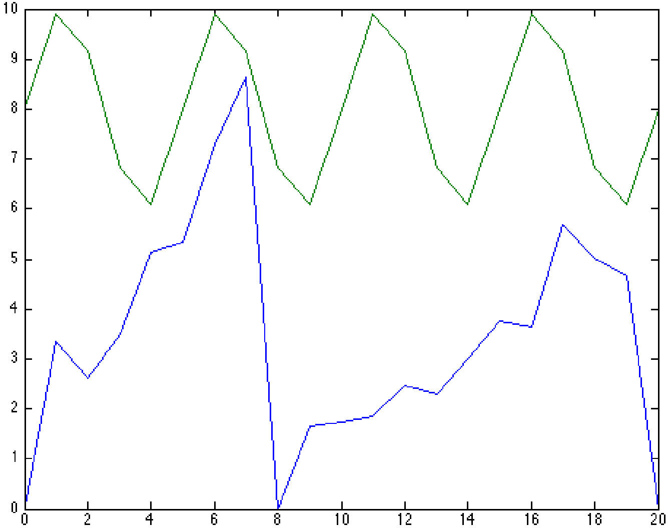
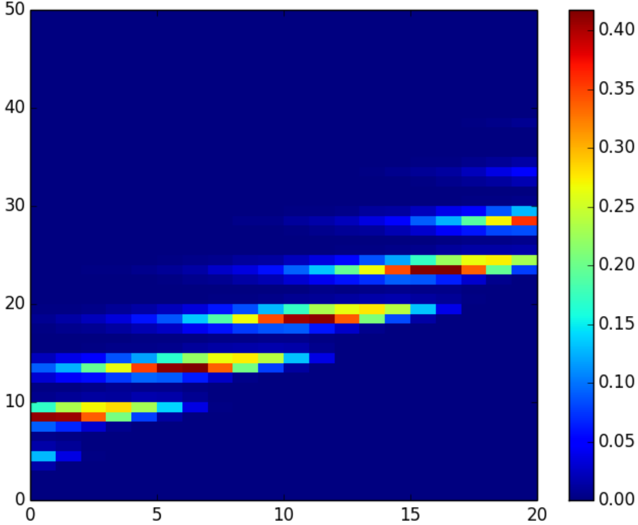
Since T is periodic,

$$g(t + N) = g(t) + N$$

Small Example Run



Example of Return Map



Huybers' Made Continuous

Theorem:

Let $X \sim \mathcal{N}(\mu_1, \sigma_1^2), Y \sim \mathcal{N}(\mu_2, \sigma_2^2)$.

Then $X + Y \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

Huybers' Made Continuous

Suppose $\eta_t \sim \mathcal{N}(\mu, \sigma^2)$. Then $\eta_1 + \eta_2 + ... + \eta_t \sim \mathcal{N}(t\mu, t\sigma^2)$

If $V_0 = 0$, $V_t \sim \mathcal{N}(t\mu, t\sigma^2)$

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What continuous stochastic process has this property?

- Brownian motion with a drift (A Gaussian Lévy Process)

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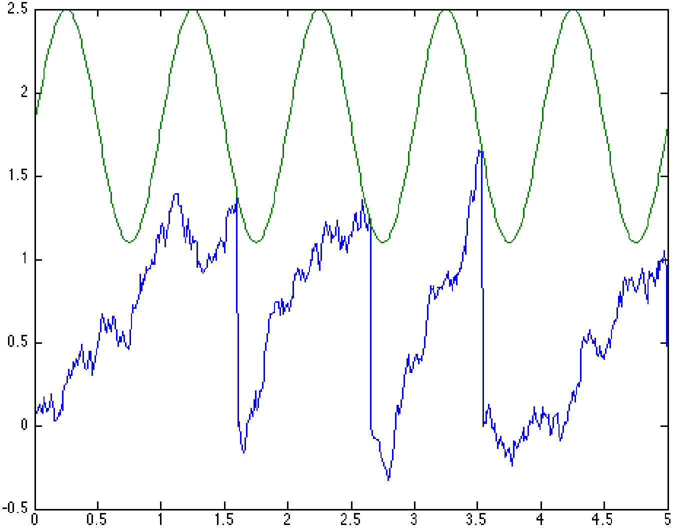
What continuous stochastic process has this property?

- Brownian motion with a drift (A Gaussian Lévy Process)
- $V_t = \mu t + \sigma W_t$ where W_t is Brownian motion.

Example Run



Circle Map

If $V_t = \mu t + \sigma W_t$, we define the return map g by

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g induces a discrete Markov process $(X_0, X_1, X_2, ...)$ of termination times:

$$\mathbb{P}[X_{n+1} \in A | X_n] = \mathbb{P}[g(X_n) \in A]$$

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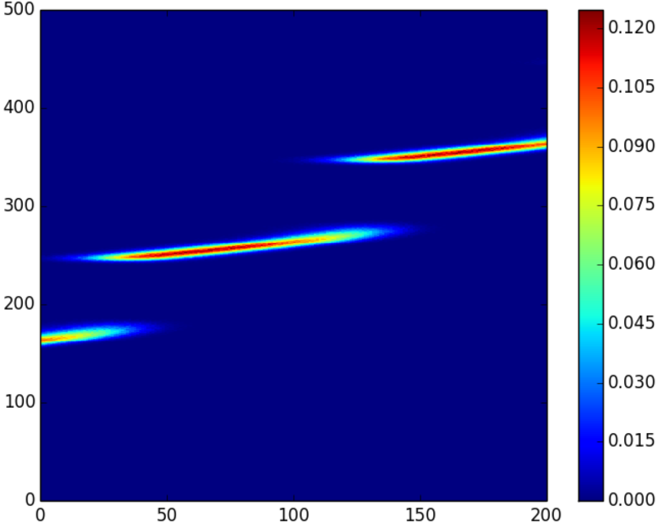
"Rotation number" for g:

$$\rho(g) = \lim_{n\to\infty} \frac{X_n - X_0}{n}$$

“pdf” of g


500
400
300
200
100
0
0
50
100
150
200
0.120
0.105
0.090
0.075
0.060
0.045
0.030
0.015
0.000

?

Does this stochastic process circle map make sense?
How does $\rho(g)$ depend on T, μ, and σ?
Can we reconcile these definitions of random circle maps?
What about random circle maps which are not homeomorphisms?